LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN | MATHEMATISCHES INSTITUT

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Mathematics for Physicists II

Sheet 4

Exercise 1. Let X, Y, Z be linear spaces, $f \in \mathcal{L}(X, Y)$ and $g \in \mathcal{L}(Y, Z)$.
(i) The composition function $g \circ f$ is in $\mathcal{L}(X, Z)$.
[0.5 point]
(ii) $\mathrm{id}_X \in \mathcal{L}(X)$.
[0.5 point]
(iii) The constant function $\overline{\mathbf{0}} : X \to Y$, $x \mapsto \mathbf{0}$, is in $\mathcal{L}(X, Y)$.
[0.5 point]
(iv) $f(\mathbf{0}) = \mathbf{0}$.
[0.5 point]
(v) if $x \in X$, then $f(-x) = -f(x)$.
[0.5 point]
(vi) If $n \geq 1$, $a_1, \ldots a_n \in \mathbb{R}$, and $x_1, \ldots x_n \in X$, then

$$f\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i f(x_i).$$

[0.5 point]
(vii) Suppose that f satisfies the following property: if $x_1, \ldots, x_n$ are linearly independent in X, then $f(x_1), \ldots, f(x_n)$ are linearly independent in Y, for every $x_1, \ldots, x_n \in X$, and $n \geq 1$. Show that f is an injection.
[0.5 point]
(viii) If we define

$$\texttt{Ker}(f) := \{x \in X \mid f(x) = \mathbf{0}\},$$
$$\texttt{Im}(f) := \{y \in Y \mid \exists_{x \in X}(f(x) = y)\},$$

show that $\texttt{Ker}(f) \preceq X$, $\texttt{Im}(f) \preceq Y$, and $\texttt{Ker}(f) = \{\mathbf{0}\}$ if and only if f is an injection.
[0.5 point]

Exercise 2. Let X, Y be linear spaces with $\dim(X) = n$ and $\dim(Y) = m$, for some $n, m \geq 1$. Let the following linear operations defined on $X \times Y$:

$$(x, y) + (x', y') := (x + x', y + y'),$$
$$a \cdot (x, y) := (a \cdot x, a \cdot y),$$
$$\mathbf{0} := (\mathbf{0}, \mathbf{0}).$$

Show the following:

(i) $X \times Y$ is a linear space.
[0.5 point]
(ii) If $\{v_1, \ldots, v_n\}$ is a basis of X and $\{w_1, \ldots, w_m\}$ is a basis of Y, then

$$\big\{(v_1, \mathbf{0}), \ldots, (v_n, \mathbf{0}), (\mathbf{0}, w_1), \ldots, (\mathbf{0}, w_m)\big\}$$

is a basis of $X \times Y$.
[1 point]
(iii) $\dim(X \times Y) = n + m$.
[0.5 point]
(iv) The projections $\mathtt{pr}_X : X \times Y \to X$ and $\mathtt{pr}_Y : X \times Y \to Y$, defined by

$$\mathtt{pr}_X(x, y) := x \;\;\&\;\; \mathtt{pr}_Y(x, y) := y,$$

are linear maps.
[0.5 point]
(v) The product linear space $X \times Y$ satisfies the *universal property of products* i.e., for every linear space Z, and every linear map $f : Z \to X$, and every linear map $g : Z \to Y$ there is a unique linear map $h : Z \to X \times Y$ such that the following inner diagrams commute

$$\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[d, dashed, "h"'] \arrow[dr, "g"] & \\
X & X \times Y \arrow[l, "\mathtt{pr}_X"] \arrow[r, "\mathtt{pr}_Y"'] & Y
\end{tikzcd}$$

i.e., $f = \mathtt{pr}_X \circ h$ and $g = \mathtt{pr}_Y \circ h$.
[1.5 points]

Exercise 3. If X is a linear space, and $n \in \mathbb{N}$, such that $\dim(X) = n$, then if $Y \preceq X$ and $Z \preceq X$, show that

$$\dim(Y) + \dim(Z) = \dim(Y + Z) + \dim(Y \cap Z).$$

[4 points]

Exercise 4. Let X, Y be linear spaces, and $f \in \mathcal{L}(X, Y)$.
(i) If $n \in \mathbb{N}$, and $\dim(X) = n = \dim(Y)$, the following are equivalent:
(a) $\mathtt{Ker}(f) = \{\mathbf{0}\}$.
(b) $\mathtt{Im}(f) = Y$.
(c) f is a bijection i.e., f is an injection and a surjection.

(a) $\Rightarrow$ (b) **[0.5 point]**
(b) $\Rightarrow$ (c) **[0.5 point]**
(c) $\Rightarrow$ (a) **[0.5 point]**
(ii) If f is a linear isomorphism, then $g := f^{-1} \in \mathcal{L}(Y, X)$, and if $\dim(X) = n$, for some $n \in \mathbb{N}$, then $\dim(Y) = n$.
[1.5 points]
(iii) If $n \geq 1$, then $\dim(X) = n$ if and only if X is isomorphic to $\mathbb{R}^n$.
[1 point]

Exercise 5. If X is a linear space, and $T \in \mathcal{L}(X)$, we define

$$T^n := \begin{cases} \mathrm{id}_X & , n = 0 \\ T \circ T^{n-1} & , n > 0. \end{cases}$$

(i) If X is a linear space, and $P \in \mathcal{L}(X)$, such that $P^2 = P$, then

$$X = \mathtt{Ker}(P) \oplus \mathtt{Im}(P).$$

[2 points]
(ii) Let X be a linear space, $T \in \mathcal{L}(X)$, such that $T^2 = \mathrm{id}_X$, and let

$$P := \frac{1}{2}(\mathrm{id}_X + T) \quad \& \quad Q := \frac{1}{2}(\mathrm{id}_X - T).$$

Show the following:
(a) $P + Q = \mathrm{id}_X$.
[0.5 point]
(b) $P^2 = P$, and $Q^2 = Q$.
[1 point]
(iii) $PQ = QP = \mathbf{0}$.
[0.5 point]

Submission. Wednesday 22. May 2019, **16:00**.

Discussion. Wednesday 22. May 2019, in the Exercise-session.