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Mathematics for Physicists II

Sheet 15 (Probeklausur)

Exercise 1. Let X be a linear space and $x_1, x_2, x_3, x_4, x_5 \in X$. Show that if

$$x_1, x_2, x_3, x_4, x_5,$$

are linearly independent, then

$$x_1, x_2, x_3 + 2019x_4, x_4, x_5$$

are linearly independent.

[6 Points]

Exercise 2. Let X, Y be linear spaces, $f : X \to Y$ a linear map, Y a subspace of X, and $C \subseteq X$.

(i) If $x_1, x_2 \in X$, let

$$[x_1, x_2] := \big\{tx_1 + (1-t)x_2 \mid t \in [0,1]\big\}.$$

Show that

$$f\big([x_1, x_2]\big) = [f(x_1), f(x_2)].$$

[3 Points]

(ii) If C is convex in X, then

$$C + Y := \{c + Y \mid c \in C\}$$

is convex in X/Y.

[3 Points]

Exercise 3. Let X be a linear space, $n \geq 1$, and $B_X = \{v_1, \ldots, v_n\}$ a basis of X. Let the sets

$$X^* := \{f : X \to \mathbb{R} \mid f \text{ is linear}\},$$

$$X^{**} := \big(X^*\big)^* := \{g : X^* \to \mathbb{R} \mid g \text{ is linear}\}.$$

(i) Suppose that $n \geq 2$. Prove or disprove the following:

"there is $f \in X^*$ such that $\mathtt{Ker}(f) = \{\mathbf{0}\}$".

[1.5 Points]

(ii) Let $\phi : X \to X^{**}$, defined by

$$x \mapsto \phi_x$$

$$\phi_x(f) := f(x),$$

for every $f \in X^*$ and every $x \in X$.

(a) If $x \in X$, show that $\phi_x \in X^{**}$.

[0.5 Point]

(b) Show that ϕ is a linear isomorphism between X and X^{**}.

[4.5 Points]

Exercise 4. Let $U \subseteq \mathbb{R}^n$ be open and $f \in (\mathbb{R}^n)^*$ such that $f \neq \mathbf{0}$.

(i) Show that there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) = 1$.

[1 Point]

(ii) Let $x \in U$ and $\epsilon > 0$ such that $\mathcal{B}(x, \epsilon) \subseteq U$. If $\lambda \in \mathbb{R}$ such that

$$|\lambda| < \frac{\epsilon}{|x_0|},$$

show that $x + \lambda x_0 \in \mathcal{B}(x, \epsilon)$ and $f(x) + \lambda \in f(U)$.

[2 Points]

(iii) Show that $f(U)$ is open in $\mathbb{R}^n$.

[Hint: use (ii)]

[3 Points]

Exercise 5. Let $(X, \langle\langle \cdot, \cdot \rangle\rangle)$ be a positive definite, inner product space of dimension $n \geq 1$, $B_X := \{v_1, \ldots, v_n\}$ a basis of X, and

$$x = \sum_{i=1}^{n} \lambda_i v_i \quad \& \quad y = \sum_{i=1}^{n} \mu_i v_i,$$

where $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n \in \mathbb{R}$. Let

$$a_{ij} := \langle\langle v_i, v_j \rangle\rangle,$$

for every $i, j \in \{1, \ldots, n\}$. Show the following:

(i) $\langle\langle x, y \rangle\rangle = \sum_{i,j=1}^{n} \lambda_i \mu_j a_{ij}$.

[2 Points]

(ii) If B_X is orthogonal, then $\langle\langle x, y \rangle\rangle = \sum_{i=1}^{n} \lambda_i \mu_i a_{ii}$.

[2 Points]

(iii) If B is orthonormal, then $\langle\langle x, y \rangle\rangle = \sum_{i=1}^{n} \lambda_i \mu_i$.

[2 Points]

Exercise 6. **(i)** Let X be a finite-dimensional linear space, $S, T \in \mathcal{L}(X)$ invertible, and let $\lambda \in \mathbb{R} \setminus \{0\}$.

(a) Show that $S \circ T$ is invertible and find $(S \circ T)^{-1}$.

[1 Point]

(b) Show that λS is invertible.

[0.5 Point]

(ii) Let $n \geq 1$, and let $A \in M_n(\mathbb{R})$ be invertible. Show that if $\lambda \neq 0$ is an eigenvalue of A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

[4.5 Points]

Discussion. Thursday 25. July 2019, in the last lecture.