LMU
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UNIVERSITÄT
MÜNCHEN

MATHEMATISCHES INSTITUT

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Mathematics for Physicists II

Sheet 8

Exercise 1. Let $A, B \in M_{m,n}(\mathbb{R})$, $C \in M_n(\mathbb{R})$, and $a \in \mathbb{R}$.
(i) $(A + B)^t = A^t + B^t$.
[1 Point]
(ii) $(a \cdot B)^t = a \cdot B^t$.
[1 Point]
(iii) $\left(A^t\right)^t = A$.
[1 Point]
(iv) $C + C^t$ is symmetric.
[1Point]

Exercise 2. Let $A \in M_{m,n}(\mathbb{R})$, $B, C \in M_{n,l}(\mathbb{R})$, and $D \in M_{l,s}(\mathbb{R})$.
i) $AI_n = A$ and $I_m A = A$.
[0.5 Point]
(ii) $A(B + C) = AB + AC$.
[1 Point]
(iii) If $a \in \mathbb{R}$, then $A(a \cdot B) = a \cdot (AB)$.
[0.5 Point]
(iv) $A(BD) = (AB)D$.
[1 Point]
(v) The multiplication $B^t A^t$ is well-defined, and $(AB)^t = B^t A^t$.
[1 Point]

Exercise 3. Let $A, B, C \in M_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Show the following.
(i) $\mathtt{Tr}(A + B) = \mathtt{Tr}(A) + \mathtt{Tr}(B)$.
[0.5 Point]
(ii) $\mathtt{Tr}(\lambda A) = \lambda \mathtt{Tr}(A)$.
[0.5 Point]
(iii) $\mathtt{Tr}(AB) = \mathtt{Tr}(BA)$.
[0.5 Point]
(iv) If B is invertible, then $\mathtt{Tr}(B^{-1}AB) = \mathtt{Tr}(A)$.
[0.5 Point]
(v) $\mathtt{Tr}(A(B + C)) = \mathtt{Tr}(AB) + \mathtt{Tr}(AC)$.
[0.5 Point]
(vi) $\mathtt{Tr}((\lambda A)B) = \lambda \mathtt{Tr}(AB)$.
[0.5 Point]
(vii) There are no matrices $A, B \in M_n(\mathbb{R})$ such that

$$AB - BA = I_n.$$

[0.5 Point]
(viii) If $A \in M_n(\mathbb{R})$ such that for every $B \in M_n(\mathbb{R})$, we have that $\mathtt{Tr}(AB) = 0$, then $A = \mathbf{0}_n$.
[0.5 Point]

Exercise 4. Show the following.
(i) The set of symmetric matrices $\mathtt{Sym}_n(\mathbb{R})$ is a linear subspace of $M_n(\mathbb{R})$, and determine its dimension.
[1 Point]
(ii) Determine the dimension of all $n \times n$-matrices $A := [a_{ij}]$ such that

$$a_{11} + a_{22} + \ldots + a_{nn} = 0.$$

[1 Point]
(iii) If $A \in \mathtt{Sym}_n(\mathbb{R})$, then $\mathtt{Tr}(AA) \geq 0$.
[1 Point]
(iv) If $A \in \mathtt{Sym}_n(\mathbb{R})$ and $A \neq \mathbf{0}_n$, then $\mathtt{Tr}(AA) > 0$.
[1 Point]

Exercise 5. If $\theta \in \mathbb{R}$, let the matrix

$$R(\theta) := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

(i) Show that $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$.
[1 Point]
(ii) Show that the matrix $R(\theta)$ has an inverse, and write down this inverse.
[1 Point]

(iii) If $(x, y) \in \mathbb{R}^2$, its *length*, or its *norm*, is defined by

$$|(x, y)| := \sqrt{x^2 + y^2}.$$

Show that the linear map $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$R_\theta(x, y) := R(\theta) \begin{bmatrix} x \\ y \end{bmatrix},$$

preseves the length of vectors i.e.,

$$|R_\theta(x, y)| = |(x, y)|.$$

[1 Point]

(iv) Show that

$$R^2(\theta) := \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}.$$

[1 Point]

Submission. Wednesday 19. June 2019, **16:00**.

Discussion. Wednesday 19. June 2019, in the Exercise-session.