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Mathematics for Physicists II

Sheet 14

Exercise 1. **(i)** Show the following:

$$\begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix} = adf.$$

[2 Points]

(ii) Show the following:

$$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = (x_2 - x_1)(x_3 - x_2)(x_3 - x_1).$$

[2 Points]

Exercise 2. Let X be a linear space, $T \in \mathcal{L}(X)$, $n \geq 1$, and $B_X := \{v_1, \ldots, v_n\}$ is a basis of X consisting of eigenvectors of T having distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, respectively. Show that an eigenvector of T is a scalar multiple of some v_i, where $i \in \{1, \ldots, n\}$.

[4 Points]

Exercise 3. **(i)** Find the eigenvalues and a basis for the eigenspaces of the matrix

$$A = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{vmatrix}.$$

[2 Points]

(ii) Find the eigenvalues and a basis for the eigenspaces of the matrix

$$B = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 5 & -1 \\ 0 & 0 & 7 \end{vmatrix}.$$

[2 Points]

Exercise 4. Let $(X, \langle\langle\cdot,\cdot\rangle\rangle)$ be a non-trivial, finite-dimensional, positive definite inner product space. If $T \in \mathcal{L}(X)$ is symmetric with respect to $\langle\langle\cdot,\cdot\rangle\rangle$, we call X *positive definite*, if

$$\forall_{x\in X}\big(x \neq \mathbf{0} \Rightarrow \langle\langle T(x), x\rangle\rangle > 0\big).$$

Show that if T is positive definite, then all eigenvalues of T are strictly positive real numbers.
[4 Points]

Exercise 5. Let $(X, \langle\langle\cdot,\cdot\rangle\rangle)$ be a non-trivial, finite-dimensional, positive definite inner product space, $T \in \mathcal{L}(X)$ symmetric with respect to $\langle\langle\cdot,\cdot\rangle\rangle$, and v_1, v_2 eigenvectors of T with eigenvalues λ_1, λ_2, respectively. Show that

$$\lambda_1 \neq \lambda_2 \Rightarrow v_1 \perp v_2.$$

[4 Points]