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Summer term 19
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Mathematics for Physicists II

Sheet 11

Exercise 1. Let $(X, \langle\langle \cdot, \cdot \rangle\rangle)$ be a positive definite, inner product space, $x, y, x_1, \ldots, x_n \in X$, and let $|| \cdot ||$ be the norm on X induced by $\langle\langle \cdot, \cdot \rangle\rangle$.
(i) If $x_i \perp x_j$, for every $i, j \in \{1, \ldots, n\}$ such that $i \neq j$, then

$$||x_1 + \ldots + x_n||^2 = ||x_1||^2 + \ldots + ||x_n||^2.$$

[1 Point]
(ii) $||x + y||^2 + ||x - y||^2 = 2\big(||x||^2 + ||y||^2\big)$.
[0.5 Point]
(iii) $||x + y|| \leq ||x|| + ||y||$. The equality holds if and only if either $x = \mathbf{0} \vee y = \mathbf{0}$ or there is some $\lambda > 0$ such that $x = \lambda y$.
[1 Point]
(iv) $\big| ||x|| - ||y|| \big| \leq ||x - y||$.
[1 Point]
(v) $||x|| - ||y|| \leq \big| ||x|| - ||y|| \big| \leq ||x + y||$.
[0.5 Point]

Exercise 2. Complete the proof of the Theorem 4.2.2.
[Hint: Use the fact that every real number is the limit of a sequence of rational numbers.]
[4 Points]

Exercise 3. (i) Find an orthonormal basis for the vector subspace Y of the Euclidean space $\mathbb{R}^4$, where

$$Y := \big\langle (1, 1, 0, 1), (1, -2, 0, 0), (1, 0, -1, 2) \big\rangle.$$

[1 Point]
(ii) If $(X, \langle\langle \cdot, \cdot \rangle\rangle)$ is a positive definite, inner product space of dimension $n \geq 1$, and Y is a subspace of X of dimension r, where $0 \leq r \leq n$, then

$$X = Y \oplus Y^{\perp}.$$

[2 Points]
(iii) Find a subspace Y of the Minkowski linetime such that $\{\mathbf{0}\} \subsetneq Y^{\perp} \cap Y$ and $\mathbb{R}^2$ is not equal to $Y + Y^{\perp}$.
[1 Point]

Exercise 4. Let $(X, \langle\langle \cdot, \cdot \rangle\rangle)$ be a positive definite, inner product space, and $|| \cdot ||$ the norm on X induced by $\langle\langle \cdot, \cdot \rangle\rangle$. Let $B := \{v_1, \ldots, v_n\} \subseteq X$, such that

(a) $v_i \perp v_j$, for every $i, j \in \{1, \ldots, n\}$ such that $i \neq j$, and

(b) $v_i \neq \mathbf{0}$, for every $i \in \{1, \ldots, n\}$.

(c) $||v_i|| = 1$, for every $i \in \{1, \ldots, n\}$.

Show that the following are equivalent.

(i) For every $x \in X$ it holds

$$||x||^2 = \sum_{i=1}^{n} \lambda_i(x)^2,$$

where, for every $i \in \{1, \ldots, n\}$

$$\lambda_i(x) := \frac{\langle\langle x, v_i \rangle\rangle}{\langle\langle v_i, v_i \rangle\rangle} = \langle\langle x, v_i \rangle\rangle.$$

(ii) B is a basis of X.

[4 Points]

Exercise 5. (i) If $A := [a_{ij}] \in M_{m,n}(\mathbb{R})$ and $(x_1, \ldots, x_n) \in \mathbb{R}^n$, show that the following are equivalent.

(a) $(x_1, \ldots, x_n)$ is a solution of the following system of linear equations

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= 0 \\ \vdots \qquad\qquad \vdots \quad &\;\; \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= 0. \end{aligned}$$

(b) $(x_1, \ldots, x_n)$ is orthogonal to the row vectors $A_1, \ldots, A_m$ of A.

[2 Points]

(ii) If $A \in M_{m,n}(\mathbb{R})$, show that

$$\texttt{cRank}(A) = \texttt{rRank}(A).$$

[2 Points]

Submission. Wednesday 10. Juli 2019, **16:00**.

Discussion. Wednesday 10. Juli 2019, in the Exercise-session.