Dr. Iosif Petrakis

Winter term 19/20
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Mathematics for Natural Scientists I

Sheet 12

Exercise 1. **(i)** Let the function $\tan : D_{\tan} \to \mathbb{R}$, where

$$D_{\tan} = \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \mid k \in \mathbb{Z} \right\},$$

and

$$\tan(x) = \frac{\sin(x)}{\cos(x)},$$

for every $x \in D_{\tan}$. With the use of Proposition 2.4.3. show that

$$\tan'(x_0) = \frac{1}{\cos(x_0)^2},$$

for every $x_0 \in D_{\tan}$.

[2 points]

(ii) Let the funktion $\cot : D_{\cot} \to \mathbb{R}$, where

$$D_{\cot} = \mathbb{R} \setminus \left\{ k\pi \mid k \in \mathbb{Z} \right\},$$

and

$$\cot(x) = \frac{\cos(x)}{\sin(x)},$$

for every $x \in D_{\cot}$. Find the derivative $\cot'(x_0)$, where $x_0 \in D_{\cot}$.

[2 points]

[**Hint**. Use the equality $\sin(x)^2 + \cos(x)^2 = 1$.]

Exercise 2. Let $f, g : D \to \mathbb{R}$ be n-times differentiable functions at $x_0 \in D$, where $n \in \mathbb{N}$. If $f^{(0)} = f$, show the following equality:

$$(f \cdot g)^{(n)}(x_0) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x_0) \cdot g^{(k)}(x_0).$$

[4 points]

Exercise 3. **(i)** Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable and even function (see Exercise 3, Sheet 11). Show that its derivative f' is an odd function.

[2 points]

(ii) Let $a \in \mathbb{R}$ and $g : \mathbb{R}^{+*} \to \mathbb{R}$ be defined by

$$g(x) = x^a,$$

for every $x \in \mathbb{R}$. Show that

$$g'(x_0) = a x_0^{a-1},$$

for every $x_0 \in \mathbb{R}^{+*}$.

[2 points]

[**Hint**. Use the equality $x^a = \exp(a \ln(x))$ and the chain-rule.]

Exercise 4. **(i)** Let $f : C \to \mathbb{R}$, $g : D \to \mathbb{R}$, and $h : E \to \mathbb{R}$ such that $f(C) \subseteq D$ and $F(D) \subseteq E$. If f is differentiable at $x_0 \in E$, g is differentiable at $y_0 = f(x_0) \in E$, and h is differentiable at $z_0 = g(y_0)$, then the composite function $h \circ g \circ f : C \to \mathbb{R}$ is differentiable at x_0 with

$$(h \circ g \circ f)'(x_0) = h'(g(f(x_0))) \cdot g'(f(x_0)) \cdot f'(x_0).$$

[2 points]

(ii) Let the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sin\left(\sqrt{x^2 + 1}\right),$$

for every $x \in \mathbb{R}$. Find the derivative $f'(x_0)$ of f, where $x_0 \in \mathbb{R}$.

[2 points]

[**Hint**. Use the fact that $\sqrt{x} = x^{\frac{1}{2}}$.]

Submission. Wednesday 22. January 2020, in the Exercise-session.

Discussion. Wednesday 22. January 2020, in the Exercise-session.