LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Dr. Iosif Petrakis
Leonid Kolesnikov

Summer term 19
09.05.2019

Mathematics for Physicists II

Sheet 3

Exercise 1. Let $n \geq 1$, and let $v_1, \ldots, v_n$ be linearly independent elements of a linear space X.

(i) If their set $M := \{v_1, \ldots, v_n\}$ is a maximal set of linearly independent elements of X i.e., for every $x \in X$ we have that

$$x, v_1, \ldots, v_n$$

are linearly dependent elements of X, then M is a basis of X.
[1 point]

(ii) If $\dim(X) = n$, and $w_1, \ldots, w_n$ are linearly independent elements of X, then $B := \{w_1, \ldots, w_n\}$ is a basis of X.
[1 point]

(iii) If Y is a subspace of X with $\dim(Y) = \dim(X) = n$, then $Y = X$.
[1 point]

(iv) If $\dim(X) = n$, $1 \leq r < n$, and $w_1, \ldots, w_r$ are linearly independent elements of X, then there are elements $v_{r+1}, \ldots, v_n$ of X such that the set

$$\{w_1, \ldots, w_r, v_{r+1}, \ldots, v_n\}$$

is a basis of X.
[1 point]

Exercise 2. If X is a linear space, and $Y, Z \preceq X$, such that

$$\forall_{x \in X} \exists_{!y \in Y} \exists_{!z \in Z} \big(x = y + z\big),$$

we write $X := Y \oplus Z$. The following are equivalent:
(a) $X = Y \oplus Z$.
(b) $X = Y + Z$ and $Y \cap Z = \{\mathbf{0}\}$.

(a) $\Rightarrow$ (b) **[2 points]**,
(b) $\Rightarrow$ (a) **[2 points]**

Exercise 3. Let X be a linear space, $n \in \mathbb{N}$, and $\dim(X) = n$.
(i) If $Y \preceq X$, there is some $Z \preceq X$ such that $X = Y \oplus Z$.
[1.5 points]

(ii) Is this Z in case (i) unique?
[0.5 point]

(iii) If $Y, Z \preceq X$ such that $X = Y \oplus Z$, then $\dim(X) = \dim(Y) + \dim(Z)$.
[2 points]

Exercise 4. Let Y be a linearly independent subset of a linear space X, and $x_0 \in X$. If $x_0 \notin \langle Y \rangle$, then $Y \cup \{x_0\}$ is a linearly independent subset of X.
[4 points]

Exercise 5. If Y is a linearly independent subset of a non-trivial linear space X, there is a basis B of X, such that $Y \subseteq B$.
[4 points]

Submission. Wednesday 15. May 2019, **16:00**.

Discussion. Wednesday, 15. May 2019, in the Exercise-session.