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Summer term 19
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Mathematics for Natural Scientists II

Sheet 1

Exercise 1. Let $\mathcal{V} := (X; +, \mathbf{0}, \cdot)$ be a linear space, $a, b \in \mathbb{R}$, and $x, y, z, w \in X$.

(i) If $z = w$ and $x = y$, then $z + x = w + y$.
[0.5 point]

(ii) If $x = y$ and $a = b$, then $a \cdot x = b \cdot y$.
[0.5 point]

(iii) If $x + y = x + z = \mathbf{0}$, then $y = z$.
[1 point]

(iv) $0 \cdot x = \mathbf{0}$.
[0.5 point]

(v) $(-1) \cdot x = -x$, where, because of case (iii), $-x$ is the unique element y of X in condition (LS_3) such that $x + y = \mathbf{0}$.
[0.5 point]

(vi) If $x \neq \mathbf{0}$ and $a \cdot x = \mathbf{0}$, then $a = 0$.
[1 point]

Exercise 2. Let $\mathbb{F}(\mathbb{R})$ be the linear space of all real-valued functions on $\mathbb{R}$.

(i) Show that the set $C(\mathbb{R})$ of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ is a linear subspace of $\mathbb{F}(\mathbb{R})$.
[2 points]

(ii) Show that the set $D(\mathbb{R})$ of all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ is a linear subspace of $C(\mathbb{R})$.
[1.5 points]

(iii) Is $D(\mathbb{R})$ a linear subspace of $\mathbb{F}(\mathbb{R})$?
[0.5 point]

Exercise 3. Let $\mathcal{V} := (X; +, \mathbf{0}, \cdot)$ be a linear space, and let $(U_i)_{i\in I}$ be a family of subspaces of $\mathcal{V}$ indexed by the set I.

(i) Show that the intersection

$$\bigcap_{i\in I} U_i := \Big\{x \in X \mid \forall_{i\in I}\big(x \in U_i\big)\Big\}$$

is a linear subspace of $\mathcal{V}$.

[0.5 point]

(ii) If we define

$$\langle Y\rangle := \bigcap \big\{U \preceq X \mid Y \subseteq U\big\} := \big\{x \in X \mid \forall_{U\preceq X}(Y \subseteq U \Rightarrow x \in U)\big\},$$

then $\langle Y\rangle$ is the least linear subspace of X that includes Y.

[1.5 point]

(iii) If $Y \neq \emptyset$, then

$$\langle Y\rangle = \bigg\{\sum_{i=1}^{n} a_i y_i \mid n \geq 1 \ \&\ i \in \{1, \ldots, n\} \ \&\ a_i \in \mathbb{R} \ \&\ y_i \in Y\bigg\}.$$

[1.5 point]

(iv) Prove, or find a counterexample to the inclusion $\langle Y \cup Z\rangle \subseteq \langle Y\rangle \cup \langle Z\rangle$.

[0.5 point]

Exercise 4. Let $D(\mathbb{R})$ be the linear space of all differentiable functions $f : \mathbb{R} \to \mathbb{R}$.

(i) Show that the functions $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$, defined by $p(t) := t$, and $q(t) := t^2$, for every $t \in \mathbb{R}$, respectively, are linearly independent in $D(\mathbb{R})$.

[2 points]

(ii) Show that the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, defined by $f(t) := e^t$, and $g(t) := e^{2t}$, for every $t \in \mathbb{R}$, respectively, are linearly independent in $D(\mathbb{R})$.

[2 points]

Submission. Monday 13. May 2019, in the Exercise-session.

Discussion. Wednesday 13. May 2019, in the Exercise-session.