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Summer term 19
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Mathematics for Physicists II

Sheet 12

Exercise 1. (i) Find a basis B_2 of the Minkowski linetime and a basis B_3 of the Minkowski planetime consisting of lightlike vectors only. Are these bases B_2 and B_3 orthogonal?
[1 Point]
(ii) Is there an orhogonal basis of the Minkowski spacetime consisting of spacelike vectors only?
[0.5 Point]
(iii) Is there an orhogonal basis of the Minkowski spacetime consisting of timelike vectors only?
[0.5 Point]
(iv) Find an orthogonal basis of the Minkowski linetime and an orthogonal basis of the Minkowski planetime, and determine the indices of nullity, positivity, and negativity of the corresponding Minkowski inner products.
[2 Points]

Exercise 2. Let X be a non-trivial, n-dimensional linear space, and let $\langle\langle \cdot, \cdot \rangle\rangle$ be an inner product on X.
(i) Show that there is a direct sum decomposition

$$X = X^+ \oplus X^- \oplus X_0$$

of X, where

$$X_0 :=:= \big\{x \in X \mid \forall_{z \in X}\big(\langle\langle x, z \rangle\rangle = 0\big\},$$

and X^+ and X^- satisfy the following conditions:

$$\forall_{x \in X^+}\big(x \neq \mathbf{0} \Rightarrow Q(x) > 0\big),$$

$$\forall_{x \in X^-}\big(x \neq \mathbf{0} \Rightarrow Q(x) < 0\big).$$

[2 Points]
(ii) Prove or disprove the following assertions:
The sets

$$\{x \in X \mid Q(x) > 0\} \cup \{\mathbf{0}\},$$

$$\{x \in X \mid Q(x) < 0\} \cup \{\mathbf{0}\}$$

are subspaces of X.
[2 Points]

Exercise 3. **(i)** If $x_0 := (x_0^1, x_0^2, x_0^3) \in \mathbb{R}^3$, show that $\{x_0\}$ is closed in $\mathbb{R}^3$.
[1.5 Points]
(i) If $(U_i)_{i\in I}$ is a family of open sets in $\mathbb{R}^n$ i.e., U_i is open for every $i \in I$, show that their union

$$\bigcup_{i\in I} U_i := \big\{x \in \mathbb{R}^n \mid \exists_{i\in I}\big(x \in U_i\big)\big\}$$

is open.
[1.5 Points]
(ii) If $(F_i)_{i\in I}$ is a family of closed sets in $\mathbb{R}^n$ i.e., U_i is closed for every $i \in I$, show that their intersection

$$\bigcap_{i\in I} F_i := \big\{x \in \mathbb{R}^n \mid \forall_{i\in I}\big(x \in F_i\big)\big\}$$

is closed.
[1 Point]

Exercise 4. Let the function $f : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$, defined by

$$f(x, y) := x^y,$$

for every $x \in (0, +\infty)$ and for every $y \in \mathbb{R}$.
(i) Show that the set $(0, +\infty) \times \mathbb{R}$ is open in $\mathbb{R}^2$.
[2 Points]
(ii) Find the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.
[Hint: Use the equality $x^y = e^{y \ln x}$]
[2 Points]

Exercise 5. Let X be an n-dimensional linear space, and let Y be a subspace of X. Let

$$Y^{\mathrm{perp}} := \big\{\phi \in X^* \mid \forall_{y\in Y}\big(\phi(y) = 0\big)\big\}.$$

(i) Show that $\dim(Y) + \dim(Y^{\mathrm{perp}}) = n$.
[2 Points]
(ii) If $\langle\langle \cdot, \cdot \rangle\rangle$ is a non-degenerate inner product on X, and if $L : X \to X^*$ is the isomorphism between X and X^* defined in the Theorem 4.7.1, then the restriction $L_{|Y^\perp} : Y^\perp \to Y^{\mathrm{perp}}$, defined by

$$x \mapsto L_x,$$

for every $x \in Y^\perp$, is an isomorphism.
[2 Points]

Submission. Wednesday 17. Juli 2019, **16:00**.

Discussion. Wednesday 17. Juli 2019, in the Exercise-session.