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Summer term 19
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Mathematics for Physicists II

Sheet 9

Exercise 1. If $A \in M_n(\mathbb{R})$ and $A^1, \ldots, A^n$ are the columns of A, show that A is invertible if and only if the vectors $A^1, \ldots, A^n$ are linearly independent in $\mathbb{R}^n$.
[4 Points]

Exercise 2. If $A \in M_{m,n}(\mathbb{R})$, show that

$$\mathtt{Rank}(T_A) = \mathtt{cRank}(A) \quad \& \quad \dim(\mathtt{Ker}(T_A)) = n - \mathtt{cRank}(A).$$

[4 Points]

Exercise 3. Let $A := [a_{ij}] \in M_{m,n}(\mathbb{R})$, $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and $B = (b_1, \ldots, b_m) \in \mathbb{R}^m$. Let X_0 be in the set of solutions $\mathtt{Sol}$ of the system of linear equations

$$AX = B.$$

If $\mathtt{Sol}_0$ is the set of solutions of the corresponding homogeneous system of linear equations

$$AX = \mathbf{0}_m,$$

show that

$$\mathtt{Sol} = X_0 + \mathtt{Sol}_0.$$

[4 Points]

Exercise 4. Let X be a linear space, $n \geq 1$, $B_X := \{v_1, \ldots, v_n\}$ and $C_X := \{w_1, \ldots, w_n\}$ are bases of X.
(i) If $x \in X$, show that $x_{C_X} = A_{(\mathrm{id}_X)_{B_X C_X}} x_{B_X}$ and

$$A_{(\mathrm{id}_X)_{B_X C_X}} = \begin{bmatrix} (v_1)_{C_X} & \ldots & (v_n)_{C_X} \end{bmatrix}$$

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"] & X \arrow[d, "e_{C_X}"] \\
\mathbb{R}^n \arrow[u, "e_{B_X}^{-1}"] \arrow[r, dashed, "(\mathrm{id}_X)_{B_X C_X}"'] & \mathbb{R}^n.
\end{tikzcd}$$

[1 Point]

(ii) Show that $(\mathrm{id}_X)_{B_X B_X} = \mathrm{id}_{\mathbb{R}^n}$ and $A_{(\mathrm{id}_X)_{B_X B_X}} = I_n$

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"] & X \arrow[d, "e_{B_X}"] \\
\mathbb{R}^n \arrow[u, "e_{B_X}^{-1}"] \arrow[r, dashed, "(\mathrm{id}_X)_{B_X B_X}"'] & \mathbb{R}^n.
\end{tikzcd}$$

[1 Point]

(iii) Find a 2-dimensional linear space X and bases B_X, C_X of X such that

$$(\mathrm{id}_X)_{B_X C_X} \neq \mathrm{id}_{\mathbb{R}^2} \quad \& \quad A_{(\mathrm{id}_X)_{B_X C_X}} \neq I_2.$$

[2 Points]

Exercise 5. Let X, Y be linear spaces, $n, m \geq 1$, $B_X := \{v_1, \ldots, v_n\}$ a basis of X, $B_Y := \{w_1, \ldots, w_m\}$ a basis of Y, and let $f, g : X \to Y$ be linear maps. Show the following:

(i) $A_{(f+g)_{B_X B_Y}} = A_{f_{B_X B_Y}} + A_{g_{B_X B_Y}}$.

[1 Point]

(ii) If $\lambda \in \mathbb{R}$, then $A_{(\lambda f)_{B_X B_Y}} = \lambda A_{f_{B_X B_Y}}$.

[1 Point]

(iii) The function $e_{B_X B_Y} : \mathcal{L}(X, Y) \to M_{m,n}(\mathbb{R})$, defined by $f \mapsto A_{f_{B_X B_Y}}$, for every $f \in \mathcal{L}(X, Y)$, is a linear isomorphism.

[2 Points]

Submission. Wednesday 26. June 2019, **16:00**.

Discussion. Wednesday 26. June 2019, in the Exercise-session.