Dr. Iosif Petrakis

Winter term 19/20
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Mathematics for Natural Scientists I

Sheet 4

Exercise 1. With the use of the Archimedean axiom show the following:

(i) $\forall_{x\in\mathbb{R}}\exists_{!m\in\mathbb{Z}}\big(m-1<x\leq m\big)$.

[2 points]

(ii) $\forall_{\varepsilon>0}\exists_{n\in\mathbb{N}^+}\big(\frac{1}{n}<\varepsilon\big)$.

[2 points]

Exercise 2. Let $a\in\mathbb{R}$. With the use of the Archimedean axiom and the Bernoulli inequality show the following.

(i) If $a>1$, then $\forall_{x\in\mathbb{R}}\exists_{n\in\mathbb{N}}\big(a^n>x\big)$.

[2 points]

(ii) If $0<a<1$, then $\forall_{\varepsilon>0}\exists_{n\in\mathbb{N}}\big(a^n<\varepsilon\big)$.

[2 points]

Exercise 3. Let the sequence $\delta:\mathbb{N}\to\mathbb{R}$, defined by

$$\delta_n=\frac{n}{n+1},\qquad \text{for every } n\in\mathbb{N},$$

and let the sequence $\zeta:\mathbb{N}\to\mathbb{R}$, defined by

$$\zeta_n=\frac{n}{2^n},\qquad \text{for every } n\in\mathbb{N}.$$

(i) Show that $\delta_n\xrightarrow{n}1$.

[1 point]

(ii) Using the corresponding induction principle on the set $\{n\in\mathbb{N}\mid n\geq 4\}$, show that

$$\forall_{n\geq 4}\big(n^2\leq 2^n\big).$$

[1 point]

(iii) Show that $\zeta_n\xrightarrow{n}0$.

[2 points]

Exercise 4. **(i)** Prove or disprove the following:

"The sequence of the Fibonacci numbers `Fib` is convergent"

[1 point]

(ii) Let $x \in \mathbb{R}$ be a non-zero real number, and let the sequence $\alpha : \mathbb{N} \to \mathbb{R}$, defined by $\alpha_n = x^n$, for every $n \in \mathbb{N}$. Show the following.

(a) If $|x| < 1$, then $\alpha_n \xrightarrow{n} 0$.

[1 point]

(b) If $x = 1$, then $\alpha_n \xrightarrow{n} 1$, and if $x = -1$, then α is divergent.

[1 point]

(c) If $|x| > 1$, then α is divergent.

[1 point]

Submission. Wednesday 13. November 2019, in the Exercise-session.

Discussion. Wednesday 13. November 2019, in the Exercise-session.