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Summer term 19
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Mathematics for Physicists II

Sheet 1

Exercise 1. Let $\mathcal{S} := (X; S), \mathcal{T} := (Y; T)$, and $\mathcal{U} := (Z; U)$ be S-spaces.

(i) If $(f, g) : \mathcal{S} \simeq \mathcal{T}$, then f and g are bijections (a function $f : X \to Y$ is a bijection, if it is an injection i.e., $\forall_{x,x' \in X}\big(f(x) = f(x') \Rightarrow x = x'\big)$, and a surjection i.e., $\forall_{y \in Y}\exists_{x \in X}\big(f(x) = y\big)$).

[2 points]

(ii) Replace the question marks with the appropriate functions, and prove the following:

(a) $(?, ?) : \mathcal{S} \simeq \mathcal{S}$.

(b) If $(f, g) : \mathcal{S} \simeq \mathcal{T}$, then $(?, ?) : \mathcal{T} \simeq \mathcal{S}$.

(c) If $(f, g) : \mathcal{S} \simeq \mathcal{T}$ and $(f', g') : \mathcal{T} \simeq \mathcal{U}$, then $(?, ?) : \mathcal{S} \simeq \mathcal{U}$.

[2 points]

Exercise 2. **(i)** Show that the structure **Rel**, defined in the Example 1.0.4, is a category.

[2 points]

(ii) Show that the structure **Pos**, defined in the Example 1.0.6, is a category.

[2 points]

Exercise 3. **(i)** Show that the pair $F := (F_0, F_1)$, defined in the example 1.0.9, is a covariant functor from **Set** to **Rel**.

[2 points]

(ii) If $\boldsymbol{C}, \boldsymbol{D}$ and $\boldsymbol{E}$ are categories, $F : \boldsymbol{C} \to \boldsymbol{D}$ and $G : \boldsymbol{D} \to \boldsymbol{E}$, define the composition of these functors $G \circ F := \big((G \circ F)_0, (G \circ F)_1\big)$ and show that $G \circ F : \boldsymbol{C} \to \boldsymbol{E}$.

[2 points]

Exercise 4. **(i)** Show that the pair $H := (H_0, H_1)$, defined in the Example 1.0.12, is a covariant functor from **Set** to **Set**.

[1 point]

(ii) Show that the the family of arrows (τ_X) in **Set**, defined in the Example 1.0.12, is a natural transformation from $\mathrm{Id}_{\mathbf{Set}}$ to H.

[3 points]

Exercise 5. Let $\boldsymbol{C}$ be a category and A in C_0. An object of $\boldsymbol{C}/A$ is an arrow f in C_1 such that $\mathtt{cod}(f) = A$. If $g : C \to A$ and $h : D \to A$ are objects of $\boldsymbol{C}/A$, an arrow α from $g : C \to A$ to $h : D \to A$ is an arrow $\alpha : C \to D$ in C_1 such that the following diagram commutes

$$\begin{tikzcd} C \arrow[rr, "\alpha"] \arrow[dr, "g"'] & & D \arrow[dl, "h"] \\ & A. & \end{tikzcd}$$

If $g : C \to A$ is an object of $\boldsymbol{C}/A$, the identity arrow of $g : C \to A$ is the arrow $\mathbf{1}_C$

$$\begin{tikzcd} C \arrow[rr, "\mathbf{1}_C"] \arrow[dr, "g"'] & & C \arrow[dl, "g"] \\ & A. & \end{tikzcd}$$

If $g : C \to A, h : D \to A$, and $k : E \to A$ are objects of $\boldsymbol{C}/A$, and if α is an arrow from $g : C \to A$ to $h : D \to A$, and if β is an arrow from $h : D \to A$ to $k : E \to A$, their composition is the arrow $\beta \circ \alpha$ in C_1

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[dr, "g"'] & D \arrow[r, "\beta"] \arrow[d, "h"] & E \arrow[dl, "k"] \\ & A. & \end{tikzcd}$$

(i) Show that $\boldsymbol{C}/A$ is a category.
[1 point]
(ii) If B is in C_0 and $f : A \to B$ is in C_1, define (and show that it is) a functor

$$F : \boldsymbol{C}/A \to \boldsymbol{C}/B.$$

[3 points]

Submission. Wednesday 08. May 2019, **16:00**.

Discussion. Wednesday, 8. May 2019, in the Exercise-session.