Dr. Iosif Petrakis
Leonid Kolesnikov

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Mathematics for Physicists II

Sheet 7

Exercise 1. Let X, Y be linear spaces, $C, C' \subseteq X$, and $D \subseteq Y$. Show the following:
(i) If C is convex in X, and D is convex in Y, then $C \times D$ is convex in $X \times Y$.
[2 Points]
(ii) If C, C' are convex, then $C + C' := \{c + c' \mid c \in C \; \& \; c' \in C'\}$ is convex.
[1 Point]
(iii) If C is convex, and $a \in \mathbb{R}$, then $aC := \{a \cdot c \mid c \in C\}$ is convex.
[1 Point]

Exercise 2. Let X be a linear space, and $Y, Z \subseteq X$. Show the following:
(i) $Y \subseteq \mathtt{Conv}(Y)$.
[0.5 Point]
(ii) If $Y \subseteq Z$, then $\mathtt{Conv}(Y) \subseteq \mathtt{Conv}(Z)$.
[0.5 Point]
(iii) $\mathtt{Conv}(\mathtt{Conv}(Y)) = \mathtt{Conv}(Y)$.
[1 Point]
(iv) $\mathtt{Conv}(Y \cap Z) \subseteq \mathtt{Conv}(Y) \cap \mathtt{Conv}(Z)$. Does the converse inclusion hold?
[1 Point]
(v) $\mathtt{Conv}(Y) \cup \mathtt{Conv}(Z) \subseteq \mathtt{Conv}(Y \cup Z)$. Does the converse inclusion hold?
[0.5 Point]
(vi) Y is convex if and only if $\mathtt{Conv}(Y) = Y$.
[0.5 Point]

Exercise 3. Let X, Y be linear spaces, $f \in \mathcal{L}(X, Y)$, $C \subseteq X$, $D \subseteq Y$, and $Z \preceq X$. Show the following:

(i) If D is convex in Y, then $f^{-1}(D) := \{x \in X \mid f(x) \in D\}$ is convex in X.
[1 Point]

(ii) If C is convex in X, then $f(C) := \{f(c) \mid c \in C\}$ is convex in Y.
[1 Point]

(iii) If C is convex in X, then $C + Z := \{c + Z \mid c \in C\}$ is convex in X/Z.
[1 Point]

(iv) If $n \geq 1$, and $x_1, \ldots, x_n \in X$, then

$$f\big(\texttt{Conv}(x_1, \ldots, x_n)\big) = \texttt{Conv}(f(x_1), \ldots, f(x_n)).$$

[1 Point]

Exercise 4. If X is a linear space, $f, g \in X^* := \mathcal{L}(X, \mathbb{R})$, and $a \in \mathbb{R}$, show that the following sets

$$\begin{aligned}
[f > a] &:= \{x \in X \mid f(x) > a\}, \\
[f \geq a] &:= \{x \in X \mid f(x) \geq a\}, \\
[f = a] &:= \{x \in X \mid f(x) = a\}, \\
[f < a] &:= \{x \in X \mid f(x) > a\}, \\
[f \leq a] &:= \{x \in X \mid f(x) \geq a\}, \\
[f > g] &:= \{x \in X \mid f(x) > g(x)\}, \\
[f \geq g] &:= \{x \in X \mid f(x) \geq g(x)\}, \\
[f = g] &:= \{x \in X \mid f(x) = g(x)\},
\end{aligned}$$

are convex in X.

(each case **[0.5 Point]**)

Exercise 5. **(i)** Let C_1, C_2 be convex sets in $\mathbb{R}^2$, and let $x \in \texttt{Conv}(C_1 \cup C_2)$. Show that there are $y, y' \in C_1 \cup C_2$ such that $x \in \texttt{Conv}(y, y')$ (note that y and y' can be equal).
[2 Points]

(ii) If $1 < k \leq n$, and $C_1, \ldots, C_k$ are convex subsets of $\mathbb{R}^n$, show that for every

$$x \in \texttt{Conv}\left(\bigcup_{i=1}^{k} C_i\right)$$

there is $m \leq k$, and there are $y_1, \ldots, y_m \in \bigcup_{i=1}^{k} C_i$, such that

$$x \in \texttt{Conv}(y_1, \ldots, y_m).$$

[2 Points]

Submission. Wednesday 12. June 2019, **16:00**.

Discussion. Wednesday 12. June 2019, in the Exercise-session.