LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Mathematics for Physicists II

Sheet 6

Exercise 1. **(i)** Find a set X such that $\varepsilon(\varepsilon X)$ is not linearly isomorphic to εX.
[2 Points]
(ii) If X is a linear space, show that there is a unique linear map $\pi_X : \varepsilon X \to X$ such that $\pi_X \circ i_X = \mathrm{id}_X$

$$\begin{tikzcd} X \arrow[r, hook, "i_X"] \arrow[dr, "\mathrm{id}_X"'] & \varepsilon X \arrow[d, dashed, "\pi_X"] \\ & X. \end{tikzcd}$$

[2 Points]

Exercise 2. Let W be a linear space and $j_X : X \to W$ an injection, such that for every linear space Y and every function $h : X \to Y$ there is a unique linear map $h_W : W \to Y$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, hook, "j_X"] \arrow[dr, "h"'] & W \arrow[d, dashed, "h_W"] \\ & Y. \end{tikzcd}$$

Show that W is linearly isomorphic to εX.
[4 Points]

Exercise 3. Let X, Y, Z be sets, and $h : X \to Y$, $g : Y \to Z$ functions.
(i) There is a unique linear map $\varepsilon h : \varepsilon X \to \varepsilon Y$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, hook, "i_X"'] & Y \arrow[d, hook, "i_Y"] \\ \varepsilon X \arrow[r, dashed, "\varepsilon h"'] & \varepsilon Y. \end{tikzcd}$$

[1.5 Points]

(ii) The following lower outer diagram commutes i.e., $\varepsilon(g \circ h) = \varepsilon g \circ \varepsilon h$

$$\begin{tikzcd}
X \arrow[r, "h"'] \arrow[rr, bend left, "g \circ h"] \arrow[d, hook, "i_X"'] & Y \arrow[r, "g"'] \arrow[d, hook, "i_Y"] & Z \arrow[d, hook, "i_Z"] \\
\varepsilon X \arrow[r, dashed, "\varepsilon h"] \arrow[rr, bend right, "\varepsilon(g \circ h)"'] & \varepsilon Y \arrow[r, dashed, "\varepsilon g"] & \varepsilon Z.
\end{tikzcd}$$

[1.5 Points]

(iii) If X is a set, let $E_0(X) := \varepsilon X$, and, if $h : X \to Y$ is a function from the set X to the set Y, let $E_1(h) := \varepsilon h : \varepsilon X \to \varepsilon Y$ is the linear map determined in the case (i).

Show that the pair $E := (E_0, E_1)$ is a covariant functor from the category **Set** to the category **Lin**.

[1 Point]

Exercise 4. Let X, Y be linear spaces and $h : X \to Y$ a function.
The function h is a linear map if and only if $\pi_Y \circ \varepsilon h = h \circ \pi_X$

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, hook, "i_X"] & Y \arrow[d, hook, "i_Y"'] \\
\varepsilon X \arrow[r, dashed, "\varepsilon h"'] \arrow[u, bend left, "\pi_X"] & \varepsilon Y, \arrow[u, bend right, "\pi_Y"']
\end{tikzcd}$$

where the functions π_X and π_Y are from the Exercise 1(ii).
(if) **[3 Points]**
(only if) **[1 Point]**

Exercise 5. Let X be a linear space, and let the linear map $\pi_X : \varepsilon X \to X$ from the Exercise 1(ii). Moreover, let

$$N(X) := \big\{ f_{\lambda x + \mu y} - \lambda f_x - \mu f_y \mid x, y \in X \ \& \ \lambda, \mu \in \mathbb{R} \big\}.$$

(i) Show that $\langle N(X) \rangle \subseteq \mathtt{Ker}(\pi_X)$.
[0.5 Point]
(ii) Show that $f_{\mathbf{0}} \in N(X)$, and show that if $f = f(x_1) f_{x_1} \in \mathtt{Ker}(\pi_X)$, then $f \in \langle N(X) \rangle$.
[0.5 Point]
(iii) Show that if $f = f(x_1) f_{x_1} + f(x_2) f_{x_2} \in \mathtt{Ker}(\pi_X)$, then $f \in \langle N(X) \rangle$.
[0.5 Point]
(iv) Show that if $f = f(x_1) f_{x_1} + f(x_2) f_{x_2} + f(x_3) f_{x_3} \in \mathtt{Ker}(\pi_X)$, then $f \in \langle N(X) \rangle$.
[1 Point]
(iv) Show that if $f = f(x_1) f_{x_1} + f(x_2) f_{x_2} + f(x_3) f_{x_3} + f(x_4) f_{x_4} \in \mathtt{Ker}(\pi_X)$, then $f \in \langle N(X) \rangle$.
[1.5 Points]

Submission. Wednesday 05. June 2019, **16:00**.

Discussion. Wednesday 05. June 2019, in the Exercise-session.