Dr. Iosif Petrakis
Leonid Kolesnikov

Summer term 19
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Mathematics for Physicists II

Sheet 2

Exercise 1. Let $\mathbb{Q}(\sqrt{18}) := \{a + b\sqrt{18} \mid a \in \mathbb{Q} \ \& \ b \in \mathbb{Q}\}$, and $a, a', b, b' \in \mathbb{Q}$.
(i) If $a + b\sqrt{18} = a' + b'\sqrt{18}$, then $a = a'$ and $b = b'$.
[1 point]

(ii) Show that $0, 1 \in \mathbb{Q}(\sqrt{18})$ and that $\mathbb{Q}(\sqrt{18})$ is closed under the operations of addition and multiplication of real numbers.
[1.5 points]

(iii) If $a + b\sqrt{18} \in \mathbb{Q}(\sqrt{18})$ and $a + b\sqrt{18} \neq 0$, calculate its inverse

$$z := \frac{1}{a + b\sqrt{18}}$$

and show that $z \in \mathbb{Q}(\sqrt{18})$.
[1.5 points]

Exercise 2. Let $\mathcal{V} := (X; +, \mathbf{0}, \cdot)$ be a linear space, $a, b \in \mathbb{R}$, and $x, y, z, w \in X$.
(i) If $z = w$ and $x = y$, then $z + x = w + y$.
[0.5 points]

(ii) If $x = y$ and $a = b$, then $a \cdot x = b \cdot y$.
[0.5 points]

(iii) If $x + y = x + z = \mathbf{0}$, then $y = z$.
[1 point]

(iv) $0 \cdot x = \mathbf{0}$.
[0.5 points]

(v) $(-1) \cdot x = -x$, where, because of case (iii), $-x$ is the unique element y of X in condition (LS_3) such that $x + y = \mathbf{0}$.
[0.5 points]

(vi) If $x \neq \mathbf{0}$ and $a \cdot x = \mathbf{0}$, then $a = 0$.
[1 point]

Exercise 3. Let $\mathcal{V} := (X; +, \mathbf{0}, \cdot)$ be a linear space, $Y, Z \subseteq X$, and let $U, V \preceq X$.
(i) If $U + V := \{u + v \mid u \in U \ \& \ v \in V\}$, then $U + V \preceq X$.
[0.5 point]

(ii) If $U \cap V := \{x \in X \mid x \in U \ \& \ x \in V\}$, then $U \cap V \preceq X$.
[0.5 point]

(iii) If we define

$$\langle Y \rangle := \bigcap \big\{U \preceq X \mid Y \subseteq U\big\} := \big\{x \in X \mid \forall_{U \preceq X}(Y \subseteq U \Rightarrow x \in U)\big\},$$

then $\langle Y \rangle$ is the least linear subspace of X that includes Y.
[0.5 point]

(iv) If $Y \neq \emptyset$, then $\langle Y \rangle = \{\sum_{i=1}^{n} a_i y_i \mid n \geq 1 \ \& \ i \in \{1, \ldots, n\} \ \& \ a_i \in \mathbb{R} \ \& \ y_i \in Y\}$.
[0.5 point]

(v) If $Y \subseteq Z$, then $\langle Y \rangle \subseteq \langle Z \rangle$.
[0.5 point]

(vi) $\langle \langle Y \rangle \rangle = \langle Y \rangle$.
[0.5 point]

(vii) $\langle Y \rangle \cup \langle Z \rangle \subseteq \langle Y \cup Z \rangle$.
[0.5 point]

(viii) Prove, or find a counterexample to the inclusion $\langle Y \cup Z \rangle \subseteq \langle Y \rangle \cup \langle Z \rangle$.
[0.5 point]

Exercise 4. For every $n \geq 1$, the functions

$$f_1(t) := e^t, \ \ldots, \ f_n(t) := e^{nt}$$

are linearly independent in $\mathbb{F}(\mathbb{R})$.
[4 points]

Exercise 5. Prove the following implications:
(i) $(P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow \neg P)$.
[0.5 point]

(ii) $(\neg\neg\neg P) \Rightarrow \neg P$ (prove it without using the rule DNS).
[0.5 point]

(iii) $\neg \forall_{x \in X} \phi(x) \Rightarrow \exists_x \big(\neg \phi(x)\big)$ (use the rule DNS).
[2 points]

(iv) $\exists_x \big(\neg \phi(x)\big) \Rightarrow \neg \forall_{x \in X} \phi(x)$.
[1 point]

Submission. Wednesday 08. May 2019, **16:00**.

Discussion. Wednesday, 8. May 2019, in the Exercise-session.