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Summer term 19 18.06.2019

Mathematics for Physicists II Sheet 9

Exercise 1. If $A \in M_n(\mathbb{R})$ and A^1, \ldots, A^n are the columns of A, show that A is invertible if and only if the vectors A^1, \ldots, A^n are linearly independent in \mathbb{R}^n . [4 Points]

Exercise 2. If $A \in M_{m,n}(\mathbb{R})$, show that

$$\operatorname{Rank}(T_A) = \operatorname{cRank}(A)$$
 & $\dim(\operatorname{Ker}(T_A)) = n - \operatorname{cRank}(A).$

[4 Points]

Exercise 3. Let $A := [a_{ij}] \in M_{m,n}(\mathbb{R}), X = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and $B = (b_1, \ldots, b_m) \in \mathbb{R}^m$. Let X_0 be in the set of solutions Sol of the system of linear equations

$$AX = B.$$

If Sol_0 is the set of solutions of the corresponding homogeneous system of linear equations

$$AX = \mathbf{0}_m,$$

show that

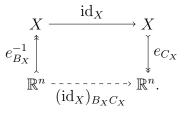
$$Sol = X_0 + Sol_0.$$

[4 Points]

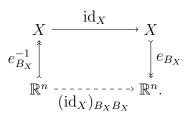
Exercise 4. Let X be a linear space, $n \ge 1$, $B_X := \{v_1, \ldots, v_n\}$ and $C_X := \{w_1, \ldots, w_n\}$ are bases of X.

(i) If $x \in X$, show that $x_{C_X} = A_{(\mathrm{id}_X)_{B_X C_X}} x_{B_X}$ and

$$A_{(\mathrm{id}_X)_{B_X C_X}} = \begin{bmatrix} (v_1)_{C_X} & \dots & (v_n)_{C_X} \end{bmatrix}$$



[1 Point] (ii) Show that $(id_X)_{B_XB_X} = id_{\mathbb{R}^n}$ and $A_{(id_X)_{B_XB_X}} = I_n$



[1 Point]

(iii) Find a 2-dimensional linear space X and bases B_X, C_X of X such that

$$(\mathrm{id}_X)_{B_X C_X} \neq \mathrm{id}_{\mathbb{R}^2} \quad \& \quad A_{(\mathrm{id}_X)_{B_X C_X}} \neq I_2.$$

[2 Points]

Exercise 5. Let X, Y be linear spaces, $n, m \ge 1, B_X := \{v_1, \ldots, v_n\}$ a basis of $X, B_Y := \{w_1, \ldots, w_m\}$ a basis of Y, and let $f, g : X \to Y$ be linear maps. Show the following: (i) $A_{(f+g)_{B_XB_Y}} = A_{f_{B_XB_Y}} + A_{g_{B_XB_Y}}$. [1 Point] (ii) If $\lambda \in \mathbb{R}$, then $A_{(\lambda f)_{B_XB_Y}} = \lambda A_{f_{B_XB_Y}}$. [1 Point] (iii) The function $e_{B_XB_Y} : \mathcal{L}(X,Y) \to M_{m,n}(\mathbb{R})$, defined by $f \mapsto A_{f_{B_XB_Y}}$, for every $f \in \mathcal{L}(X,Y)$, is a linear isomorphism. [2 Points]

Submission. Wednesday 26. June 2019, 16:00.

Discussion. Wednesday 26. June 2019, in the Exercise-session.