LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Summer term 19
17.07.2019

Mathematics for Physicists II

Sheet 13

Exercise 1. Let $(X, \langle\langle \cdot, \cdot \rangle\rangle)$ be a finite-dimensional, non-degenerate inner product space, $S, T \in \mathcal{L}(X)$ and $\lambda \in \mathbb{R}$. Show the following.
(i) $(S + T)^t = S^t + T^t$.
[1 Point]
(ii) $(\lambda S)^t = \lambda S^t$.
[1 Point]
(iii) $(S \circ T)^t = T^t \circ S^t$.
[1 Point]
(iv) $(S^t)^t = S$.
[1 Point]

Exercise 2. (i) Give an example of a finite-dimensional, positive definite inner product space $(X, \langle\langle \cdot, \cdot \rangle\rangle)$ and of an operator $T \in \mathcal{L}(X)$ such that T preserves orthogonality with respect to $\langle\langle \cdot, \cdot \rangle\rangle$, but T is not unitary.
[1 Point]
(ii) Let $(X, \langle\langle \cdot, \cdot \rangle\rangle)$ be a finite-dimensional, positive definite inner product space, and $U \in \mathcal{L}(X)$. Show the following.
(a) U is unitary if and only if $U^t \circ U = \mathrm{id}_X$.
[2 Points]
(b) If U is unitary, then U is invertible and $U^{-1} = U^t$.
[1 Point]

Exercise 3. (i) Let $(X, \langle\langle \cdot, \cdot \rangle\rangle)$ be a finite-dimensional, positive definite inner product space, and $U \in \mathcal{L}(X)$. If U is unitary, then U^t is unitary.
[2 Points]
(ii) If X is a linear space with $\dim(X) = n \geq 1$, $T \in \mathcal{L}(X)$, and $v_1, \ldots, v_n$ are eigenvectors of T with eigenvalues $\lambda_1, \ldots, \lambda_n$, respectively, such that $\lambda_i \neq \lambda_j$ for every $i, j \in \{1, \ldots, n\}$ with $i \neq j$, show that $\{v_1, \ldots, v_n\}$ is a basis of X.
[2 Points]

Exercise 4. Let X be a non-trivial finite-dimensional linear space, $B_X = \{v_1, \ldots, v_n\}$ a basis of X, and $T \in \mathcal{L}(X)$. Show that B_X diagonalises T if and only if $v_1, \ldots, v_n$ are eigenvectors of T.
[4 Points]

Exercise 5. **(i)** If $a \in \mathbb{R}$ and $x : J \to \mathbb{R}$ is differentiable, where J is an interval of $\mathbb{R}$, show that the ordinary differential equation

$$x'(t) = ax(t) \tag{1}$$

has as set of solutions the set

$$\texttt{Sol}(1) = \big\{s : J \to \mathbb{R} \mid \exists_{C \in \mathbb{R}} \forall_{t \in J} \big(s(t) = Ce^{at}\big)\big\}.$$

[2 Points]
(ii) Let the system of odes (5.15), and let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that

$$A = \texttt{Diag}(\lambda_1, \ldots, \lambda_n).$$

If $\texttt{Sol}$ is the set of solutions of the system (5.15), then $\texttt{Sol}$ is a linear space and

$$\texttt{Sol} = \big\langle \big[e^{\lambda_1 t}\big], \ldots, \big[e^{\lambda_n t}\big] \big\rangle,$$

where

$$\big[e^{\lambda_1 t}\big] := \big(e^{\lambda_1 t}, \mathbf{0}, \ldots, \mathbf{0}\big), \ \ldots, \ \big[e^{\lambda_n t}\big] := \big(\mathbf{0}, \mathbf{0}, \ldots, e^{\lambda_n t}\big).$$

[2 Points]

Submission. Wednesday 24. Juli 2019, **16:00**.

Discussion. Wednesday 24. Juli 2019, in the Exercise-session.