Introduction to Topos Theory

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Contents.

Section 1 Introduction

· **Definition 1.1.1** (Lawvere–Tierney, 1969) An *elementary topos* is a category $\mathcal{E}$ s.t.

(ET1) $\mathcal{E}$ is finitely complete ($\overset{\text{not def}}{\Leftrightarrow}$ $\mathcal{E}$ has terminal object and pullbacks)

(ET2) $\mathcal{E}$ is finitely co-complete ($\Leftrightarrow$ $\mathcal{E}$ has initial object and pushouts)

(ET3) $\mathcal{E}$ has exponentials

(ET4) $\mathcal{E}$ has a subobject classifier

· $ET_1 + ET_3 \Leftrightarrow \mathcal{E}$ is cartesian closed

· **Theorem 1.1.2** (Mikkelsen) $ET_1 + ET_3 + ET_4 \vdash ET_2$

· **Corollary 1.1.3** A *topos* (the term elementary can be omitted) is a cartesian-closed category with a subobject classifier

There is another def based on a cat-characterization of power sets

Theorem 1.1.4 (Mikkelsen) A cat $\mathcal{C}$ is a topos $\Leftrightarrow$ $\mathcal{C}$ is finitely complete and has power objects

· Eilenberg – MacLane 1945 : Introduction of the basic notions of Cat Theory

Origin of Cat Th: Homology theory (Alg Topology)

· The concept of arrow (an abstraction of the notion of function) replaces "$x \in A$" and becomes the building block of math

· arrow $f : a \to b$, $x \in A$; notion of function

point-free approach — point-approach to math

external approach to the notion of math objects — internal structure of the set

· A function-arrow is fundamental and not reduced, independent, from the notion of set

· In ZF a function f is a set of ordered pairs st $\langle x, y\rangle \in f \wedge \langle x, z\rangle \in f \to y = z$

f is identified to its graph

· f is a function from A to B if $A = \mathrm{dom}(f)$ and $B \supseteq \mathrm{Im}(f)$

$\{x \mid \exists y : \langle x,y\rangle \in f\}$ $\quad$ $\{y \mid \exists x : \langle x,y\rangle \in f\}$

Hence no unique codomain

· Ex $\quad id_A : A \to A$, $A \subseteq B$ $\quad$ conceptually different objects, but equal sets of ordered pairs

$id_A : A \hookrightarrow B$

· Way out : $f = (A, B, R)$ where $R = Gr(f)$ i.e. $\mathrm{dom}(f)$, $\mathrm{cod}(f)$ part of the def of f

- Spec f in this way is static. It doesn't convey the "operational" character of f. f is a dynamical concept, which generally represents a kind of algorithm $x \mapsto f(x)$.

- BHK, MLTT, Cat Th., Th. Comp. Science: function is fundamental, independent from sets, types, spaces

- **Set**: the cat of sets with arrows the functions between them
The basic set-theoretic notions (ZF/ZFC) can be described through the arrows in Set, and these descriptions can be interpreted in any category.

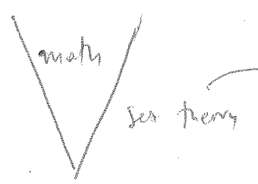
- <u>Main Question</u> (<u>Lawvere</u> 1964) What conditions must a cat satisfy in order to be "essentially the same" to Set? First attempt to answer (not satisfying, not without Set)

- Late 60's: <u>Grothendieck</u> (alg. geometry) The collection of sheaves over a top. space forms a cat. They called a generalization of this notion of sheaf over a top. space, topos

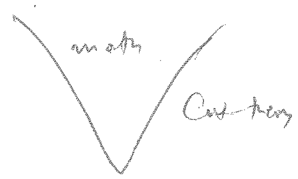
- <u>Lawvere + Tierney</u> 1969:
  - They studied cats with a special arrow: the <u>subobject classifier</u>, generalizing characteristic functions in Set.
    The ~~main~~ key to the main question
  - They discovered the Grothendieck topoi had subobject classifiers, and took over the name topos.
  - The notion of topos was formulated (Def 1971) within Cat-Th. and independently from set theory.

- <u>Mitchell</u> (1972), <u>Cole</u> (1973): Full answer to Main Question.

<u>Consequence 1</u>:



characterization of Set within CatTheory
axiomatization of set th. within Cat (notion of topos)



"Categorical foundation of math".

<u>Consequence 2</u>: TT (Topos Theory) unified set-theory and alg. geometry (Groth).

<u>Consequence 3</u>: Each topos comes with a mathematical algebra (Heyt-Bool), its own logic. In general this logic is INTUITIONISTIC!!

Section 1.2 Categories

Definition 1.2.1: A category is a structure $\mathcal{C} = (\overset{Ob_{\mathcal{C}}}{\mathcal{C}\text{-objects}}, \overset{Arr_{\mathcal{C}}}{\mathcal{C}\text{-arrows}}, dom_{\mathcal{C}}, cod_{\mathcal{C}}, \circ_{\mathcal{C}})$

where: (i) $Ob_{\mathcal{C}}$ is a collection of $\mathcal{C}$-objects, the objects of $\mathcal{C}$.

(ii) $Arr_{\mathcal{C}}$ — " — $\mathcal{C}$-arrows, the arrows of $\mathcal{C}$

(iii) $dom_{\mathcal{C}} : Arr_{\mathcal{C}} \to Ob_{\mathcal{C}} \qquad f \mapsto dom_{\mathcal{C}}(f)$

$cod_{\mathcal{C}} : Arr_{\mathcal{C}} \to Ob_{\mathcal{C}} \qquad f \mapsto cod_{\mathcal{C}}(f)$

If $a, b : Ob_{\mathcal{C}}$ (to denote that these collections are not sets)

and $a \equiv dom_{\mathcal{C}}(f)$, $b \equiv cod_{\mathcal{C}}(f)$, $f : Arr_{\mathcal{C}}$ we use the notation

$f : a \to b$, or $a \xrightarrow{f} b$

(iv) If $g, f : Arr_{\mathcal{C}}$ s.t. $dom_{\mathcal{C}}(g) = cod_{\mathcal{C}}(f)$

$\circ_{\mathcal{C}} : Arr_{\mathcal{C}} \times Arr_{\mathcal{C}} \to Arr_{\mathcal{C}}$

$(g, f) \mapsto g \circ_{\mathcal{C}} f$ (or simply $g \circ f$)

s.t. $dom_{\mathcal{C}}(g \circ_{\mathcal{C}} f) \equiv dom_{\mathcal{C}}(f)$

$cod_{\mathcal{C}}(g \circ f) \equiv cod_{\mathcal{C}}(g)$

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "g\circ f"'] & b \arrow[d, "g"] \\ & c \end{tikzcd}$$

associative law

s.t. the following condition holds:

If $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$, then $h \circ (g \circ f) = (h \circ g) \circ f$

or

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, shift right, "h\circ(g\circ f)"'] \arrow[d, shift left, "(h\circ g)\circ f"] \arrow[dr, "g\circ f"] & b \arrow[d, "g"] \arrow[dl, "h\circ g"'] \\ d & c \arrow[l, "h"] \end{tikzcd}$$

commutes

(v) $1 : Ob_{\mathcal{C}} \to Arr_{\mathcal{C}}$ (assignment)

$a \mapsto 1_a$, where $1_a : a \to a$ the identity arrow on a

s.t. the following identity law is satisfied:

If $f: a \to b$ and $g: c \to a$

$$f \circ 1_a = f \quad \text{and} \quad 1_a \circ g = g$$

i.e., the following diagram commutes "TFDC":

$$\begin{tikzcd}
c \arrow[r, "g"] \arrow[dr, "g"'] & a \arrow[d, "1_a"] \arrow[dr, "f"] & \\
 & a \arrow[r, "f"'] & b
\end{tikzcd}
\quad , \text{ or } \quad
\begin{tikzcd}
c \arrow[r, "g"] \arrow[rr, bend left, "g"] & a \arrow[r, "1_a"] \arrow[rr, bend right, "f"'] & a \arrow[r, "f"] & b
\end{tikzcd}$$

Note: Ob_C need not be sets, Arr_C need not be functions, $\circ_C$ need not be composition of functions

Examples 1.2.2: (a) $\mathbb{1} = (\{a\}, \{1_a\}, dom_1, cod_1, \circ_1)$

$$\begin{tikzcd}
a \arrow[loop above, "1_a"]
\end{tikzcd}$$

- a can be set and 1_a its identity function
- a any kind object and 1_a an arrow with $dom(1_a) = cod(1_a) = a$.

There is one only category with one object and one arrow.

(b) $\mathbb{2} = (\overset{\text{collection}}{\{a,b\}}, \{1_a, 1_b, f\}, cod_2, dom_2, \circ_2)$

$$\begin{tikzcd}
a \arrow[loop above, "1_a"] \arrow[r, "f"] & b \arrow[loop above, "1_b"]
\end{tikzcd}$$

The arrows can be represented by $\overset{1_a}{\overset{\parallel}{(a,a)}}, \overset{f}{\overset{\parallel}{(a,b)}}, \overset{1_b}{\overset{\parallel}{(b,b)}}$, the obvious defs of cod_2, dom_2, $\circ_2$

(c) $\mathbb{3} = (\{a,b,c\}, \{1_a, 1_b, 1_c, f, g, h\}, dom_3, cod_3, \circ_3)$ —"—

$$\begin{tikzcd}
a \arrow[loop above, "1_a"] \arrow[r, "f"] \arrow[dr, "h"'] & b \arrow[loop above, "1_b"] \arrow[d, "g"] \\
 & c \arrow[loop below, "1_c"]
\end{tikzcd}$$

In (a) – (c) between any two objects there is one at most arrow

$\Rightarrow$ composition $\circ_C$ is defined in a unique way.

Definition 1.2.3 A category $\mathcal{C}$ is called a <u>pre-order</u> iff between any two $a, b : Ob_{\mathcal{C}}$ there is <u>at most one</u> arrow $f : a \to b$.

- If $\mathcal{C}$ is a pre-order we define $\leq \; \subseteq Ob_{\mathcal{C}} \times Ob_{\mathcal{C}}$

$$\boxed{a \leq_{\mathcal{C}} b \Leftrightarrow \exists_{f : Arr_{\mathcal{C}}} (f : a \to b)}$$

$\leq_{\mathcal{C}}$ is a <u>pre-ordering</u> : $a \leq_{\mathcal{C}} a$

$$a \leq_{\mathcal{C}} b \to b \leq_{\mathcal{C}} c \to a \leq_{\mathcal{C}} c$$

If a pre-ordering is antisymmetric $(a \leq b \to \underbrace{b \leq a}_{} \to a = b)$ it is called a <u>partial ordering</u> and $(P, \leq)$ is a <u>poset</u>

↳ Its categorical expression? $(p. 42)$

- If $(P, \leq)$ is a pre-ordering structure we get a pre-order category

$$\mathcal{C}_P \equiv (P, Arr_{\mathcal{C}_P}, dom_{\mathcal{C}_P}, cod_{\mathcal{C}_P}, \circ_{\mathcal{C}_P})$$

an arrow $f : a \to b$ is a pair (a, b) s.t. $a \leq b$, and

$$(a,b) \circ_{\mathcal{C}} (b,c)$$

$$\begin{tikzcd} a \arrow[r] \arrow[rr, bend right, "{(a,c)}"'] & b \arrow[r] & c \end{tikzcd}$$

- $0 = (\{0\}, \{(0,0)\})$ is a poset and $\mathcal{C}_0 = \mathbf{1}$
- $(\{0,1\}, 0 \leq 1, 0 \leq 0, 1 \leq 1) \sim \mathcal{C} = \mathbf{2}$
- $\{0,1,2\} \sim \mathcal{C} = \mathbf{3}$

$\omega = \{0, 1, 2, 3, \ldots\}$ is a poset

$$\mathcal{C}_\omega : \begin{tikzcd} 0 \arrow[r] \arrow[rr, bend right] \arrow[rrr, bend right] & 1 \arrow[r] \arrow[rr, bend left] & 2 \arrow[r] & 3 \arrow[r] & \cdots \end{tikzcd}$$

Example 1.2.4: Of a preorder, which is not partially ordered i.e. $f : a \to b$ ($a \leq b$), $g : b \to a$ ($b \leq a$) but not $a = b$:

$$\begin{tikzcd} a \arrow[loop above, "1_a"] \arrow[r, bend left, "f"] & b \arrow[loop above, "1_b"] \arrow[l, bend left, "g"] \end{tikzcd}$$

Proposition 1.2.5 $\mathcal{C}$, $a : \mathrm{Ob}_{\mathcal{C}}$ (uniqueness of the identity arrow)

If $f : a \to a$ satisfies the identity law, then $f = 1_a$.

Proof: By the right triangle of the identity law for f we get

$$\begin{tikzcd} a \arrow[d, "f"'] \arrow[dr, "1_a"] & \\ a \arrow[r, "1_a"'] & a \end{tikzcd} \qquad 1_a = 1_a \circ f = f$$

□

(Hence; $b \cong 1_b$).

Definition 1.2.6 $\mathcal{C}$ is called discrete, if the only arrows are the identity arrows.

A discrete category is just a collection of objects. Conversely, a set X is turned into a discrete category by adding $1_x : x \to x$ for every $x \in X$.

Clearly $x \leq_X y \leftrightarrow x = y$.

Example 1.2.7 $\mathbb{N} = (\underbrace{\{\mathbb{N}\}}_{\text{unique obj}}, \underbrace{\{0, 1, 2, \dots\}}_{\text{infinitely many arrows}}, \mathrm{dom}, \mathrm{cod}, \circ)$

$n : \mathbb{N} \to \mathbb{N}$

$$\begin{tikzcd} \mathbb{N} \arrow[r, "m"] \arrow[rr, bend right, "n+m"'] & \mathbb{N} \arrow[r, "n"] & \mathbb{N} \end{tikzcd}$$

associativity of $\circ_{\mathbb{N}}$ is associativity of $+$.

$n + m = n \circ_{\mathbb{N}} m$

$$\begin{tikzcd} \mathbb{N} \arrow[r, "m"] \arrow[rr, bend right] & \mathbb{N} \arrow[r, "1_{\mathbb{N}}"] \arrow[rr, bend left] & \mathbb{N} \arrow[r, "n"] & \mathbb{N} \end{tikzcd} \qquad 1_{\mathbb{N}} \equiv 0$$

Example 1.2.8 (Generalization of Ex. 1.2.7.)

A monoid is a structure $M = (M, *, e)$ s.t.

- $x * (y * z) = (x * y) * z$
- $e * x = x * e = x$

$$\mathcal{C}_M = (\{M\}, \{x \in M\}, x \circ y \equiv x * y)$$

$1_M \equiv e$

Conversely, if $\mathcal{C}$ is a cat with one only object and M is its collection of arrows, then $(M, \circ, 1_a)$ is a monoid

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New categories from given ones

Definition 1.2.9: $\mathcal{C}$, $a, b : \mathrm{Ob}_{\mathcal{C}}$. We define:

$Arr_{\mathcal{C}}(a,b) \equiv \llbracket f : Arr_{\mathcal{C}} \mid dom(f) = a$ and $cod(f) = b \rrbracket$ (as a collection)

$f : a \to b$

- $\mathcal{A} = (Ob_{\mathcal{A}}, Arr_{\mathcal{A}}, dom_{\mathcal{A}}, cod_{\mathcal{A}}, \circ_{\mathcal{A}})$ is a <u>subcategory</u> of $\mathcal{C}$ $(\mathcal{A} \subseteq \mathcal{C})$ if

  (i) $a : Ob_{\mathcal{A}} \Rightarrow a : Ob_{\mathcal{C}}$

  (ii) $a, b : Ob_{\mathcal{A}} \Rightarrow f : Arr_{\mathcal{A}}(a,b) \Rightarrow f : Arr_{\mathcal{C}}(a,b)$

  eg, FinSet $\subseteq$ Set, and actually a full subcategory of Set

- $\mathcal{A}$ is a <u>full</u> subcategory of $\mathcal{C}$, if $\mathcal{A} \subseteq \mathcal{C}$ and $Arr_{\mathcal{C}}(a,b) \sqsubseteq Arr_{\mathcal{A}}(a,b)$ (subcollection) hence $=$

  i.e. for every $a, b : Ob_{\mathcal{A}} \Rightarrow f : Arr_{\mathcal{C}}(a,b) \Rightarrow f : Arr_{\mathcal{A}}(a,b)$

  (Ob_C : types Arr_C cont
  Ob_A : count. types Arr_A = [illegible])

- <u>**Definition 1.2.10:**</u> $\mathcal{C}, \mathcal{D}$ cats. The <u>product</u> category $\mathcal{C} \times \mathcal{D}$ has objects the pairs (a,b) with $a : Ob_{\mathcal{C}}$, $b : Ob_{\mathcal{D}}$, and if $(a,b), (c,d) : Ob_{\mathcal{C} \times \mathcal{D}}$ then $(f,g) : (a,b) \to (c,d)$ if $f : a \to c$ and $g : b \to d$

  $1_{(a,b)} = (1_a, 1_b)$, and $(f,g) \circ (f',g') \equiv (f \circ f', g \circ g')$.

- <u>**Definition 1.2.11:**</u> If $\mathcal{C}$ is a cat, then $\mathcal{C}^{\to}$ the <u>arrow category</u> of $\mathcal{C}$ has objects the Arr.

  Let $f : a \to b$, $g : c \to d$: An "arrow" between f, g is a pair of arrows $h : a \to c$ and $i : b \to d$ s.t. $g \circ h = i \circ f$

$$\begin{tikzcd} a \arrow[r, "h"] \arrow[d, "f"'] & c \arrow[d, "g"] \\ b \arrow[r, "i"'] & d \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[r, "id_a"] \arrow[d, "f"'] & a \arrow[d, "f"] \\ b \arrow[r, "id_b"'] & b \end{tikzcd} \qquad 1_{(f,f)} = (1_a, 1_b)$$

$$\begin{tikzcd} a \arrow[r, "i"] \arrow[d, "f"'] & c \arrow[r, "k"] \arrow[d, "g"] & e \arrow[d, "h"] \\ b \arrow[r, "j"'] & d \arrow[r, "l"'] & x \end{tikzcd} \qquad (k,l) \circ (i,j) = (k \circ i, l \circ j)$$

(Does it correspond to suspension of type?)

• **Definition 1.2.12**: If X is a set, then $\mathbf{Set}\downarrow X$ is the category of X-valued functions

$\mathcal{C}$ is a cat, $a : \mathrm{Ob}\,\mathcal{C}$, $\mathcal{C}\downarrow a$ the category of arrows over a, has objects

(Slice Category)

arrows in $\mathcal{C}$ $f : b \to a$. An arrow $f \to g$ is an arrow $h : b \to c$ s.t. $g \circ h = f$ $\quad \boxed{(b, f) : \mathrm{Ob}_{\mathcal{C}\downarrow a}}$

$$\begin{tikzcd} b \arrow[rr, "h"] \arrow[dr, "f"'] & & c \arrow[dl, "g"] \\ & a & \end{tikzcd}$$

$$\begin{tikzcd} b \arrow[rr, "1_b"] \arrow[dr, "f"'] & & b \arrow[dl, "f"] \\ & a & \end{tikzcd}$$

$1_{(b,f)} = 1_b$

$$\begin{tikzcd} b \arrow[r, "i"] \arrow[rr, bend left, "j\circ i"] \arrow[dr, "f"'] & c \arrow[r, "j"] \arrow[d, "g"] & d \arrow[dl, "h"] \\ & a & \end{tikzcd}$$

$i : (b, f) \to (c, g)$, $g \circ i = f$

$j : (c, g) \to (d, h)$, $h \circ j = g$

$j \circ i : (b, f) \to (c, g)$, $h \circ (j \circ i) = f$

since $h \circ (j \circ i) = (h \circ j) \circ i$
$= g \circ i$
$= f$.

Dually one defines $\mathcal{C}\uparrow a$ the category of arrows under a

a-slice cat

$$\begin{tikzcd} & a \arrow[dl, "f"'] \arrow[dr, "g"] & \\ b \arrow[rr, "h"'] & & c \end{tikzcd}$$

$$\begin{tikzcd} & a \arrow[dl, "f"'] \arrow[dr, "f"] & \\ b \arrow[rr, "1_b"'] & & b \end{tikzcd}$$

$$\begin{tikzcd} & a \arrow[dl, "f"'] \arrow[d, "g"] \arrow[dr, "h"] & \\ b \arrow[r, "i"'] \arrow[rr, bend right, "j\circ i"'] & c \arrow[r, "j"'] & d \end{tikzcd}$$

or C/a or $C|a$ ← C slice functor

$\mathcal{C}\downarrow a$, $\mathcal{C}\uparrow a$: comma categories (special case of ↓ the arrow category) : the slice cats. They provide examples of topoi and important in the development of the general theory. (introduced by Lawvere in 1963)

• Fundamental Theorem of Topoi (1.30) : $a : \mathrm{Ob}\,\mathcal{C}$. If $\mathcal{C}$ is a topos, then $\mathcal{C}\downarrow a$ is a topos ($\mathbf{Bn}(I) = \mathbf{Set}\downarrow I$)

- $\mathcal{A}, \mathcal{B}, \mathcal{C}$ cats

  F, G functors

$$\begin{tikzcd} & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$$

- The comma cat $(F \downarrow G)$ the term from Lawvere's original notation

: Objects (a, b, f) where $a : Ob_A$, $b : Ob_{\mathcal{B}}$

$f : F(a) \longrightarrow G(b)$ in $\mathcal{C}$

Arrows between (a, b, f) and (a', b', f') are pairs

(g, h)

$g : a \to a'$, $h : b \to b'$ st the following commutes $j : b' \to b''$, $i : a' \to a''$

$$\begin{tikzcd} F(a) \arrow[r, "F(g)"] \arrow[d, "f"'] & F(a') \arrow[r, dashed, "F(i)"] \arrow[d, "f'"] & F(a'') \arrow[d, dashed, "f''"] \\ G(b) \arrow[r, "G(h)"'] & G(b') \arrow[r, dashed, "G(j)"'] & G(b'') \end{tikzcd}$$

Comp. $(i, j) \circ (g, h) = (i \circ g, j \circ h)$

$1_{(a,b,f)} = (1_a, 1_b)$

- Slice Category :

$$\begin{tikzcd} & \mathbb{1} \arrow[d, "T"] \\ A \arrow[r, "Id"'] & A \end{tikzcd}$$

$(\bullet, \bullet)^{1_\bullet}$ $T(\bullet) : A_0 : ob_A$

the identity functor

$(Id \downarrow T) \equiv (\mathcal{C} \downarrow A_0)$ the slice category over A_0

$(a, *, f) \sim (A, f)$ $f : A \to A_0$

. If $A, \mathcal{B} = \mathcal{C}$ $(\mathcal{C} \downarrow_{\mathcal{C}})$ and F, G identity functors we get $\mathcal{C}^{\to}$!!

- A $\mathcal{C}$ is locally Cartesian closed, if every slice cat of $\mathcal{C}$ is cart. closed.

- Lawvere thesis: $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ are adjoint iff

$$(F \downarrow id_{\mathcal{D}}) \underset{\text{isomorphic cats}}{\cong} (id_{\mathcal{C}} \downarrow G)$$

and equivalent elements in the comma cat can be projected onto the same element of $\mathcal{C} \times \mathcal{D}$.

Hence adjunctions are described without sets, and this was the original motivation for introducing comma cats.

Section 1.3 : Special arrows

Definition 1.3.1 $a, b : \mathrm{Ob}\,\mathcal{C}$

$f : a \to b$ is called <u>monic</u>, if for every $c : \mathrm{Ob}\,\mathcal{C}$, $g, h : c \to a$

$$c \overset{g}{\underset{h}{\rightrightarrows}} a \xrightarrow{f} b$$

$$f \circ g = f \circ h \implies g = h$$

If f is monic we denote it by $f : a \rightarrowtail b$ (monomorphism)

Remark 1.3.2 A monic arrow is the point-free characterization of an injective function:

Let $f : A \to B$. TFAE

(i) f is injective i.e., $f(x) = f(y) \to x = y$

(ii) $\forall_{C,\, g,\, h : C \to A} \left(f \circ g = f \circ h \to g = h \right)$

Proof: (i) $\Rightarrow$ (ii) let $z \in C$ $\quad f(g(z)) = f(h(z)) \xrightarrow{(i)} g(z) = h(z)$

i.e. $\forall_{z \in C} \left(g(z) = h(z) \right)$

hence $g = h$.

(ii) $\Rightarrow$ (i) let $x, y \in A$ s.t. $f(x) = f(y)$

Let $g : A \to A$, $g(a) = x$, $\forall a \in A$.

$h : A \to A$, $h(a) = y$, $\forall a \in A$

By our hypothesis $f \circ g = f \circ h$. Hence $g = h$, hence $x = y$.

Example 1.3.3 : (a) In $\mathbb{N}$ (Ex 1.2.7) every arrow $m : \mathbb{N} \to \mathbb{N}$ is monic: $m + n = m + k \to n = k$

(b) In a pre-order every arrow is monic: $c \overset{g}{\underset{h}{\rightrightarrows}} a \xrightarrow{f} b$ $\quad f \circ g, f \circ h : c \to b$

Always $f \circ g = f \circ h$.

Always $g = h$.

(c) In $\mathcal{C}/a$ an arrow f from (b, i) to (c, j) is monic, if for every g, h : from (d, k) to (b, i)

$$\begin{tikzcd} b \arrow[rr, "f"] \arrow[dr, "i"] & & c \arrow[dl, "j"'] \\ & a & \\ d \arrow[uu, shift left, "g"'] \arrow[uu, shift right, "h"] \arrow[ur, "k"'] & & \end{tikzcd}$$

$f \circ g = f \circ h$ from (d, k) to (c, j) $\Rightarrow$

$g = h$

i.e. f is monic in $\mathcal{C}$.

• **Proposition 1.3.4** $\mathcal{C}$ cat, $g, f : \mathrm{Arr}_{\mathcal{C}}$ (cannot show the set-theoretic version of this prop.)

(or Exercise)

(i) $g \circ f$ is monic $\Leftarrow$ f and g are monic

(ii) $g \circ f$ is monic $\Rightarrow$ f is monic

Proof: (i) $d \underset{h'}{\overset{g'}{\rightrightarrows}} a \xrightarrow{f} b \xrightarrow{g} c$

Let $(g \circ f) \circ g' = (g \circ f) \circ h'$ $\Leftrightarrow$

$g \circ (f \circ g') = g \circ (f \circ h') \overset{g \text{ monic}}{\Longrightarrow} f \circ g' = f \circ h'$

$\overset{f \text{ monic}}{\Longrightarrow} g' = h'$

(ii) $f \circ g' = f \circ h' \Rightarrow g \circ (f \circ g') = g \circ (f \circ h')$

$\Leftrightarrow (g \circ f) \circ g' = (g \circ f) \circ h'$

$\underset{\text{monic}}{\overset{g \circ f}{\Longrightarrow}} g' = h'$

• **Definition 1.3.4**: $a, b : \mathrm{Ob}_{\mathcal{C}}$

$f : a \to b$ is called <u>epic</u>, if for every $g, h : b \to c$

$$a \xrightarrow{f} b \underset{h}{\overset{g}{\rightrightarrows}} c$$

$g \circ f = h \circ f \Rightarrow g = h$

We write $f : a \twoheadrightarrow b$ for an epic arrow

• **Remark 1.3.5**: An epic arrow is the point-free characterization of a surjective function.

(Ex) Let $f : A \to B$. TFAE:

(i) f is a surjection i.e. $\forall_{y \in B} \exists_{x \in A} (f(x) = y)$

(ii) $\forall_{C, g, h : B \to C} (g \circ f = h \circ f \Rightarrow g = h)$

• Exercise: epic in $\mathbb{N}$

epic in preorder

• **Definition 1.3.6.** $f : a \to b$ is an <u>"iso"</u>, or <u>invertible</u>, if there is $g : b \to a$ s.t. $g \circ f = 1_a$ and $f \circ g = 1_b$

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "1_a"'] & b \arrow[d, "g"] \arrow[dr, "1_b"] & \\ & a \arrow[r, "f"'] & b \end{tikzcd}$$

• **Proposition 1.3.7**

• If g exists, it is unique: $\begin{tikzcd} b \arrow[d, "g'"] \arrow[dr, bend left, "1_b"] & \\ a \arrow[r, "f"'] & b \end{tikzcd}$ again start from the right triangle.

Proof:

$\cancel{f \circ g' = 1_b \to (g \circ f) \circ g' = 1_b \circ g \to g' \circ (f \circ g) \circ g}$

$\cancel{\to g' \circ 1_b = g}$

$\cancel{\to g' \circ g}$

Proof of Prop. 1.3.7.

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[dr, "1_a"'] & b \arrow[d, "g'"] \arrow[dr, "1_b"] & \\
& a \arrow[r, "f"'] & b
\end{tikzcd}$$

$$1_b = f \circ g' \quad \text{and} \quad g' \circ f = 1_a$$

$$1_b = f \circ g \quad \text{and} \quad g \circ f = 1_a$$

$$\overline{\quad f \circ g' = f \circ g \quad} \qquad \overline{\quad g' \circ f = g \circ f \quad}$$

$$g \circ (f \circ g') = g \circ (f \circ g) \Leftrightarrow$$

$$(g \circ f) \circ g' = (g \circ f) \circ g \Leftrightarrow$$

$$1_a \circ g' = 1_a \circ g \Leftrightarrow$$

$$g' = g$$

$\square$

- g is then denoted by $f^{-1}: b \to a$
- $f: a \cong b$: "f is an iso"

- **Proposition 1.3.8** An iso arrow f is monic (dually it is also an epic)

  Proof:

$$\begin{tikzcd} c \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & a \arrow[r, "f"] \arrow[dr, "1_a"'] & b \arrow[d, "f^{-1}"] \\ & & a \end{tikzcd}$$

We use only the left triangle of the iso-diagram

$$\begin{aligned} h &= 1_a \circ h \\ &= (f^{-1} \circ f) \circ h \\ &= f^{-1} \circ (f \circ h) \\ &= f^{-1} \circ (f \circ g) \\ &= (f^{-1} \circ f) \circ g \\ &= 1_a \circ g \\ &= g. \qquad \square \end{aligned}$$

- Example: The only iso in $\mathbb{N}$ is 0 (p.40).

(Go)
- Ex : In a poset every arrow is monic and epic, but the only isos are the identity arrows!! (p.41)
  (i.e. monic & epic $\not\Rightarrow$ iso generally)

- **Definition 1.3.9** $a \cong b \Leftrightarrow \exists f: a \to b$ (f is an iso)
  $\Leftrightarrow \exists f \, (f: a \cong b)$

- A property is categorical if it is invariant under iso
- A $\mathcal{C}$ is called skeletal, if $a \cong b \Rightarrow a = b$
- In a poset the only isos are the identity arrows: categorical account of antisymmetry in pre-orders:
  A poset is precisely a skeletal pre-order category
- $\cong$ is eq rel.

Section 1.9 Special objects

Definition 1.9.1 $0 : ob_{\mathcal{C}}$ is called *initial*, if $\forall a : ob_{\mathcal{C}}\ \exists! f : 0 \to a$

We denote $0_a : 0 \to a$ (or $!$)

Hence $\boxed{0_0 = 1_0}$

Proposition 1.9.2 $a, b : \mathcal{C}$, a, b initial. Then $a \cong b$. (use in Theorem 1.10.6 (i))

Proof:

$$\begin{tikzcd} a \arrow[r, "!_b"] \arrow[d, "1_a"'] \arrow[d, shift left, "!_a"] & b \arrow[d, "1_b"] \\ a \arrow[r, "!_b"'] & b \end{tikzcd}$$

$1_a = !_a \circ !_b$ on the unique arrow from a to a

$1_b = !_b \circ !_a$ " " b to b. □

See also Th 1.10 6(ii).

Here we use the symbol 0 ($\emptyset$ is initial in Set) for the initial obj. of $\mathcal{C}$.

$\emptyset_A : \emptyset \to A$ the $\emptyset$ function
$\forall x\,(x \in \emptyset \to \exists y \in A\,((x,y) \in \emptyset))$
while $f : A \to \emptyset$ $\ \forall x\,(x \in A \to \exists y \in \emptyset\,(\ ))$
if $x \in \neq \emptyset$ this is not true

Examples 1.9.3: In a preorder an initial object is a minimal element
(+ 1.9.3) (exe)

Definition 1.9.4 $1 : ob_{\mathcal{C}}$ is called *terminal*, if $\forall a : ob_{\mathcal{C}}\ \exists! f : a \to 1$

We denote $!_a : a \to 1$

Similar uniqueness (*)

In Set $1 = \{x\}$ singletons, Examples, more ...
(the unique function is the constant x.)

Ex Prop. 1.9.5 1 terminal in $\mathcal{C}$

$f : 1 \to a$

Then f is monic

Proof: $c \underset{h}{\overset{g}{\rightrightarrows}} 1 \xrightarrow{f} a$ $\quad f \circ g = f \circ h$. Even without this assumption, we get $g = h$ as there is one only arrow from c to 1.

Add <u>Initiality is a categorical property</u>. I did it in the class

(*) or by Duality.

Definition 1.9.6

(i) Σ formula in the basic language of cats $\rightsquigarrow \Sigma^{op}$ (the const. of Σ applied to $\mathcal{C}^{op}$)

$\emptyset$ is terminal in Set^{op}.

dom $\rightsquigarrow$ cod

cod $\rightsquigarrow$ dom

$h = f \circ g \rightsquigarrow h = g \circ f$

epic is dual to monic, terminal dual to initial

(ii) Definition $\mathcal{C} \rightsquigarrow \mathcal{C}^{op}$ dual or opposite cat.

$(Ob_{\mathcal{C}}, Arr_{\mathcal{C}}, dom_{\mathcal{C}}, cod_{\mathcal{C}}, \circ_{\mathcal{C}}) \rightsquigarrow (Ob_{\mathcal{C}}, f^{op} : b \to a, \quad f^{op} \circ g^{op}$ only when $g \circ f$ is defined in $\mathcal{C}$

$f : a \to b$ $\quad (g \circ f)^{op}$ $\quad a \xrightarrow{f} b \xrightarrow{g} c$

Prop. 1.4.7:

(i) $\mathcal{C}$ discrete $\Rightarrow$ $\mathcal{C}^{op} = \mathcal{C}$

(ii) $\left(\mathcal{C}^{op}\right)^{op} = \mathcal{C}^{op}$

(meta)

* Theorem 1.4.8 : If Σ is true, then Σ^{op} is true

(DUALITY PRINCIPLE)

Proof: If Σ is true, i.e., derivable from the cat-axioms, then Σ^{op} holds in all cats in all cats of the form $\mathcal{C}^{op}$. Any cat $\mathcal{D}$ is of the form $\mathcal{C}^{op}$ ($\mathcal{C} = \mathcal{D}^{op}$), hence Σ^{op} holds in all cats.

* Corollary: Any two terminal $\mathcal{C}$-objects are isomorphic

Proof: Σ : any two initial $\mathcal{C}$-objects are isomorphic

"f is an iso" is self dual : the dual of an invertible arrow is an inv. arrow

$$\left(f^{op}\right)^{-1} = \left(f^{-1}\right)^{op}$$

By DP Σ^{op} is true, which is what we want

Section 1.5 Products and Co-products

Definition 1.5.1: $a, b : Ob_{\mathcal{C}}$

A product $a \times b : Ob_{\mathcal{C}}$ s.t. there are $pr_a : a\times b \to a$, $pr_b : a \times b \to b$ s.t. for any $f : c \to a$ and $g : c \to b$, there is unique $\langle f, g\rangle : c \to a\times b$ s.t. the following diagram commutes

$$
\begin{tikzcd}
 & c \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g\rangle"] \arrow[dr, "g"] & \\
a & a\times b \arrow[l, "pr_a"] \arrow[r, "pr_b"'] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[d, dashed] \\
a\times b
\end{tikzcd}
\text{ there is ! arrow}
$$

i.e., $f = pr_a \circ \langle f, g\rangle$ and $g = pr_b \circ \langle f, g\rangle$

Proposition 1.5.2: $a\times b$ is unique up to isomorphism.

Proof: Let (d, p_a, p_b) $p_a : d \to a$, $p_b : d \to b$ with the above property, and apply it first above for $a\times b$ and then bottom for d we get

$$
\begin{tikzcd}
 & d \arrow[dl, "p_a"'] \arrow[d, dashed, "\langle p_a,p_b\rangle"] \arrow[dr, "p_b"] & \\
a & a\times b \arrow[l, "pr_a"] \arrow[r, "pr_b"'] & b \\
 & d \arrow[ul, "p_a"] \arrow[ur, "p_b"'] \arrow[from=u, dashed, "\langle pr_a,pr_b\rangle"] &
\end{tikzcd}
$$

$p_a = pr_a \circ \langle p_a, p_b\rangle$

$p_b = pr_b \circ \langle p_a, p_b\rangle$

$pr_a = p_a \circ \langle pr_a, pr_b\rangle$

$pr_b = p_b \circ \langle pr_a, pr_b\rangle$

$$
\begin{tikzcd}
 & d \arrow[dl, "p_a"'] \arrow[d, dashed, "s"] \arrow[dr, "p_b"] & \\
a & d \arrow[l, "p_a"] \arrow[r, "p_b"'] & b
\end{tikzcd}
$$

There is unique s s.t. $p_b = p_b \circ s$

$p_a = p_a \circ s$

Hence $s = id_d$, since id_d has this property.

Hence $\langle pr_a, pr_b\rangle \circ \langle p_a, p_b\rangle = 1_d$

Working similarly with the diagrams

$$
\begin{tikzcd}
 & a\times b \arrow[dl, "pr_a"'] \arrow[d, dashed, "\langle pr_a,pr_b\rangle"] \arrow[dr, "pr_b"] & \\
a & d \arrow[l, "p_a"] \arrow[r, "p_b"'] & b \\
 & a\times b \arrow[ul, "pr_a"] \arrow[ur, "pr_b"'] \arrow[from=u, dashed, "\langle p_a,p_b\rangle"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & a\times b \arrow[dl, "pr_a"'] \arrow[d, dashed, "\tau"] \arrow[dr, "pr_b"] & \\
a & a\times b \arrow[l, "pr_a"] \arrow[r, "pr_b"'] & b
\end{tikzcd}
$$

i.e. $\boxed{\langle p_a, p_b\rangle \circ \langle pr_a, pr_b\rangle = 1_{a\times b}}$

we get $\langle p_a, p_b\rangle \circ \langle pr_a, pr_b\rangle = 1_{a\times b}$. Hence $a\times b \cong d$. $\square$

this emerges from the cat. props of them

Exercise: (i)

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g\rangle"] \arrow[dr, "g"] & \\ a & a\times b \arrow[l, "pr_a"] \arrow[r, "pr_b"'] & b \end{tikzcd} \qquad \begin{tikzcd} & c \arrow[dl, "f'"'] \arrow[d, dashed, "\langle f',g'\rangle"] \arrow[dr, "g'"] & \\ a & a\times b \arrow[l, "pr_a"] \arrow[r, "pr_b"'] & b \end{tikzcd}$$

$\langle f', g'\rangle = \langle f, g\rangle \longrightarrow f = f'$ and $g' = g$

(ii)

$\langle f\circ h, g\circ h\rangle = \langle f, g\rangle \circ h$

$$\begin{tikzcd} & d \arrow[d, "h"] \arrow[ddl, bend right, "f\circ h"'] \arrow[ddr, bend left, "g\circ h"] & \\ & c \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g\rangle"] \arrow[dr, "g"] & \\ a & a\times b \arrow[l, "pr_a"] \arrow[r, "pr_b"'] & b \end{tikzcd}$$

(iii) If $\mathcal{C}$ has 1 and products, then

(a) $a \cong a\times 1$

(b) $\langle 1_a, !_a\rangle$ is an iso

Solution (i) $f' = pr_a \circ \langle f', g'\rangle = pr_a \circ \langle f, g\rangle = f$

(ii) $pr_a \circ (\langle f, g\rangle \circ h) = (pr_a \circ \langle f, g\rangle)\circ h = f\circ h$.

Definition 1.5.3 $f : a \to b$, $g : c \to d$, $a \times c, b \times d \in Ob_{\mathcal{C}}$

$f \times g : a \times c \to b \times d$

$f \times g \equiv \langle f \circ pr_a, g \circ pr_c \rangle$

$$\begin{tikzcd}
 & a \times c \arrow[dl, "pr_a"'] \arrow[dd, dashed] \arrow[dr, "pr_c"] & \\
a \arrow[d, "f"'] & & c \arrow[d, "g"] \\
b & b \times d \arrow[l, "pr_b"] \arrow[r, "pr_d"'] & d
\end{tikzcd}$$

Exercise (i) $1_a \times 1_b = 1_{a \times b}$, if $a \times b \in \mathcal{C}_0$

(ii) $a \times b \cong b \times a$

(iii) $(a \times b) \times c \cong (a \times b) \times c$

(iv) $(f \times h) \circ \langle g, k \rangle = \langle f \circ g, h \circ k \rangle$

(v) $(f \times h) \circ (g \times k) = (f \circ g) \times (h \circ k)$

Definition 1.5.4 $a = a_1, \ldots, a_m \in Ob_{\mathcal{C}}$

$a^m = a \times \cdots \times a$ By (iii) unique up to iso w.r.t. parentheses

$$\begin{tikzcd}
 & & c \arrow[dll, "f_1"'] \arrow[dl, "f_2"'] \arrow[d, dashed, "\langle f_1, \ldots, f_m \rangle"] \arrow[dr, "f_m"] & \\
a & a & & a \\
 & & a^m \arrow[ull, "pr_1^m"] \arrow[ul, "pr_2^m"] \arrow[ur, "pr_m^m"'] &
\end{tikzcd}$$

$a_1 \times \cdots \times a_m$

$f_1 \times \cdots \times f_m$

• **Definition 1.5.5** . $a, b : Ob_{\mathcal{C}}$

A **coproduct** $a+b : Ob_{\mathcal{C}}$ (is an object of $\mathcal{C}$) with $i_a : a \to a+b$, $i_b : b \to a+b$ s.t. tfdc:

$$\begin{tikzcd} a \arrow[r, "i_a"] \arrow[dr, "f"'] & a+b \arrow[d, dashed, "{[f,g]}"] & b \arrow[l, "i_b"'] \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

$f = [f,g] \circ i_a$

$g = [f,g] \circ i_b$

In Set $A+B$ is the disjoint union.

• Dual to product.

Section 1.6 Equalizers and Co-equalizers

Definition 1.6.1: Let $f, g : a \to b \in \mathrm{Arr}_{\mathcal{C}}$

$i : e \to a$ is an <u>equalizer</u> of f, g, if

(a) $f \circ i = g \circ i$

(b) If $h : c \to a$ and $f \circ h = g \circ h$, there is <u>unique</u> $x : c \to e$ s.t. $i \circ x = h$

$$\begin{tikzcd} e \arrow[r, "i"] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \\ c \arrow[u, dashed, "x"] \arrow[ur, "h"'] & & \end{tikzcd}$$

"h factors uniquely through i".

Motivation (In Set) In Set $f, g : A \to B$

$E = \{x \in A \mid f(x) = g(x)\}$

$i : E \to A$ the identity on E

Then $f \circ i = g \circ i$: the functions f, g are "equalized" by i

Moreover, i is a <u>canonical</u> equalizer of f, g : if $h : C \to A$ s.t. $f \circ h = g \circ h$

$$\begin{tikzcd} E \arrow[r, hook, "i"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ C \arrow[u, dashed, "x"] \arrow[ur, "h"'] & & \end{tikzcd}$$

We need $i \circ x = h \to x(c) = h(c)$, for every $c \in A$. We define $x = h$ (the codomain changes)

Since $f(h(c)) = g(h(c)) \to h(c) \in E$, hence $x(c)$ is well-defined.

This also shows the uniqueness of x.

Proposition 1.6.2: An equalizer is monic

<u>Proof:</u> Let $\begin{tikzcd} c \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & e \arrow[r, "i"] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \end{tikzcd}$

Let $i \circ f' = i \circ g'$. We show $f' = g'$. Since $f \circ i = g \circ i$,

$$\begin{aligned} f \circ (i \circ f') &= (f \circ i) \circ f' \\ &= (g \circ i) \circ f' \\ &= g \circ (i \circ f') \end{aligned}$$

There is unique x, $i \circ x = i \circ f'$. Clearly f', g' have the property of x

Hence $f' = g'$

Ex. (p. 59) Find monic, which is NOT equalizer of some pair

* Proposition 1.63 An epic equalizer is an iso.

Proof:

$$\begin{tikzcd} e \arrow[r, "i"] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \\ d \arrow[u, "k"] \arrow[ur, "h"'] & & \end{tikzcd}$$

Since $f \circ i = g \circ i$ and i is an epic, we get $f = g$.
Hence $f \circ h = g \circ h$, for every $h: d \to a$
Hence there is ! k s.t. $h = i \circ k$.

[Take (f,g) the pair for which i equalizes

$$\begin{tikzcd} e \arrow[r, "i"] & a \\ a \arrow[u, dashed] \arrow[ur, "1_a"'] & \end{tikzcd}$$

There is ! $k: a \to e$ s.t. $1_a = i \circ k$.

We need to show $k \circ i = 1_e$.

$$\begin{tikzcd} e \arrow[r, "i"] \arrow[rr, bend right, "1_e"'] & a \arrow[r, "k"] & e \arrow[r, "i"] & a \end{tikzcd}$$

$i \circ (k \circ i) = (i \circ k) \circ i = 1_a \circ i = i = i \circ 1_e$

By Prop 1.62 i is monic $\Rightarrow$ $k \circ i = 1_e$. □

Proposition
Remark 1.64 In Set a monic is an equalizer. (in for some point) (and in every topos)

(Ex.)

Proof: Let $i: E \overset{1\text{-}1}{\to} A$. We find f,g s.t. i is an equalizer for f,g:

$$\begin{tikzcd} E \arrow[r, "i"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & 2 \\ C \arrow[u, dashed, "k"] \arrow[ur, "h"'] & & \end{tikzcd}$$

$f: A \to 2$, $f(a) = 1$, $\forall a \in A$

$g: A \to 2$, $g(a) = \begin{cases} 1, & a \in Im(i) \\ 0 & o.w. \end{cases}$

Clearly $f \circ i = g \circ i$

Let $h: C \to A$ s.t. $f(h(c)) = g(h(c))$ $\forall c \in C$

Hence $f(h(c)) = g(h(c)) = 1$. $\Rightarrow h(c) \in Im(i)$, $\forall c \in C$

i.e., $h(C) \subseteq Im(i)$ $(Im(h) \subseteq Im(i))$

Existence of k: Let $h(c) = a \in A$. Since $h(c) \in Im(i)$ there is (unique by 1-1 of i) $e \in E$ s.t. $i(e) = h(c)$.
We define $k(c) = e$. Hence $h = i \circ k$ holds by definition of k.

Uniqueness of k: Let $k': C \to E$ s.t. $h(c) = i(k'(c))$
$= i(k(c))$
$\overset{i \text{ is } 1\text{-}1}{\Longrightarrow} k'(c) = k(c)$, $\forall c \in C$. □

Definition 16.5 Let $f, g : a \to b \in Arr_{\mathcal{C}}$

An arrow $j : b \to e$ is a <u>co-equalizer</u> of f, g, if

(a) $j \circ f = j \circ g$

(b) If $h : b \to c$ s.t. $h \circ f = h \circ g$, there is <u>unique</u> $k : e \to c$ s.t. the d.c. i.e., $h = k \circ j$

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \arrow[r, "j"] \arrow[dr, "h"'] & e \arrow[d, dashed, "k"] \\ & & c \end{tikzcd}$$

Dually: - co-equalizers are epic

- a monic co-equalizer is iso

- In Set an epic is a co-equalizer

Motivation (In Set): • Let $R \subseteq A \times A$ an eqrel on A.

$pr_1 : R \to A \quad pr_1(\langle a, b\rangle) = a$, where $\langle a, b\rangle \in R$

$pr_2 : R \to A \quad pr_2(\langle a, b\rangle) = b$, — " —

$\pi_R : A \to A/R \quad \pi_R(a) = [a]_R = \{ b \in A \mid \langle a, b\rangle \in R \}$

$$\begin{tikzcd} R \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] & A \arrow[r, "\pi_R"] \arrow[dr, "h"'] & A/R \arrow[d, dashed, "k"] \\ & & C \end{tikzcd}$$

// π_R is the co-equalizer of pr_1, pr_2 //

We define $k([a]_R) = h(a)$. We show that k is well-defined i.e. independent from the choice of a.

If $\langle a, b\rangle \in R$, then $k([b]_R) = h(b) = h(a)$, since $h(pr_1(\langle a, b\rangle)) = h(a) = h(pr_2(\langle a, b\rangle)) = h(b)$.

Uniqueness of k is automatic.

• $f, g : A \to B$ in Set. We define their co-equalizer in Set as follows:

Let $R_0 = \{ \langle f(x), g(x)\rangle \mid x \in A \} \subseteq B \times B$

Let R be the least eqrel $\supseteq R_0$

The co-equalizer of f, g is π_R. $\quad (f \leq pr_1, \ g \leq pr_2)$