LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Summer term 19
23.05.2019

Mathematics for Physicists II

Sheet 5

Exercise 1. Let the linear space

$$C(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\},$$

and let the function $\int : C(\mathbb{R}) \to C(\mathbb{R})$, defined by

$$f \mapsto \int f,$$

$$\left(\int f\right)(x) := \int_0^x f(t)dt, \qquad x \in \mathbb{R}.$$

(i) Show that $\int$ is a linear map.
[1 Point]
(ii) Show that $\int$ is an injection.
[1 Point]
(iii) Show that $\mathtt{Im}(\int)$ is a subspace of the set $C^1(\mathbb{R})$ of all differentiable functions $\mathbb{R} \to \mathbb{R}$ that have a continuous derivative. Is $C^1(\mathbb{R})$ equal to $C(\mathbb{R})$? Is $\mathtt{Im}(\int)$ equal to $C^1(\mathbb{R})$?
[2 Points]

Exercise 2. Let X be a linear space and $Y \preceq X$. If $x, x', y, y' \in X$, let

$$x \sim x'(\text{mod}Y) :\Leftrightarrow x' - x \in Y.$$

Show the following:
(i) $x \sim x(\text{mod}Y)$.
[0.5 point]
(ii) If $x \sim x'(\text{mod}Y)$, then $x' \sim x(\text{mod}Y)$.
[0.5 point]
(iii) If $x \sim x'(\text{mod}Y)$ and $x' \sim x''(\text{mod}Y)$, then $x \sim x''(\text{mod}Y)$.
[1 point]
(iv) If $x \sim x'(\text{mod}Y)$ and $y \sim y'(\text{mod}Y)$, then $x + y \sim x' + y'(\text{mod}Y)$.
[1 point]
(b) If $x \sim x'(\text{mod}Y)$ and $a \in \mathbb{R}$, then $a \cdot x \sim a \cdot x'(\text{mod}Y)$.
[1 point]

Exercise 3. **(i)** Let X be a linear space, Y a subspace of X and $x_1, \ldots, x_n \in X$. Show that $x_1 + Y, \ldots, x_n + Y$ are linearly dependent in X/Y if and only if

$$\exists_{y \in Y} \left(\sum_{i=1}^{n} a_i x_i = y \right),$$

for some $n \geq 1$, and $a_1, \ldots, a_n \in \mathbb{R}$ such that $a_i \neq 0$, for some $i \in \{1, \ldots, n\}$.
[2 points]

(ii) Let L be a line in $\mathbb{R}^2$, given by the equation $y = ax$, for some $a \in \mathbb{R}$. Show that the function $e : X/L \to \mathbb{R}$, defined in the Example 2.5.4 of the script, by

$$(x, y) + L \mapsto a_{(x,y)+L} := y - ax,$$

for every $(x, y) + L \in X/L$, is an isomorphism.
[2 points]

Exercise 4. If X is a linear space and $Z \preceq Y \preceq X$, then

$$(X/Z)/(Y/Z) \simeq X/Y.$$

[4 points]

Exercise 5. Let $n > 1$ and let X be a linear space with $\dim(X) = n$. Let Y be a subspace of X such that $\dim(Y) = n - 1$. Let also the set-theoretic complement $X \setminus Y$ of Y, defined by

$$X \setminus Y := \{x \in X \mid x \notin Y\}.$$

If $x, x' \in X \setminus Y$, we define

$$x \sim x' :\Leftrightarrow [x, x'] \cap Y = \emptyset,$$

where

$$[x, x'] := \big\{tx + (1 - t)x' \mid t \in [0, 1]\big\}$$

is the *linear segment* connecting x and x' in X.
Show that the relation $x \sim x'$ is an equivalence relation on $X \setminus Y$ and there are precisely two equivalence classes with respect to $\sim$.
[4 points]

Submission. Wednesday 29. May 2019, **16:00**.

Discussion. Wednesday 29. May 2019, in the Exercise-session.