LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Mathematics for Physicists II

Sheet 10

Exercise 1. Let X, Y, Z and W be linear spaces, $n, m, l, s \geq 1$, $B_X := \{v_1, \ldots, v_n\}$ a basis of X, $B_Y := \{w_1, \ldots, w_m\}$ a basis of Y, $B_Z := \{u_1, \ldots, u_l\}$ a basis of Z, $B_W := \{\rho_1, \ldots, \rho_s\}$ a basis of W, and let $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$ be linear maps. Show the following:

(i) $(h \circ g \circ f)_{B_X B_W} = h_{B_Z B_W} \circ g_{B_Y B_Z} \circ f_{B_X B_Y}$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rrr, bend left, "h \circ g \circ f"] & Y \arrow[r, "g"] \arrow[d, bend right, "e_{B_Y}"'] & Z \arrow[r, "h"] \arrow[d, bend right, "e_{B_Z}"'] & W \arrow[d, "e_{B_W}"] \\
\mathbb{R}^n \arrow[u, "e_{B_X}^{-1}"] \arrow[r, dashed, "f_{B_X B_Y}"] \arrow[rrr, dashed, bend right, "(h \circ g \circ f)_{B_X B_W}"'] & \mathbb{R}^m \arrow[u, bend right, "e_{B_Y}^{-1}"'] \arrow[r, dashed, "g_{B_Y B_Z}"'] & \mathbb{R}^l \arrow[u, bend right, "e_{B_Z}^{-1}"'] \arrow[r, dashed, "h_{B_Z B_W}"] & \mathbb{R}^s.
\end{tikzcd}$$

[2 Points]

(ii) $A_{(h \circ g \circ f)_{B_X B_W}} = A_{h_{B_Z B_W}} A_{g_{B_Y B_Z}} A_{f_{B_X B_Y}}$.

[2 Points]

Exercise 2. **(i)** If $n \geq 1$, X is an n-dimensional linear space, B_X is a basis of X, and $T : X \to X$ is a linear map, show that T is diagonalisable if and only if there exists some invertible matrix $B \in M_n(\mathbb{R})$ such that $B^{-1} A_{T_{B_X}} B$ is a diagonal matrix.

[3 Points]

(ii) If $A, A' \in M_n(\mathbb{R})$, we say that A and A' are *similar*, in symbols $A \sim A'$, if there is some invertible matrix $B \in M_n(\mathbb{R})$ such that $A' = B^{-1}AB$. Show that the relation $A \sim A'$ is an equivalence relation on $M_n(\mathbb{R})$.

[1 Point]

Exercise 3. Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space.

(i) If $x \in X$, then $\langle \mathbf{0}, x \rangle = 0$.

[1 Point]

(ii) If $\langle \cdot, \cdot \rangle$ is positive definite, then $\langle \cdot, \cdot \rangle$ is positive, and non-degenerate.

[1 Point]

(iii) For every $x, y \in X$ we have that

$$\langle x, y \rangle = \frac{1}{4}\big(\langle x+y, x+y \rangle - \langle x-y, x-y \rangle\big).$$

[1 Point]

(iv) If $\langle\langle \cdot, \cdot \rangle\rangle$ is an inner product on X such that $Q_{\langle \cdot, \cdot \rangle}(x) = Q_{\langle\langle \cdot, \cdot \rangle\rangle}(x)$, for every $x \in X$, then $\langle\langle x, y \rangle\rangle = \langle x, y \rangle$, for every $x, y \in X$.

[1 Point]

Exercise 4. Show the following:

(i) The Euclidean norm is a norm on $\mathbb{R}^n$.

[1 Point]

(ii) The Minkowski product is an indefinite, non-degenerate inner product on $\mathbb{R}^4$.

[2 Points]

(iii) Find timelike vectors u, w in the Minkowski spacetime satisfying

$$\langle u, w \rangle^2 > \langle u, u \rangle \langle w, w \rangle.$$

[1 Point]

Exercise 5. Let $x, y \in \mathbb{R}^4$ such that x and y are lightlike with respect to the Minkowski product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^4$. Then x and y are $\langle \cdot, \cdot \rangle$-orthogonal if and only if x, y are linearly dependent.

[Hint: Use the inequality of Cauchy]

[4 Points]

Submission. Wednesday 03. Juli 2019, **16:00**.

Discussion. Wednesday 03. Juli 2019, in the Exercise-session.