Logic: Lecture Notes

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Chapter 1

Derivations in Minimal Logic

Mathematical logic (ML), or simply logic, is concerned with the study of formal systems related to the foundations and practice of mathematics. ML is a very broad field encompassing various theories, like the following. *Proof theory*, the main object of study of which is the concept of (formal) derivation, or (formal) proof (see e.g., [18]). *Model theory* studies interpretations, or models, of formal theories (see e.g., [6]). Axiomatic *set theory* is the formal theory of sets that underlies most of the standard mathematical practice (see e.g., [12]). It is also called Zermelo-Fraenkel set theory (ZF). The theory ZFC is ZF together with the axiom of choice. ML has strong connections to *category theory*, a theory developed first by Eilenberg and Mac Lane within homology and homotopy theory (see e.g. [2]). *Categorical logic* is that part of category theory connected to logic (see [14]). *Computability theory* is the theory of computable functions, or in general of algorithmic objects (see e.g., [17]).

An alternative to the notion of set is the concept of type (or data-type). *Type theory*, which has its origins to Russell's so called ramified theory of types, is evolved in modern times to *Martin-Löf type theory* (MLTT), which also has many applications to theoretical computer science (see [15], [16]). Recently, the late Fields medalist V. Voevodsky revealed unexpected connections between homotopy theory and logic, developing *Homotopy Type Theory* (HoTT), an extension of MLTT with his axiom of univalence and higher inductive types (see [21]).

In this chapter we develop the basics of first-order theories and we study derivations in minimal logic. Although standard mathematics is done within classical logic, great mathematicians, have developed mathematics within constructive logic. The aforementioned theories MLTT and HoTT are within intuitionistic logic. There is also constructive set theory (see [1]), constructive computability theory (see [5]). For basic mathematical theories within constructive logic see [3], [4]. Minimal logic is the most general (constructive) logic that we study here.

1.1 Inductive definitions in metatheory

In order to define the fundamental concepts of a first-order language, we need a so-called *metatheory* $\mathcal{M}$ that permits such definitions. This metatheory $\mathcal{M}$ is, in principle, a formal theory the exact description of which is left here open. What we ask from $\mathcal{M}$ is to include some theory of natural numbers, of sets and functions and of rather simple inductively defined sets. For example, one could take $\mathcal{M}$ to be the whole Zermelo-Fraenkel set theory (ZF), but smaller parts of ZF would also suffice. One could use a constructive theory of sets as a metatheory $\mathcal{M}$. Next we explain the kind of inductive definitions that must be possible in $\mathcal{M}$.

An *inductively defined set*, or an *inductive set*, X is determined by two kinds of rules (or axioms); the *introduction rules*, which determine the way the elements of X are formed, or introduced, and the *induction principle* Ind_X for X (or *elimination rule* for X) which guarantees that X is the *least* set satisfying its introduction rules.

Example 1.1.1. The most fundamental example of an inductive set is that of the set of *natural numbers* $\mathbb{N}$. Its introduction rules are:

$$\frac{}{0 \in \mathbb{N}}\ , \quad \frac{n \in \mathbb{N}}{\mathrm{Succ}(n) \in \mathbb{N}}\ .$$

According to these rules, the elements of $\mathbb{N}$ are formed by the element 0 and by the primitive, or given successor-function $\mathrm{Succ}\colon \mathbb{N} \to \mathbb{N}$. These rules alone do not determine a unique set; for example the rationals $\mathbb{Q}$ and the reals $\mathbb{R}$ satisfy the same rules. We determine $\mathbb{N}$ by postulating that $\mathbb{N}$ is the least set satisfying the above rules. This we do in a " bottom-up" way[1] with the induction principle for $\mathbb{N}$. If P, Q, R are formulas in our metatheory $\mathcal{M}$, the $\mathcal{M}$-formula $P \Rightarrow Q \Rightarrow R$ is the formula $P \Rightarrow (Q \Rightarrow R)$ or $(P \ \&\ Q) \Rightarrow R$ i.e., 'if P and if Q, then R.

The induction principle $\mathrm{Ind}_{\mathbb{N}}$ for $\mathbb{N}$ is the following formula (in $\mathcal{M}$): for every formula $A(n)$ on $\mathbb{N}$ in $\mathcal{M}$,

$$A(0) \Rightarrow \forall_{n\in\mathbb{N}}(A(n) \Rightarrow A(\mathrm{Succ}(n))) \Rightarrow \forall_{n\in\mathbb{N}}(A(n)).$$

The interpretation of $\mathrm{Ind}_{\mathbb{N}}$ is the following: the hypotheses of $\mathrm{Ind}_{\mathbb{N}}$ say that A satisfies the two formation rules for $\mathbb{N}$ i.e., $A(0)$ and $\forall_{n\in\mathbb{N}}(A(n) \to A(\mathrm{Succ}(n)))$. In this case A is a "competitor" predicate to $\mathbb{N}$. Then, if we view A as the set of all objects such that $A(n)$ holds, the conclusion of $\mathrm{Ind}_{\mathbb{N}}$ guarantees that $\mathbb{N} \subseteq A$, i.e., $\forall_{n\in\mathbb{N}}(A(n))$. In other words, $\mathbb{N}$ is "smaller" than A, and this is the case for any such A.

Notice that we use the following conventions in $\mathcal{M}$:

$$\forall_{x\in X}\big(\phi(x)\big) :\Leftrightarrow \forall_x\big(x \in X \Rightarrow \phi(x)\big),$$

$$\exists_{x\in X}\big(\phi(x)\big) :\Leftrightarrow \exists_x\big(x \in X \ \&\ \phi(x)\big).$$

The induction principle in an inductive definition is the main tool for proving properties of the defined set. In the case of $\mathbb{N}$, one can prove (exercise) its corresponding *recursion theorem* $\mathrm{Rec}_{\mathbb{N}}$, which determines the way one defines functions on $\mathbb{N}$. According to a simplified version of it, if X is a set, $x_0 \in X$ and $g : X \to X$, there exists a unique function $f : \mathbb{N} \to X$ such that

$$f(0) = x_0,$$

$$f(\mathrm{Succ}(n)) = g(f(n)); \qquad n \in \mathbb{N},$$

To show e.g., the uniqueness of f with the above properties, let $h : \mathbb{N} \to X$ such that $h(0) = x_0$ and $h(\mathrm{Succ}(n)) = g(h(n))$, for every $n \in \mathbb{N}$. Using $\mathrm{Ind}_{\mathbb{N}}$ on $A(n) :\Leftrightarrow (f(n) = h(n))$, we get $\forall_n(A(n))$. As an example of a function defined through $\mathrm{Rec}_{\mathbb{N}}$, let $\mathtt{Double} : \mathbb{N} \to \mathbb{N}$ defined by

$$\mathtt{Double}(0) = 0,$$

$$\mathtt{Double}(\mathrm{Succ}(n)) = \mathrm{Succ}(\mathrm{Succ}(\mathtt{Double}(n)))$$

i.e., $X = \mathbb{N}, x_0 = 0$ and $g = \mathrm{Succ} \circ \mathrm{Succ}$.

[1]If we work inside the set of real numbers $\mathbb{R}$, we can also do this in a "top-down" way by defining $\mathbb{N}$ to be the intersection of all sets satisfying these introduction rules. Notice that also in this way $\mathbb{N}$ is the least subset of $\mathbb{R}$ satisfying its introduction rules.

Example 1.1.2. Let A be a non-empty set that we call *alphabet*. The set A^* of *words* over A is introduced by the following rules

$$\frac{}{\mathrm{nil}_{A^*} \in A^*}\ , \quad \frac{w \in A^*, a \in A}{w \star a \in A^*}\ .$$

The symbol nil_{A^*} denotes the empty word, while the word $w \star s$ denotes the concatenation of the word w and the letter $a \in A$. The induction principle Ind_{A^*} for A^* is the following: if $P(w)$ is any formula on A^* in $\mathcal{M}$, then

$$P(\mathrm{nil}_{A^*}) \Rightarrow \forall_{w \in A^*} \forall_{a \in A} \big(P(w) \Rightarrow P(w \star a)\big) \Rightarrow \forall_{w \in A^*}(P(w)).$$

A simplified version of the corresponding recursion theorem Rec_{A^*} is the following: If X is a set, $x_0 \in X$, and if $g_a : X \to X$, for every $a \in A$, there is a function $f : A^* \to X$ such that

$$f(\mathrm{nil}_{A^*}) = x_0,$$

$$f(w \star a) = g_a(f(w)); \qquad w \in A^*,\ a \in A.$$

As an example of a function defined through Rec_{A^*}, if $X = A^*$, $w_0 \in A^*$ and if $g_a(w) = w \star a$, for every $a \in A$, let the function $f_{w_0} : A^* \to A^*$ defined by

$$f_{w_0}(\mathrm{nil}_{A^*}) = w_0,$$

$$f_{w_0}(w \star a) = g_a(f_{w_0}(w))$$

i.e., $f_{w_0}(w) = w_0 \star w$ is the concatenation of the words w_0 and w (we use the same symbol for the concatenation of a word and a symbol and for the concatenation of two words).

If ZF is our metatheory $\mathcal{M}$, then the proof of the recursion theorem that corresponds to an inductive definition can be complicated. If as metatheory we use a theory like Martin-Löf's type theory MLTT, there is a completely mechanical, hence trivial, way to recover the corresponding recursion rule from the induction rule of an inductive definition.

1.2 First-order languages

Definition 1.2.1. *Let* $\mathtt{Var} = \{v_n \mid n \in \mathbb{N}\}$ *be a fixed countably infinite set of variables. We also denote the elements of* $\mathtt{Var}$ *by* x, y, z, *etc. Let* $L = \{\to, \wedge, \vee, \forall, \exists, (,), ,\}$, *where each element of* L *is called a logical symbol. A first-order language over* $\mathtt{Var}$ *and* L *is a pair* $\mathcal{L} = (\mathtt{Rel}, \mathtt{Fun})$, *where* $\mathtt{Var}, L, \mathtt{Rel}, \mathtt{Fun}$ *are pairwise disjoint sets*[2] *such that*

$$\mathtt{Rel} = \bigcup_{n \in \mathbb{N}} \mathtt{Rel}^{(n)},$$

where for every $n \in \mathbb{N}$, $\mathtt{Rel}^{(n)}$ *is a, possibly empty, set of* n*-ary relation symbols or predicate symbols. Moreover,* $\mathtt{Rel}^{(n)} \cap \mathtt{Rel}^{(m)} = \emptyset$, *for every* $n \neq m$. *A* 0*-ary relation symbol is called a propositional symbol. The symbol* $\bot$ (*read falsum*) *is required as a fixed propositional symbol*

[2]For simplicity, we do no use the more accurate notations $\mathtt{Rel}_{\mathcal{L}}, \mathtt{Fun}_{\mathcal{L}}$.

i.e., $\texttt{Rel}^{(0)}$ *is inhabited by* $\bot$. *The language will not, unless stated otherwise, contain the equality symbol* $=$, *which is a* 2*-ary relation symbol. Moreover,*

$$\texttt{Fun} = \bigcup_{n \in \mathbb{N}} \texttt{Fun}^{(n)},$$

where for every $n \in \mathbb{N}$, $\texttt{Fun}^{(n)}$ *is a, possible empty, set of* n*-ary function symbols. Moreover,* $\texttt{Fun}^{(n)} \cap \texttt{Fun}^{(m)} = \emptyset$, *for every* $n \neq m$. *A* 0*-ary function symbol is called constant, and let*

$$\text{Const} = \texttt{Fun}^{(0)}.$$

Clearly, the above definition rests on some theory of sets, and of natural numbers, which, as we have already said, are presupposed for our metaheory $\mathcal{M}$. The equality symbol used in Definition 1.2.1 is the equality (of sets, or objects) in $\mathcal{M}$. If our formal language includes one more fixed countably infinite set of variables $\texttt{VAR} = \{V_n \mid n \in \mathbb{N}\}$, where V_i is a variable of another sort, e.g., a set-variable, then one could define the notion of a *second-order* language over $\texttt{Var}, \texttt{VAR}$ and L in a similar fashion.

Example 1.2.2. The first-order language of *arithmetic* is the pair $(\{\bot, =\}, \{0, S, +, \cdot\})$, which is written for simplicity as $(\bot, =, 0, S, +, \cdot)$, where $0 \in \text{Const}, S \in \texttt{Fun}^{(1)}$, and $+, \cdot \in \texttt{Fun}^{(2)}$. The first-order language of Zermelo-Fraenkel set theory (ZF) is the pair $(\{\bot, =, \in\}, \emptyset)\})$, which is written for simplicity as $(\bot, =, \in)$, where $\in$ is in $\texttt{Rel}^{(2)}$.

1.3 Terms

The set $\texttt{Term}_{\mathcal{L}}$ of terms of a first-order language $\mathcal{L}$ is inductively defined. For simplicity we omit the subscript $\mathcal{L}$. $\mathbb{N}^+$ denotes the set of strictly positive natural numbers.

Definition 1.3.1. *The set* $\texttt{Term}$ *of terms of a first-order language* $\mathcal{L}$ *is defined by the following introduction rules:*

$$\frac{x \in \texttt{Var}}{x \in \texttt{Term}}, \qquad \frac{c \in \text{Const}}{c \in \texttt{Term}},$$

$$\frac{n \in \mathbb{N}^+, \quad t_1, \ldots, t_n \in \texttt{Term}, \quad f \in \texttt{Fun}^{(n)}}{f(t_1, \ldots, t_n) \in \texttt{Term}},$$

to which, the following induction principle $\text{Ind}_{\texttt{Term}}$ *corresponds:*

$$\forall_{x \in \texttt{Var}}(P(x)) \Rightarrow$$

$$\forall_{c \in \text{Const}}(P(c)) \Rightarrow$$

$$\forall_{n \in \mathbb{N}^+} \forall_{t_1, \ldots, t_n \in \texttt{Term}} \forall_{f \in \texttt{Fun}^{(n)}}((P(t_1) \ \& \ \ldots \ \& \ P(t_n)) \Rightarrow P(f(t_1, \ldots, t_n)) \Rightarrow$$

$$\forall_{t \in \texttt{Term}}(P(t)),$$

where $P(t)$ *is any formula (in* $\mathcal{M}$*) on* $\texttt{Term}$.

In words, every variable is a term, every constant is a term, and if $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol with $n \geq 1$, then $f(t_1, \ldots, t_n)$ is a term. If r, s are terms and $\circ$ is a binary function symbol, we usually write $r \circ s$ instead of $\circ(r, s)$. E.g.,

$$0, \ S(0), \ S(S(0)), \ S(0) + S(S(0))$$

are terms of the language of arithmetic. As in the case of the induction principle for natural numbers, the induction principle for $\texttt{Term}$ expresses that $\texttt{Term}$ is the *least* set satisfying its defining rules. A formula $P(t)$ on $\texttt{Term}$ could be "the number of left parentheses, (, occurring in t is equal to the number of right parentheses,), occurring in t". We need to express this formula in mathematical terms. For that we need the following recursion theorem for $\texttt{Term}$.

Proposition 1.3.2 (Recursion theorem for $\texttt{Term}$ ($\mathrm{Rec}_{\texttt{Term}}$)). *Let X be a set. If*

$$F_{\texttt{Var}} : \texttt{Var} \to X,$$

$$F_{\mathrm{Const}} : \mathrm{Const} \to X,$$

$$F_{f,n} : X^n \to X,$$

for every $n \in \mathbb{N}^+$ and $f \in \texttt{Fun}^{(n)}$, are given functions, there is a unique function $F : \texttt{Term} \to X$ such that, for every $n \in \mathbb{N}^+, t_1, \ldots, t_n \in \texttt{Term}$, and $f \in \texttt{Fun}^{(n)}$,

$$F(x) = F_{\texttt{Var}}(x), \quad x \in \texttt{Var},$$

$$F(c) = F_{\mathrm{Const}}(c), \quad c \in \mathrm{Const},$$

$$F(f(t_1, \ldots, t_n)) = F_{f,n}(F(t_1), \ldots, F(t_n)).$$

Proof. The proof of the existence of F is similar to the corresponding existence-proof in the recursion theorem for $\mathbb{N}$, and it is an exercise. The uniqueness of F is also shown with the use of $\mathrm{Ind}_{\texttt{Term}}$. If $G : \texttt{Term} \to X$ satisfies the defining properties of F, we show that $\forall_{t \in \texttt{Term}}\big(F(t) = G(t)\big)$, by using $\mathrm{Ind}_{\texttt{Term}}$ on the formula $P(t) :\Leftrightarrow F(t) = G(t)$. □

Using the recursion theorem for $\texttt{Term}$ one can define e.g., the function $P_{\texttt{left}} : \texttt{Term} \to \mathbb{N}$ such that $P_{\texttt{left}}(t)$ is the number of left parentheses occurring in $t \in \texttt{Term}$. It suffice to define it on the variables, the constants, and the complex terms $f(t_1, \ldots, t_n)$ supposing that $P_{\texttt{left}}$ is defined on the terms $t_1, \ldots, t_n$. Namely, we define

$$P_{\texttt{left}}(u_i) = 0,$$

$$P_{\texttt{left}}(c) = 0,$$

$$P_{\texttt{left}}(f(t_1, \ldots, t_n)) = 1 + \sum_{i=1}^{n} P_{\texttt{left}}(t_i).$$

Here we used the recursion theorem for $\texttt{Term}$ with respect the functions $F_{\texttt{Var}} \to \texttt{Var} \to \mathbb{N}$, F_{Const}: Const $\to \mathbb{N}$, and $F_{f,n}$: $\mathbb{N}^n \to \mathbb{N}$, where $n \in \mathbb{N}^+$ and $f \in \texttt{Fun}^{(n)}$, defined by the rules:

$$F_{\texttt{Var}}(x) = 0 = F_{\mathrm{Const}}(c),$$

$$F_{f,n}(m_1, \ldots, m_n) = 1 + \sum_{i=1}^{n} m_i.$$

In exactly the same way, one defines the function $P_{\texttt{right}} : \texttt{Term} \to \mathbb{N}$ such that $P_{\texttt{right}}(t)$ is the number of right parentheses occurring in $t \in \texttt{Term}$. Now we can show the following.

Proposition 1.3.3. $\forall_{t \in \texttt{Term}}\big(P_{\texttt{left}}(t) = P_{\texttt{right}}(t)\big)$.

Proof. We apply $\mathrm{Ind}_{\mathtt{Term}}$ on the formula $P(t) :\Leftrightarrow P_{\mathtt{left}}(t) = P_{\mathtt{right}}(t)$. The validity of $P(x)$, for every $x \in \mathtt{Var}$, and $P(c)$, for every $c \in \mathrm{Const}$ is trivial. If $f(t_1, \ldots, t_n)$ is a complex term, such that $P(t_i)$ holds, for every $i \in \{1, \ldots, n\}$, then by the inductive hypothesis we get

$$\begin{aligned} P_{\mathtt{left}}\big(f(t_1, \ldots, t_n)\big) &= 1 + \sum_{i=1}^{n} P_{\mathtt{left}}(t_i) \\ &= 1 + \sum_{i=1}^{n} P_{\mathtt{right}}(t_i) \\ &= P_{\mathtt{right}}\big(f(t_1, \ldots, t_n)\big). \end{aligned}$$

□

1.4 Formulas

Definition 1.4.1. *The set of formulas* $\mathtt{Form}$ *of a first-order language* $\mathcal{L}$ *is defined by the following introduction rules:*

$$\frac{n \in \mathbb{N}, \quad t_1, \ldots, t_n \in \mathtt{Term}, \quad R \in \mathtt{Rel}^{(n)}}{R(t_1, \ldots, t_n) \in \mathtt{Form}},$$

$$\frac{A, B \in \mathtt{Form}}{A \to B, \ A \wedge B, \ A \vee B \in \mathtt{Form}},$$

$$\frac{A \in \mathtt{Form}, \quad x \in \mathtt{Var}}{\forall_x A, \ \exists_x A \in \mathtt{Form}},$$

to which, the following induction principle $\mathrm{Ind}_{\mathtt{Form}}$ *corresponds:*

$$\forall_{n \in \mathbb{N}} \forall_{t_1, \ldots, t_n \in \mathtt{Term}} \forall_{R \in \mathtt{Rel}^{(n)}} (P(R(t_1, \ldots, t_n))) \Rightarrow$$

$$\forall_{A, B \in \mathtt{Form}} \big(P(A) \ \& \ P(B) \Rightarrow \big(P(A \to B) \ \& \ P(A \wedge B) \ \& \ P(A \vee B)\big)\big) \Rightarrow$$

$$\forall_{A \in \mathtt{Form}} \forall_{x \in \mathtt{Var}} \big(P(A) \Rightarrow P(\forall_x A) \ \& \ P(\exists_x A)\big) \Rightarrow$$

$$\forall_{A \in \mathtt{Form}} (P(A)),$$

where $P(A)$ *is any formula in* $\mathcal{M}$ *on* $\mathtt{Form}$*. The formulas of the form* $R(t_1, \ldots, t_n)$ *are called prime formulas, or atomic formulas, or just atoms. If* r, s *are terms and* $\sim$ *is a binary relation symbol, we also write* $r \sim s$ *for the prime formula* $\sim (r, s)$*. Since* $\bot \in \mathtt{Rel}^{(0)}$*, we get* $\bot \in \mathtt{Form}$*. We call* $A \to B$ *the implication from* A *to* B*,* $A \wedge B$ *the conjunction of* A, B*, and* $A \vee B$ *the disjunction of* A, B*. The negation* $\neg A$ *of a formula* A *is defined as the formula*

$$\neg A = A \to \bot.$$

As usual, we use the notational convention $A \to B \to C = A \to (B \to C)$*. The formulas generated by the prime formulas are called complex, or non-atomic formulas. A formula* $\forall_x A$ *is called a universal formula, and a formula* $\exists_x A$ *is called an existential formula.*

As usual, the induction principle $\mathrm{Ind}_{\mathtt{Form}}$ expresses that $\mathtt{Form}$ is the *least* set satisfying its introduction rules. Note that $\mathrm{Ind}_{\mathtt{Form}}$ consists of formulas in $\mathcal{M}$, where the same quantifiers and logical symbols, except from the meta-theoretic implication symbol $\Rightarrow$ and the metatheoretic conjuction symbol &, are used. Since the variables occurring in these meta-theoretic formulas are different from $\texttt{Var}$, it is easy to understand from the context the difference between the formulas in $\texttt{Form}$ and the formulas in $\mathcal{M}$. The expression $\bot \to \bot$ is a formula, and also the expressions $\forall_x(\bot \to \bot)$ and $\exists_x(R(x) \vee S(x))$. Notice that we have added two parentheses (left and right) in both last examples, in order to make them clear to read. Alternatively, one could have used the introduction rule

$$\frac{A, B \in \texttt{Form}}{(A \to B),\ (A \wedge B),\ (A \vee B) \in \texttt{Form}},$$

but then it would be cumbersome to be faithful to it all the time. It is easy to associate to each formula its formation tree i.e., the tree of all introduction rules that generate the formula, which lies in the root of this tree. A metatheoretic formula $P(A)$ on $\texttt{Form}$ could express e.g., "the number of left parentheses occurring in A is equal to the number of right parentheses occurring in A". As in the case of terms, we need a recursion theorem for $\texttt{Form}$ to formulate $P(A)$. Let $\texttt{Prime}$ be the set of all prime formulas i.e., the set

$$\texttt{Prime} = \big\{R(t_1, \ldots, t_n) \mid R \in \texttt{Rel}^{(n)}, t_1, \ldots, t_n \in \texttt{Term}, n \in \mathbb{N}\big\}.$$

Proposition 1.4.2 (Recursion theorem for $\texttt{Form}$ ($\mathrm{Rec}_{\texttt{Form}}$)). *Let X be a set. If*

$$F_{\texttt{Prime}} : \texttt{Prime} \to X$$

$$F_{\to} : X \times X \to X, \quad F_{\wedge} : X \times X \to X, \quad F_{\vee} : X \times X \to X,$$

$$F_{\forall,x} : X \to X, \qquad F_{\exists,x} : X \to X,$$

for every $x \in \texttt{Var}$, are given functions, there is a unique function $F : \texttt{Form} \to X$ such that

$$F(R(t_1, \ldots, t_n)) = F_{\texttt{Prime}}(R(t_1, \ldots, t_n)),$$

$$F(A \to B) = F_{\to}(F(A), F(B)),$$

$$F(A \wedge B) = F_{\wedge}(F(A), F(B)),$$

$$F(A \vee B) = F_{\vee}(F(A), F(B)),$$

$$F(\forall_x A) = F_{\forall,x}(F(A)),$$

$$F(\exists_x A) = F_{\exists,x}(F(A)).$$

Proof. We proceed similarly to the proof of Proposition 1.3.2. □

It is a simple exercise to define recursively the functions $P_{\texttt{left}} : \texttt{Form} \to \mathbb{N}, P_{\texttt{right}} : \texttt{Form} \to \mathbb{N}$, and show inductively that $\forall_{A \in \texttt{Form}}\big(P_{\texttt{left}}(A) = P_{\texttt{right}}(A)\big)$. Next we define recursively the height $|A| \in \mathbb{N}$ of a formula A, which represents the "height" of the formation-tree of A with respect to the introduction rules of the set $\texttt{Form}$.

Definition 1.4.3. *The function height $|.| : \texttt{Form} \to \mathbb{N}$*

$$|P| = 0, \qquad P \in \texttt{Prime},$$

$$|A \Box B| = \max\{|A|, |B|\} + 1, \qquad \Box \in \{\to, \wedge, \vee\},$$

$$|\triangle_x A| = |A| + 1, \qquad \triangle \in \{\forall, \exists\}.$$

In Definition 1.4.3 we applied $\mathrm{Rec}_{\mathtt{Form}}$ on the following $\mathbb{N}$-valued functions, defined by

$$F_{\mathtt{Rel}}(P) = 0,$$

$$F_{\Box}(m, n) = \max\{m, n\} + 1,$$

$$F_{\triangle,x}(m) = m + 1.$$

Definition 1.4.4. *The function length* $||.|| : \mathtt{Form} \to \mathbb{N}$ *is defined recursively by the clauses*

$$||P|| = 1, \qquad P \in \mathtt{Prime},$$

$$||A \,\Box\, B|| = ||A|| + ||B||, \qquad \Box \in \{\to, \wedge, \vee\},$$

$$||\triangle_x A|| = 1 + ||A||, \qquad \triangle \in \{\forall, \exists\}.$$

Proposition 1.4.5. $\forall_{A \in \mathtt{Form}}(||A|| + 1 \leq 2^{|A|+1})$.

Proof. Exercise. $\Box$

1.5 Substitutions in terms

Next we define the set of *free* variables occurring in a term t, and the set of free variables occurring in a formula A. As prime formulas are defined by an n-relation symbol and n-number of terms, it is very often the case that in order to define a function on $\mathtt{Form}$, we need first to define a corresponding function on $\mathtt{Term}$. If X is a set, $\mathcal{P}^{\mathrm{fin}}(X)$ denotes the set of finite subsets of X. If $Y, Z \subseteq X$, then $Y \setminus Z = \{x \in X \mid x \in Y \;\&\; x \notin Z\}$.

Definition 1.5.1. *Let the function* $\mathrm{FV}_{\mathtt{Term}} : \mathtt{Term} \to \mathcal{P}^{\mathit{fin}}(\mathtt{Var})$ *defined by*

$$\mathrm{FV}_{\mathtt{Term}}(x) = \{x\},$$

$$\mathrm{FV}_{\mathtt{Term}}(c) = \emptyset,$$

$$\mathrm{FV}_{\mathtt{Term}}(f(t_1, \ldots, t_n)) = \bigcup_{i=1}^{n} \mathrm{FV}_{\mathtt{Term}}(t_i).$$

The function $\mathrm{FV}_{\mathtt{Form}} : \mathtt{Form} \to \mathcal{P}^{\mathit{fin}}(\mathtt{Var})$ *is defined by*

$$\mathrm{FV}_{\mathtt{Form}}(R) = \emptyset, \quad R \in \mathtt{Rel}^{(0)},$$

$$\mathrm{FV}_{\mathtt{Form}}(R(t_1, \ldots, t_n)) = \bigcup_{i=1}^{n} \mathrm{FV}_{\mathtt{Term}}(t_i), \quad R \in \mathtt{Rel}^{(n)}, n \in \mathbb{N}^+,$$

$$\mathrm{FV}_{\mathtt{Form}}(A \Box B) = \mathrm{FV}_{\mathtt{Form}}(A) \cup \mathrm{FV}_{\mathtt{Form}}(B),$$

$$\mathrm{FV}_{\mathtt{Form}}(\triangle_x A) = \mathrm{FV}_{\mathtt{Form}}(A) \setminus \{x\}.$$

If $\mathrm{FV}(A) = \emptyset$*, then* A *is called a sentence, or a closed formula.*

According to Definition 1.5.1, a variable y is free in a prime formula A, if just occurs in A, it is free in $A \square B$, if it is free in A or free in B, and it is free in $\triangle_x A$, if it is free in A and $y \neq x$. E.g., the formulas

$$\forall_y(R(y) \to S(y)), \quad \forall_y(R(y) \to \forall_z S(z))$$

are sentences, while y is free in the formula

$$(\forall_y(R(y)) \to S(y).$$

Definition 1.5.2. *$\mathbb{W}(\mathcal{L})$ is the set of finite lists of symbols from the set $\mathtt{Var} \cup L \cup \mathtt{Rel} \cup \mathtt{Fun}$. The set $\mathbb{W}(\mathcal{L})$ can be defined inductively as the set $[\mathtt{Var} \cup L \cup \mathtt{Rel} \cup \mathtt{Fun}]^*$ of words over the alphabet $\mathtt{Var} \cup L \cup \mathtt{Rel} \cup \mathtt{Fun}$ (see Example 1.1.2).*

Clearly, $\mathtt{Term}, \mathtt{Form} \subsetneq \mathbb{W}(\mathcal{L})$, as e.g., $fR \wedge g(\bot, u_8$ is a word neither in $\mathtt{Term}$ nor in $\mathtt{Form}$.

Definition 1.5.3. *If $s \in \mathtt{Term}$ and $x \in \mathtt{Var}$ are fixed, the function*

$$\mathrm{Sub}_{s/x} : \mathtt{Term} \to \mathbb{W}(\mathcal{L})$$

$$t \mapsto \mathrm{Sub}_{s/x}(t) = t[x := s],$$

determines the word generated by substituting x from s in t, and it is defined by the clauses

$$v_n[x := s] = \begin{cases} s & , x = v_n \\ v_n & , \textit{otherwise,} \end{cases}$$

$$c[x := s] = c,$$

$$f(t_1, \ldots, t_n)[x := s] = f(t_1[x := s], \ldots, t_n[x := s]).$$

Proposition 1.5.4. *If $s \in \mathtt{Term}$ and $x \in \mathtt{Var}$, then $\forall_{t \in \mathtt{Term}}(t[x := s] \in \mathtt{Term})$.*

Proof. It follows trivially by $\mathrm{Ind}_{\mathtt{Term}}$. □

Proposition 1.5.5. *If $s \in \mathtt{Term}$ and $x \in \mathtt{Var}$, then $\forall_{t \in \mathtt{Term}}(x \notin \mathrm{FV}(t) \Rightarrow t[x := s] = t)$.*

Proof. We use induction on $\mathtt{Term}$. If $t = v_i$, for some $v_i \in \mathtt{Var}$, then $x \notin \mathrm{FV}(v_i) \Leftrightarrow x \notin \{v_i\} \Leftrightarrow x \neq v_i$, hence $v_i[x := s] = v_i$. If $t = c$, for some $c \in \mathrm{Const}$, then $x \notin \mathrm{FV}(c) \Leftrightarrow x \notin \emptyset$, which is always the case. By definition of substitution we get $c[x := s] = c$. If $t = f(t_1, \ldots, t_n)$, for some $f \in \mathtt{Fun}^{(n)}$ and $t_1, \ldots, t_n \in \mathtt{Term}$, then $x \notin \mathrm{FV}(f(t_1, \ldots, t_n)) \Leftrightarrow x \notin \mathrm{FV}(t_i)$, for every $i \in \{1, \ldots, n\}$. By the inductive hypothesis on $t_1, \ldots, t_n$ we get $t_i[x := s] = t_i$, for every $i \in \{1, \ldots, n\}$. Hence, $f(t_1, \ldots, t_n)[x := s] = f(t_1[x := s], \ldots, t_n[x := s]) = f(t_1, \ldots, t_n)$. □

1.6 Substitutions in formulas

If we consider the formula $\exists_y(\neg(y = x))$, then the possible substitution of x from y would generate the formula $\exists_y(\neg(y = y))$, which cannot be true in any "interpretation" of these symbols i.e., when y ranges over some collection of objects and $=$ is the equality of the objects in this collection. Hence, we need to be careful with substitution on semantical, rather than syntactical, grounds. Note also that x is free in A, and if it is substituted by y, then y is not free in A (in this case we say that it is *bound* in A). This is often called a "*capture*", and we want to avoid it.

Definition 1.6.1. *Let* $s \in \texttt{Term}$*, such that* $\mathrm{FV}(s) = \{y_1, \ldots, y_m\}$*, and* $x \in \texttt{Var}$*. If* $\mathbf{2} = \{0, 1\}$*, we define a function*

$$\mathrm{Free}_{s,x} : \texttt{Form} \to \mathbf{2}$$

that determines when "the variable x *is substitutable i.e., it is free to be substituted, from* s *in some formula". Namely, if* $\mathrm{Free}_{s,x}(A) = 1$*, then* x *is substitutable from* s *in* A*, and if* $\mathrm{Free}_{s,x}(A) = 0$*, then* x *is not substitutable from* s *in* A*. From now on, when we define a function on* `Form` *that is based on a function on* `Term`*, as in the case of* $\mathrm{FV}_{\texttt{Form}}$ *and* $\mathrm{FV}_{\texttt{Term}}$*, we omit the subscripts and we understand from the context their domain of definition. The function* $\mathrm{Free}_{s,x}$ *is defined by*

$$\mathrm{Free}_{s,x}(P) = 1; \quad P \in \texttt{Prime},$$

$$\mathrm{Free}_{s,x}(A \Box B) = \mathrm{Free}_{s,x}(A) \cdot \mathrm{Free}_{s,x}(B),$$

$$\mathrm{Free}_{s,x}(\triangle_y A) = \begin{cases} 0 & , x = y \ \vee \ [x \neq y \ \& \ y \in \{y_1, \ldots, y_m\}] \\ 1, & , x \neq y \ \& \ x \notin \mathrm{FV}(A) \setminus \{y\} \\ \mathrm{Free}_{s,x}(A) & , x \neq y \ \& \ y \notin \{y_1, \ldots, y_m\} \ \& \ x \in \mathrm{FV}(A). \end{cases}$$

According to Definition 1.6.1, x is substitutable from s in a prime formula, since there are no quantifiers in it that can generate a capture. It is substitutable in $A \Box B$, if it is substitutable both in A and B. In the case of an $\exists$-, or $\forall$-formula, if x is not free in A (which is equivalent to $x \neq y \ \& \ x \notin \mathrm{FV}(A) \setminus \{y\}$), then we set $\mathrm{Free}_{s,x}(\triangle_y A) \equiv 1$, since no capture is possible to be generated.

If then we consider the formula $\exists_y(\neg(y = x))$, by Definition 1.6.1 we get

$$\mathrm{Free}_{y,x}\big(\exists_y(\neg(y = x))\big) = 0.$$

If x, y, z are distinct variables, it is easy to see that

$$\mathrm{Free}_{z,x}(R(x)) = 1,$$

$$\mathrm{Free}_{z,x}\big(\forall_z R(x)\big) = 0,$$

$$\mathrm{Free}_{f(x,z),x}\big(\forall_y S(x, y)\big) = 1, \quad \text{if } x \neq y, y \neq z,$$

$$\mathrm{Free}_{f(x,z),x}\big(\exists_z \forall_y (S(x, y) \Rightarrow R(x))\big) = 0.$$

Definition 1.6.2. *If* $s \in \texttt{Term}$ *and* $x \in \texttt{Var}$ *are fixed, the function*

$$\mathrm{Sub}_{s/x} : \texttt{Form} \to \mathbb{W}(\mathcal{L})$$

$$A \mapsto \mathrm{Sub}_{s/x}(A) = A[x := s],$$

determines the word generated by substituting x *from* s *in* A*, and it is defined as follows: If* $\mathrm{Free}_{s,x}(A) = 0$ *then* $A[x := s] = A$*. If* $\mathrm{Free}_{s,x}(A) = 1$*, then*

$$R[x := s] = R, \quad R \in \texttt{Rel}^{(0)},$$

$$R(t_1, \ldots, t_n)[x := s] = R(t_1[x := s], \ldots, t_n[x := s]), \quad R \in \texttt{Rel}^{(n)}, n \in \mathbb{N}^+,$$

$$(A \Box B)[x := s] = (A[x := s] \Box B[x := s]),$$

$$(\triangle_y A)[x := s] = \triangle_y (A[x := s]).$$

Often, we write for simplicity $A(s)$ *instead of* $A[x := s]$*.*

Note that if $\mathrm{Free}_{s,x}(A\Box B) = 1$, then $\mathrm{Free}_{s,x}(A) = \mathrm{Free}_{s,x}(B) = 1$.

Proposition 1.6.3. *If $x \in \mathtt{Var}$ and $s \in \mathtt{Term}$, then $\forall_{A\in\mathtt{Form}}(A[x := s] \in \mathtt{Form})$.*

Proof. Exercise. □

Proposition 1.6.4. *If $x \in \mathtt{Var}$ and $s \in \mathtt{Term}$, then $\forall_{A\in\mathtt{Form}}(x \notin \mathrm{FV}(A) \Rightarrow A[x := s] = A)$.*

Proof. We use induction on Form. If $A = R$, for some $R \in \mathtt{Rel}^{(0)}$, then $x \notin \mathrm{FV}(R) \Leftrightarrow x \notin \emptyset$, which is always the case. Since $\mathrm{Free}_{s,x}(R) = 1$, by definition of substitution we get $R[x := s] = R$. If $A = R(t_1, \ldots, t_n)$, for some $R \in \mathtt{Rel}^{(n)}$, $n \in \mathbb{N}^+$, and $t_1, \ldots, t_n \in \mathtt{Term}$, then $x \notin \mathrm{FV}(R(t_1, \ldots, t_n)) \Leftrightarrow x \notin \bigcup_{i=1}^n \mathrm{FV}(t_i)$, for every $i \in \{1, \ldots, n\}$. By Proposition 1.5.5 we get $t_i[x := s] = t_i$, for every $i \in \{1, \ldots, n\}$, hence, since $\mathrm{Free}_{s,x}(R(t_1, \ldots, t_n)) = 1$, we have that

$$R(t_1, \ldots, t_n)[x := s] = R(t_1[x := s], \ldots, t_n[x := s]) = R(t_1, \ldots, t_n).$$

If our formula is of the the form $A\Box B$, then $x \notin \mathrm{FV}(A\Box B) \Leftrightarrow x \notin \mathrm{FV}(A) \cup \mathrm{FV}(B) \Leftrightarrow x \notin \mathrm{FV}(A)$ and $x \notin \mathrm{FV}(B)$. If $\mathrm{Free}_{s,x}(A\Box B) = 0$, then we get immediately what we want. If $\mathrm{Free}_{s,x}(A\Box B) = 1$, then by the inductive hypothesis on A, B we get $A[x := s] = A$ and $B[x := s] = B$, hence by Definition 1.6.2 we have that

$$(A\Box B)[x := s] = (A[x := s]\Box B[x := s]) = (A\Box B).$$

If our formula is of the form $\triangle_y A$, then $x \notin \mathrm{FV}(\triangle_y A) \Leftrightarrow x \notin \mathrm{FV}(A) \setminus \{y\} \Leftrightarrow x \notin \mathrm{FV}(A)$ or $x = y$. If $x = y$, then $\mathrm{Free}_{s,x}(\triangle_y A) = 0$, hence $(\triangle_y A)[x := s] = \triangle_y A$. If $x \notin \mathrm{FV}(A) \setminus \{y\}$ and $x \neq y$, then $x \notin \mathrm{FV}(A)$, and by inductive hypothesis on A we get

$$(\triangle_y A)[x := s] = \triangle_y(A[x := s]) = \triangle_y A.$$

If $x \in \mathrm{FV}(A)$, and $x \neq y \ \wedge \ y \notin \{y_1, \ldots, y_m\}$, the required implication follows trivially. □

If $\vec{x} = (x_1, \ldots, x_n)$ is a given n-tuple of distinct variables in Var and $\vec{s} = (s_1, \ldots, s_n)$ is a given n-tuple of terms in Term, for some $n \in \mathbb{N}^+$, we can define similarly for every formula A the formula $A[\vec{x} := \vec{s}]$ generated by the substitution of x_i from s_i in A, for every $i \in \{1, \ldots, n\}$.

1.7 The Brouwer-Heyting-Kolmogorov-interpretation

The next thing to answer is "what does it mean to prove some $A \in \mathtt{Form}$?". A first informal answer was given by intuitionists like Brouwer and Heyting, and, independently, from Kolmogorov. The combination of the proof-interpretation of formlulas given by Brouwer, Heyting, and Kolmogorov is called the Brouwer-Heyting-Kolmogorov-interpretation, or the BHK-interpretation. Notice that this is interpretation presupposes an informal, or primitive, or unexplained, notion of proof. Moreover, the interpretation of a proof of a prime formula, other than $\bot$, is not addressed in BHK-interpretation.

Definition 1.7.1 (BHK-interpretation)**.** *Let $A, B \in \mathtt{Form}$, such that it is understood what it means "q is a proof (or witness, or evidence) of A" and "r is a proof of B".*

($\wedge$) *A proof of $A \wedge B$ is a pair (p_0, p_1) such that p_0 is a proof of A and p_1 is a proof of B.*

($\rightarrow$) *A proof of $A \rightarrow B$ is a rule r that associates to any proof p of A a proof $r(p)$ of B.*

($\vee$) *A proof of* $A \vee B$ *is a pair* (i, p_i)*, where if* $i = 0$*, then* p_0 *is a proof of* A*, and if* $i = 1$*, then* p_1 *is a proof of* B*.*

($\bot$) *There is no proof of* $\bot$*.*

For the next two rules let $A(x)$ *be a formula i.e.,* $\mathrm{FV}(A) \subseteq \{x\}$*, such that it is understood what it means "*q *is a proof of* $A(x)$*".*

($\forall$) *A proof of* $\forall_x A(x)$ *is a rule* R *that associates to any given* x *a proof* R_x *of* $A(x)$*.*

($\exists$) *A proof of* $\exists_x A(x)$ *is a pair* (x, q)*, where* q *is a proof of* $A(x)$*.*

We write $p : A$ *to denote that* p *is a proof of* A*.*

Usually, the BHK-interpretation of a quantified formula requires that $x \in X$, for some given set X. The extension of BHK-interpretation to formulas $\forall_x A(x)$ and $\exists_x A(x)$, where $\mathrm{FV}(A)$ is larger than some singleton $\{x\}$, is obvious. The notions of rule in the clauses ($\to$) and ($\forall$) are unclear, and are taken as primitive. As we have already said, the nature of a proof, or a witness, is also left unexplained. Despite these problems, the BHK-interpretation captures essential elements of the mathematical process of proof. Especially, it captures, informally, the notion of a *constructive proof*, as the clauses for ($\vee$) and ($\exists$) indicate. A formal version of the BHK-interpretation of `Form` is a so-called *realisability interpretation* (see [20]).

Example 1.7.2. Let the formula $D = (A \to B \to C) \to (A \to B) \to A \to C$, which, according to our notational convention, is the formula

$$(A \to (B \to C)) \to ((A \to B) \to (A \to C)).$$

According to BHK, a proof

$$p : (A \to B \to C) \to ((A \to B) \to (A \to C))$$

is a rule that sends a supposed proof $q : A \to (B \to C)$ to a proof

$$p(q) : (A \to B) \to (A \to C),$$

which, in turn, is a rule that sends a proof $r : A \to B$ to a proof $[p(q)](r) = [p(q)](r) : A \to C$. This proof is a rule that sends a proof $s : A$ to a proof $\big[[p(q)](r)\big](s) : C$. Hence we need to define the later proof through our supposed proofs. By definition $q(s) : B \to C$, and hence $[q(s)](r(s)) : C$. Thus we define

$$\big[[p(q)](r)\big](s) = [q(s)](r(s)).$$

Example 1.7.3. Let the formula

$$E = \forall_x (A \to B) \to A \to \forall_x B, \quad \text{if } x \notin \mathrm{FV}(A).$$

According to BHK, a proof $p : \forall_x (A \to B) \to A \to \forall_x B$ is a rule that sends a supposed proof $q : \forall_x (A \to B)$ to a proof

$$p(q) : A \to \forall_x B,$$

which is a rule that sends a proof $r : A$ to some proof

$$[p(q)](r) : \forall_x B.$$

The proof $q : \forall_x(A \to B)$ is understood as a family of proofs

$$q = \big(q_x : A \to B\big)_x,$$

and, similarly, the required proof $[p(q)](r) : \forall_x B$ is a family of proofs

$$[p(q)](r) = \Bigg(\big[[p(q)](r)\big]_x : B\Bigg)_x.$$

We define this family of proofs by the rule

$$\big[[p(q)](r)\big]_x = q_x(r).$$

Example 1.7.4. A BHK-proof $p : A \to A$ is a rule that associates to every $q : A$ a proof of A. Clearly the identity rule $p(q) = q$ is such a proof.

Example 1.7.5. A BHK-proof $p^* : A \to \neg\neg A$ is a rule that associates to every $q : A$ a proof $p^*(q) : (A \to \bot) \to \bot$. If $r : A \to \bot$, we need to get a proof $[p^*(q)](r) : \bot$. For that we define $[p^*(q)](r) = r(q)$.

It is easy to see that there is no straightforward method to find a BHK-proof of the converse implication $\neg\neg A \to A$, which is an instance of the so-called *double negation elimination* principle (DNE). As we will see later in the course, this principle holds only classically. There are some instances of DNE though, that can be shown constructively.

Example 1.7.6. A BHK-proof $p : \neg\neg\neg A \to \neg A$ is a rule p such that for every $q : (\neg\neg A) \to \bot$, we have that $p(q) : A \to \bot$. Let $p^* : A \to \neg\neg A$ from Example 1.7.5. If $r : A$, we define $[p(q)](r) = q(p^*(r))$.

1.8 Gentzen's derivations, a first presentation

Another informal proof of $D = (A \to B \to C) \to (A \to B) \to A \to C$ goes as follows: Assume $A \to B \to C$. To show $(A \to B) \to A \to C$, we assume $A \to B$. To show $A \to C$ we assume A. We show C by using the third assumption twice and we have $B \to C$ by the first assumption, and B by the second assumption. From $B \to C$ and B we obtain C. Then we obtain $A \to C$ by canceling the assumption on A, and $(A \to B) \to A \to C$ by canceling the second assumption; and the result follows by canceling the first assumption.

Another informal proof of $E = \forall_x(A \to B) \to A \to \forall_x B$, where $x \notin \mathrm{FV}(A)$, goes as follows: Assume $\forall_x(A \to B)$. To show $A \to \forall_x B$ we assume A. To show $\forall_x B$ let x be arbitrary; note that we have not made any assumptions on x. To show B we have $A \to B$ by the first assumption, and hence also B by the second assumption. Hence $\forall_x B$. Hence $A \to \forall_x B$, canceling the second assumption. Hence E, canceling the first assumption.

A characteristic feature of this second kind of informal proofs is that assumptions are introduced and eliminated again. At any point in time during the proof the free or "open" assumptions are known, but as the proof progresses, free assumptions may become canceled or "closed" through what we later call the "introduction rule" for $\to$.

We reserve the word *proof* for the informal level; a formal representation of a proof will be called a *derivation*. An intuitive way to communicate derivations is to view them as labeled trees each node of which denotes a rule application. The labels of the inner nodes are the formulas derived as conclusions at those points, and the labels of the leaves are formulas or terms. The labels of the nodes immediately above a node k are the *premises* of the rule application. At the root of the tree we have the conclusion (or end formula) of the whole derivation. In natural deduction systems one works with *assumptions* at leaves of the tree; they can be either *open* or *closed* (canceled). Any of these assumptions carries a *marker*. As markers we use *assumption variables* denoted $u, v, w, u_0, u_1, \dots$. The variables in `Var` will now often be called *object variables*, to distinguish them from assumption variables. If at a node below an assumption the dependency on this assumption is removed (it becomes closed), we record this by writing down the assumption variable. Since the same assumption may be used more than once (this was the case in the first example above), the assumption marked with u (written $u\colon A$) may appear many times. Of course we insist that distinct assumption formulas must have distinct markers.

An inner node of the tree is understood as the result of passing from premises to the conclusion of a given rule. The label of the node then contains, in addition to the conclusion, also the name of the rule. In some cases the rule binds or closes or cancels an assumption variable u (and hence removes the dependency of all assumptions $u\colon A$ thus marked). An application of the $\forall$-introduction rule similarly binds an object variable x (and hence removes the dependency on x). In both cases the bound assumption or object variable is added to the label of the node.

First we have an assumption rule, allowing to write down an arbitrary formula A together with a marker u:

$$u\colon A \qquad \text{assumption.}$$

The other rules of natural deduction split into introduction rules (I-rules for short) and elimination rules (E-rules) for the logical connectives. E.g., for implication $\to$ there is an introduction rule $\to^+$ and an elimination rule $\to^-$ also called *modus ponens*. The left premise $A \to B$ in $\to^-$ is called the *major* (or *main*) premise, and the right premise A the *minor* (or *side*) premise. Note that with an application of the $\to^+$-rule *all* assumptions above it marked with $u\colon A$ are canceled (which is denoted by putting square brackets around these assumptions), and the u then gets written alongside. There may of course be other uncanceled assumptions $v\colon A$ of the same formula A, which may get canceled at a later stage. We use symbols like M, N, K, for derivations.

Definition 1.8.1 (A rather simplified presentation of Gentzen's derivations)**.** *The tree*

$$\frac{a\colon A}{A}\, 1_A$$

is a derivation tree of a formula A from assumption A. We use the variable assumption $a\colon A$ only for this tree. The introduction and elimination rules for implication are:

$$\dfrac{\begin{array}{c}[u\colon A]\\ \mid M\\ B\end{array}}{A \to B}\,\to^+ u \qquad \dfrac{\begin{array}{c}\mid M\\ A \to B\end{array} \qquad \begin{array}{c}\mid N\\ A\end{array}}{B}\,\to^-$$

For the universal quantifier $\forall$ there is an introduction rule $\forall^+$ (again marked, but now with the bound variable x) and an elimination rule $\forall^-$ whose right premise is the term r to be substituted. The rule $\forall^+ x$ with conclusion $\forall_x A$ is subject to the following (eigen-)variable condition *to avoid capture: the derivation M of the premise A must not contain any open assumption having x as a free variable.*

$$\frac{\begin{array}{c}\mid M\\ A\end{array}}{\forall_x A}\,\forall^+ x \qquad \frac{\begin{array}{c}\mid M\\ \forall_x A\end{array}\qquad r \in \mathtt{Term}}{A(r)}\,\forall^-$$

For disjunction the introduction and elimination rules are

$$\frac{\begin{array}{c}\mid M\\ A\end{array}}{A \vee B}\,\vee^+_0 \qquad \frac{\begin{array}{c}\mid N\\ B\end{array}}{A \vee B}\,\vee^+_1 \qquad \frac{\begin{array}{c}\\ \mid M\\ A \vee B\end{array}\quad \begin{array}{c}[u\colon A]\\ \mid N\\ C\end{array}\quad \begin{array}{c}[v\colon B]\\ \mid K\\ C\end{array}}{C}\,\vee^- u, v$$

For conjunction we have the rules

$$\frac{\begin{array}{c}\mid M\\ A\end{array}\qquad \begin{array}{c}\mid N\\ B\end{array}}{A \wedge B}\,\wedge^+ \qquad \frac{\begin{array}{c}\\ \mid M\\ A \wedge B\end{array}\quad \begin{array}{c}[u\colon A]\quad [v\colon B]\\ \mid N\\ C\end{array}}{C}\,\wedge^- u, v$$

and for the existential quantifier we have the rules

$$\frac{r \in \mathtt{Term}\qquad \begin{array}{c}\mid M\\ A(r)\end{array}}{\exists_x A}\,\exists^+ \qquad \frac{\begin{array}{c}\\ \mid M\\ \exists_x A\end{array}\quad \begin{array}{c}[u\colon A]\\ \mid N\\ B\end{array}}{B}\,\exists^- x, u \text{ (var.cond.)}$$

Similar to $\forall^+ x$ the rule $\exists^- x, u$ is subject to an (eigen-)variable condition*: in the derivation N the variable x (i) should not occur free in the formula of any open assumption other than $u\colon A$, and (ii) should not occur free in B. Again, in each of the elimination rules $\vee^-$, $\wedge^-$ and $\exists^-$ the left premise is called* major (*or* main) *premise, and the right premise is called the* minor (*or* side) *premise.*

Notice that, as in the case of the BHK-interpretation, there is no rule for the derivation of a prime formula P, other than the trivial unit-rule 1_P. It is a nice exercise to check the compatibility of Gentzen's rules to the corresponding BHK-proofs. The rule $\vee^- u, v$

$$\frac{\begin{array}{c}\\ \mid M\\ A \vee B\end{array}\quad \begin{array}{c}[u\colon A]\\ \mid N\\ C\end{array}\quad \begin{array}{c}[v\colon B]\\ \mid K\\ C\end{array}}{C}\,\vee^- u, v$$

is understood as follows: given a derivation tree for $A \vee B$ and derivation trees for C with assumption variables $u : A$ and $v : B$, respectively, a derivation tree for C is formed, such that $u : A$ and $v : B$ are cancelled. Similarly we understand the rules $\to^+ u$, $\wedge^- u, v$ and $\exists^- x, u$. The above definition is a quite complex inductive definition. In order to rewrite it, we introduce the following notions. Note that the rules of Definition 1.8.1 are used in the presence of free assumptions in the same way. E.g., next follows a derivation tree for C with assumption formula G:

$$\dfrac{\begin{matrix} w\colon G \\ \mid M \\ A\vee B\end{matrix} \qquad \begin{matrix}[u\colon A]\\ \mid N \\ C\end{matrix} \qquad \begin{matrix}[v\colon B]\\ \mid K\\ C\end{matrix}}{C}\,\vee^- u,v$$

We now give derivations of the two example formulas D, E, treated informally above. Since in many cases the rule used is determined by the conclusion, we suppress in such cases the name of the rule. Moreover, often we write only $a\colon A$, instead of the whole tree that corresponds to 1_A. First we give the derivation of D:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[u\colon A\to B\to C] \qquad [w\colon A]}{B\to C} \qquad \cfrac{[v\colon A\to B]\qquad [w\colon A]}{B}}{C}}{A\to C}\,\to^+ w}{(A\to B)\to A\to C}\,\to^+ v}{(A\to B\to C)\to(A\to B)\to A\to C}\,\to^+ u$$

Next we give the derivation of E:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[u\colon \forall_x(A\to B)] \qquad x\in \mathtt{Var}}{A\to B} \qquad [v\colon A]}{B}\,\forall^+ x}{\forall_x B}\,}{A\to\forall_x B}\,\to^+ v}{\forall_x(A\to B)\to A\to \forall_x B}\,\to^+ u$$

Note that the variable condition is satisfied: In the derivation of B the still open assumption formulas are A and $\forall_x(A\to B)$; by hypothesis x is not free in A, and by Definition 1.5.1 it is also not free in $\forall_x(A\to B)$.

1.9 Gentzen's derivations, a more formal presentation

Next we present a more formal version of the previous, non-trivial, inductive definition in $\mathcal{M}$.

Definition 1.9.1. *Let* $\mathtt{Avar}$ *be a new infinite set of "assumption variables", and let*

$$\mathtt{Aform} = \mathtt{Avar}\times\mathtt{Form},$$

where, for every $(u, A)\in\mathtt{Aform}$ *we write* $u\colon A$*. If* V *is a non-empty finite subset of* $\mathtt{Aform}$ *i.e.,* $V = \{u_1\colon A_1, \dots, u_n\colon A_n\}$*, we define*

$$\mathtt{Form}(V) = \big\{A\in\mathtt{Form} \mid \exists_{u\in\mathtt{Avar}}\big(u\colon A\in V\big)\big\} = \{A_1,\dots,A_n\}.$$

Definition 1.9.2 (A formal presentation of Gentzen's derivations)**.** *We define inductively the set* $\mathfrak{D}_V(A)$ *of derivations of a formula* A *with assumption variables in* V*, where* V *is a finite subset of* $\mathtt{Aform}$*. If* $V=\emptyset$*, we write* $\mathfrak{D}(A)$*. The following introduction-rules are considered:*

(1_A) *The tree* 1_A *is an element of* $\mathfrak{D}_{\{a\colon A\}}(A)$*.*

$(\to^+ u)$
$$\frac{M \in \mathfrak{D}_{\{u\colon A\}}(B)}{\frac{M}{A\to B}\to^+ u \in \mathfrak{D}(A\to B)}.$$

$(\to^-)$
$$\frac{M \in \mathfrak{D}_V(A\to B) \quad N \in \mathfrak{D}_W(A)}{\frac{M\ N}{B}\to^- \in \mathfrak{D}_{V\cup W}(B)}.$$

$(\wedge^+)$
$$\frac{M \in \mathfrak{D}_V(A) \quad N \in \mathfrak{D}_W(B)}{\frac{M\ N}{A\wedge B}\wedge^+ \in \mathfrak{D}_{V\cup W}(A\wedge B)}.$$

$(\wedge^- u, w)$
$$\frac{M \in \mathfrak{D}_V(A\wedge B) \quad N \in \mathfrak{D}_{\{u\colon A, w\colon B\}\cup W}(C) \quad \{u\colon A, w\colon B\}\cap W = \emptyset}{\frac{M\ N}{C}\wedge^-u,w \in \mathfrak{D}_{V\cup W}(C)}.$$

$(\vee_0^+)$
$$\frac{M \in \mathfrak{D}_V(A)}{\frac{M}{A\vee B}\vee_0^+ \in \mathfrak{D}_V(A\vee B)}.$$

$(\vee_1^+)$
$$\frac{N \in \mathfrak{D}_W(B)}{\frac{N}{A\vee B}\vee_1^+ \in \mathfrak{D}_W(A\vee B)}.$$

$(\vee^- u, w)$
$$\frac{M \in \mathfrak{D}_V(A\vee B) \quad N \in \mathfrak{D}_{\{u\colon A\}\cup U}(C) \quad K \in \mathfrak{D}_{\{w\colon B\}\cup W}(C) \quad \{u\colon A\}\cap U=\{w\colon B\}\cap W=\emptyset}{\frac{M\ N\ K}{C}\vee^-u,w \in \mathfrak{D}_{V\cup U\cup W}(C)}.$$

$(\forall^+)$
$$\frac{M \in \mathfrak{D}_V(A) \quad x\in \mathtt{Var} \quad \forall_{B\in\mathtt{Form}(V)}(x\notin \mathrm{FV}(B))}{\frac{M}{\forall_x A}\forall^+ \in \mathfrak{D}_V(\forall_x A)}.$$

$(\forall^-)$
$$\frac{M \in \mathfrak{D}_V(\forall_x A) \quad t\in \mathtt{Term} \quad \mathrm{Free}_{t,x}(A)=1}{\frac{M\ t}{A[x:=t]}\forall^- \in \mathfrak{D}_V(A[x:=t])}.$$

$(\exists^+)$
$$\frac{t\in \mathtt{Term} \quad x\in \mathtt{Var} \quad \mathrm{Free}_{t,x}(A)=1 \quad M \in \mathfrak{D}_V(A[x:=t])}{\frac{t\ M}{\exists_x A}\exists^+ \in \mathfrak{D}_V(\exists_x A)}.$$

$(\exists^- x, u)$
$$\frac{M \in \mathfrak{D}_V(\exists_x A) \quad N \in \mathfrak{D}_{\{u\colon A\}\cup W}(B) \quad W\cap\{u\colon A\}=\emptyset \quad x\notin \mathrm{FV}(B) \quad \forall_{B\in\mathtt{Form}(W)}(x\notin \mathrm{FV}(B))}{\frac{M\ N}{B}\exists^-x,u \in \mathfrak{D}_{V\cup W}(B)}.$$

For simplicity we do not include here the corresponding induction principle[3]*. If* $V = \{u\colon A\}, W = \{v\colon A\}$, $M \in \mathfrak{D}_V(B)$ *and* $N \in \mathfrak{D}_W(B)$*, we say that* M *and* N *are equal, and we write* $M = N$.

[3]Although half of the above introduction rules are called elimination-rules, all of them are introduction rules to the definition of $\mathfrak{D}_V(A)$. They are called elimination-rules because the corresponding logical symbol in L is "eliminated" in the end-formula lying at the root of the rule.

It is easy to see that the above defined equality is an equivalence relation.

Definition 1.9.3. *A formula A is called* derivable *in* minimal logic*, or simply derivable, written* $\vdash A$*, if there is a derivation of A (without free assumptions) using the natural deduction rules of Definition 1.9.2 i.e.,*

$$\vdash A :\Leftrightarrow \exists_M (M \in \mathfrak{D}(A)).$$

A formula A is called derivable from assumptions $A_1, \ldots, A_n$*, written*

$$\{A_1, \ldots, A_n\} \vdash A, \quad \textit{or simpler} \;\; A_1, \ldots, A_n \vdash A,$$

if there is a derivation of A with free assumptions among $A_1, \ldots, A_n$ *i.e.,*

$$A_1, \ldots, A_n \vdash A :\Leftrightarrow \exists_{V \subseteq^{fin} \texttt{Aform}} \Big(\texttt{Form}(V) \subseteq \{A_1, \ldots, A_n\} \; \& \; \exists_M \big(M \in \mathfrak{D}_V(A)\big)\Big).$$

If $\Gamma \subseteq \texttt{Form}$*, a formula A is called derivable from* Γ*, written* $\Gamma \vdash A$*, if A is derivable from finitely many assumptions* $A_1, \ldots, A_n \in \Gamma$*.*

By definition we have that

$$A \vdash A :\Leftrightarrow \exists_{V \subseteq^{\text{fin}} \texttt{Aform}} \big(\texttt{Form}(V) \subseteq \{A\} \; \& \; \exists_M (M \in \mathfrak{D}_V(A))\big).$$

If $V = \{a \colon A\}$, then $\texttt{Form}(V) = \{A\}$ and $1_A \in \mathfrak{D}_V(A)$. Hence we always have that $A \vdash A$.

1.10 The preorder category of formulas

Definition 1.10.1 (Eilenberg, Mac Lane (1945))**.** *A category* $\boldsymbol{C}$ *is a structure* $(C_0, C_1, \mathrm{dom}, \texttt{cod}, \circ, \mathbf{1})$*, where*

(i) C_0 *is the collection of the objects of* $\boldsymbol{C}$*,*

(ii) C_1 *is the collection of the arrows of* $\boldsymbol{C}$*,*

(iii) *For every* f *in* C_1*,* $\mathrm{dom}(f)$*, the domain of* f*, and* $\texttt{cod}(f)$*, the codomain of* f*, are objects in* C_0*, and we write* $f : A \to B$*, where* $A = \mathrm{dom}(f)$ *and* $B = \texttt{cod}(f)$*,*

(iv) *If* $f : A \to B$ *and* $g : B \to C$ *are arrows of* $\boldsymbol{C}$ *i.e.,* $\mathrm{dom}(g) = \texttt{cod}(f)$*, there is an arrow* $g \circ f : A \to C$*, which is called the composite of* f *and* g*,*

(v) *For every A in* C_0*, there is an arrow* $\mathbf{1}_A : A \to A$*, the identity arrow of A,*

such that the following conditions are satisfied:

(a) *If* $f : A \to B$*, then* $f \circ \mathbf{1}_A = f = \mathbf{1}_B \circ f$*.*

(b) *If* $f : A \to B$*,* $g : B \to C$ *and* $h : C \to D$*, then* $h \circ (g \circ f) = (h \circ g) \circ f$*.*

If A, B *are in* C_0*, we denote by* $\mathrm{Hom}_{\boldsymbol{C}}(A, B)$*, or simply by* $\mathrm{Hom}(A, B)$*, if* $\boldsymbol{C}$ *is clear from the context, the collection of arrows* f *in* C_1 *with* $\mathrm{dom}(f) = A$ *and* $\texttt{cod}(f) = B$*.*

Example 1.10.2. The collection of sets and functions between them is the simplest example of a category, which is denoted by **Set**.

The objects of a category are not necessarily sets, and hence the arrows are not necessarily functions. This is exactly the case with the category of formulas **Form**.

Proposition 1.10.3. *The category of formulas* ***Form*** *has objects the formulas in* `Form` *and an arrow from A to B is a derivation of B from an assumption of the form $u\colon A$ i.e.,*

$$M\colon A \to B :\Leftrightarrow M \in \mathfrak{D}_{\{u\colon A\}}(B).$$

Proof. (Exercise). One has to define the composition $N \circ M$, where $N\colon B \to C$ and $M\colon A \to B$. As expected, the unit arrow 1_A is the trivial derivation of a from assumption A. To prove that **Form** satisfies properties (a) and (b) of Definition 1.10.1, one needs to use the definition of equality of derivations given in Definition 1.9.2. □

Although a formula A is not a set, we have already discussed approaches to logic, like Martin Löf's type theory MLTT, where a set, or a type, is also a formula. In BHK-interpretation one can also understand a formula A as the "set" of its proofs $p : A$. Moreover, the arrow in **Form** is a not a function, but as it is an arrow in this category, it behaves as an "abstract" function with respect to the abstract operation of composition in **Form**. Recall that the arrow $M\colon A \to B$ in **Form** is captured in BHK-interpretation by some rule that behaves like a function! So, in some sense, the category **Form** captures the "set-character" of a formula and the "function-character" of a proof $p\colon A \to B$ in BHK-interpretation.

Notice that if $L\colon A \to B$ in **Form** i.e., $L \in \mathfrak{D}_{\{u\colon A\}}(B)$, and if $M\colon A \to B$ in **Form** i.e., $M \in \mathfrak{D}_{\{v\colon A\}}(B)$, are arrows in **Form**, then by the definition of equality of derivations in Definition 1.9.2 we get $L = M$. Hence any two arrows $A \to B$ in **Form** are equal, or, in other words, there is at most one arrow from A to B. An immediate consequence of this fact is that proofs of equality of arrows in **Form** become trivial, as two arrows are always equal when they have the same domain and codomain.

Definition 1.10.4. *A category* ***C*** *is called a preorder, or a thin category, if there is at most one arrow $f \in C_1$ between objects A and B in C_0.*

Recall the following definition.

Definition 1.10.5. *preorder is a pair $(I, \preceq)$, where I is a set, and $\preceq \subseteq I \times I$ such that:*
(i) $\forall_{i \in I}\big(i \preceq i\big)$.
(ii) $\forall_{i,j,k \in I}\big(i \preceq j \ \&\ j \preceq k \Rightarrow i \preceq k\big)$.
If a preorder satisfies the condition
(iii) $\forall_{i,j \in I}\big(i \preceq j \ \&\ j \preceq i \Rightarrow i = j\big)$,
it is called a *partially ordered set*, or a *poset*.

A preorder $(I, \preceq)$ becomes a category with objects the elements of I and a unique arrow from i to j, if and only if $i \preceq j$. Conditions (i) and (ii) above ensure that I is a category. Moreover, any thin category generates a preorder.

Definition 1.10.6. *In the case of the thin category* ***Form*** *we define*

$$A \leq B :\Leftrightarrow \exists_M\big(M\colon A \to B\big).$$

Clearly, a poset is also a thin category. Many categorical notions are generalisations of order-theoretic concepts. In many cases, a category can be seen as a generalised poset, allowing more arrows between its objects.

1.11 More examples of derivations in minimal logic

Proposition 1.11.1. *The following formulas are derivable:*
(i) $A \to A$.
(ii) $A \to \neg\neg A$.
(iii) (*Brouwer*) $\neg\neg\neg A \to \neg A$.

Proof. The derivation for (i) is

$$\dfrac{\dfrac{[a\colon A]}{A}\,1_A}{A \to A}\,\to^+ a$$

The derivation for (ii) is

$$\dfrac{\dfrac{\dfrac{[u\colon A \to \bot] \qquad [a\colon A]}{\bot}}{(A \to \bot) \to \bot}\,\to^+ u}{A \to (A \to \bot) \to \bot}\,\to^+ a$$

The derivation for (iii) is an exercise. □

Note that *double negation elimination* i.e., the formula $\mathrm{DNE}_A = \neg\neg A \to A$, is in general *not* derivable in minimal logic. But this we cannot show now.

Proposition 1.11.2. *The following are derivable.*

$$\begin{aligned}
&(i)\ (A \to B) \to \neg B \to \neg A,\\
&(ii)\ \neg(A \to B) \to \neg B,\\
&(iii)\ \neg\neg(A \to B) \to \neg\neg A \to \neg\neg B,\\
&(iv)\ (\bot \to B) \to (\neg\neg A \to \neg\neg B) \to \neg\neg(A \to B),\\
&(v)\ \neg\neg\forall_x A \to \forall_x \neg\neg A.
\end{aligned}$$

Proof. Exercise. □

Proposition 1.11.3. *We consider the following formulas:*

$$\begin{aligned}
\mathtt{ax}\vee_0^+ &= A \to A \vee B,\\
\mathtt{ax}\vee_1^+ &= B \to A \vee B,\\
\mathtt{ax}\vee^- &= A \vee B \to (A \to C) \to (B \to C) \to C,\\
\mathtt{ax}\wedge^+ &= A \to B \to A \wedge B,\\
\mathtt{ax}\wedge^- &= A \wedge B \to (A \to B \to C) \to C,\\
\mathtt{ax}\exists^+ &= A \to \exists_x A,\\
\mathtt{ax}\exists^- &= \exists_x A \to \forall_x (A \to B) \to B \qquad (x \notin \mathrm{FV}(B)).
\end{aligned}$$

(i) The formulas $\mathtt{ax}\vee_0^+, \mathtt{ax}\vee_1^+$ *and* $\mathtt{ax}\vee^-$ *are equivalent, as axioms, to the rules* $\vee_0^+$, $\vee_1^+$ *and* $\vee^- u, v$ *over minimal logic.*
(ii) The formulas $\mathtt{ax}\wedge^+$ *and* $\mathtt{ax}\wedge^-$ *as axioms are equivalent, as axioms, to the rules* $\wedge^+$ *and* $\wedge^-$ *over minimal logic.*

(iii) The formulas $\texttt{ax}\exists^+$ *and* $\texttt{ax}\exists^-$ *are equivalent, as axioms, to the rules* $\exists^+$ *and* $\exists^- x, u$ *over minimal logic.*

Proof. (i) First we show that from the axiom $\texttt{ax}\lor_0^+$, a derivation of which is considered the formula itself, and a supposed derivation M of A we get the following derivation of $A \lor B$

$$\dfrac{A \to A \lor B \qquad \begin{matrix} \mid M \\ A \end{matrix}}{A \lor B}\to^-$$

Similarly we show that from the formula $\texttt{ax}\lor_1^+$ and a supposed derivation N of B we get a derivation of $A \lor B$. Next we show that from the formula $\texttt{ax}\lor^-$ and supposed derivations M of $A \lor B$, N of C with assumption A, and K of C with assumption B we get the following derivation of C

$$\dfrac{\dfrac{\dfrac{\texttt{ax}\lor^- \qquad \begin{matrix}\mid M\\ A\lor B\end{matrix}}{(A\to C)\to(B\to C)\to C}\to^- \qquad \dfrac{\begin{matrix}[u\colon A]\\ \mid N\\ C\end{matrix}}{A\to C}\to^+u}{(B\to C)\to C}\to^- \qquad \dfrac{\begin{matrix}[v\colon B]\\ \mid K\\ C\end{matrix}}{B\to C}\to^+v}{C}\to^-$$

Conversely, from the rule $\lor_0^+$ we get the following derivation of $\texttt{ax}\lor_0^+$

$$\dfrac{\dfrac{\dfrac{[a\colon A]}{A}1_A}{A\lor B}\lor_0^+}{A\to A\lor B}\to^+a$$

Similarly, from the rule $\lor_1^+$ we get a derivation of $\texttt{ax}\lor_1^+$. From the elimination rule for disjunction we get the following derivation of $\texttt{ax}\lor^-$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[u\colon A\lor B]}{A\lor B}1_{A\lor B} \qquad \dfrac{[v\colon A\to C] \qquad [v'\colon A]}{C} \qquad \dfrac{[w\colon B\to C] \qquad [w'\colon B]}{C}}{C}\lor^- v', w'}{(B\to C)\to C}\to^+w}{(A\to C)\to(B\to C)\to C}\to^+v}{A\lor B\to(A\to C)\to(B\to C)\to C}\to^+u$$

(ii) and (iii) are exercises. □

A similar result holds for axioms corresponding to the rules $\forall^+ x$ and $\forall^-$. Note that in the above derivation of C

$$\dfrac{\dfrac{u\colon A\lor B}{A\lor B}\texttt{ax} \qquad \dfrac{[v\colon A\to C] \qquad [v'\colon A]}{C} \qquad \dfrac{[w\colon B\to C] \qquad [w'\colon B]}{C}}{C}\lor^- v', w'$$

we used the rule $\lor^- v', w'$ in the "extended" way described in Definition 1.9.2, where the assumption variables $u\colon A\lor B, v\colon A\to C$ and $w\colon B\to C$ are still open. Of course, they will be canceled later in the derivation of $\texttt{ax}\lor^-$. The notation $B\leftarrow A$ means $A\to B$.

Proposition 1.11.4. *The following formulas are derivable*

$$
\begin{array}{l}
(i)\ (A\wedge B\to C)\leftrightarrow(A\to B\to C),\\
(ii)\ (A\to B\wedge C)\leftrightarrow(A\to B)\wedge(A\to C),\\[4pt]
(iii)\ (A\vee B\to C)\leftrightarrow(A\to C)\wedge(B\to C),\\
(iv)\ (A\to B\vee C)\leftarrow(A\to B)\vee(A\to C),\\[4pt]
(v)\ (\forall_x A\to B)\leftarrow\exists_x(A\to B)\quad \textit{if } x\notin \mathrm{FV}(B),\\
(vi)\ (A\to\forall_x B)\leftrightarrow\forall_x(A\to B)\quad \textit{if } x\notin \mathrm{FV}(A),\\[4pt]
(vii)\ (\exists_x A\to B)\leftrightarrow\forall_x(A\to B)\quad \textit{if } x\notin \mathrm{FV}(B),\\
(viii)\ (A\to\exists_x B)\leftarrow\exists_x(A\to B)\quad \textit{if } x\notin \mathrm{FV}(A).
\end{array}
$$

Proof. (i)-(vii) are exercise. A derivation of the final formula is

$$
\dfrac{\dfrac{\dfrac{[u\colon \exists_x(A\to B)] \qquad \dfrac{x\in\texttt{Var} \qquad \dfrac{[w\colon A\to B] \qquad [v\colon A]}{B}}{\exists_x B}}{\exists_x B}\ \exists^- x,w}{A\to\exists_x B}\ \to^+ v}{\exists_x(A\to B)\to A\to\exists_x B}\ \to^+ u
$$

The variable condition for $\exists^-$ is satisfied since the variable x (i) is not free in the formula A of the open assumption $v\colon A$, and (ii) is not free in $\exists_x B$. Of course, it is not a problem, if it occurs free in $A\to B$. □

1.12 Extension, cut, and the deduction theorem

Next we prove the extension-rule and the cut-rule.

Proposition 1.12.1. *If* $\Gamma,\Delta\subseteq\texttt{Form}$ *and* $A,B\in\texttt{Form}$*, the following rules hold:*

$$
\dfrac{\Gamma\vdash A,\quad \Gamma\subseteq\Delta}{\Delta\vdash A}\ \texttt{ext}
$$

$$
\dfrac{\Gamma\vdash A,\quad \Delta\cup\{A\}\vdash B}{\Gamma\cup\Delta\vdash B}\ \texttt{cut}
$$

Proof. The ext-rule is an immediate consequence of the definition of $\Gamma\vdash A$. Suppose next that there are $C_1,\dots,C_n\in\Gamma$ and $D_1,\dots,D_m\in\Delta$ such that $C_1,\dots,C_n\vdash A$ and $D_1,\dots,D_m,A\vdash B$. The following is a derivation of B from assumptions in $\Gamma\cup\Delta$:

$$
\dfrac{\dfrac{\begin{array}{c} u_1\colon D_1\dots u_m\colon D_m\ [u\colon A]\\ \mid M\\ B\end{array}}{A\to B}\ \to^+ u \qquad \begin{array}{c} w_1\colon C_1\dots w_n\colon C_n\\ \mid N\\ A\end{array}}{B}\ \to^-
$$

□

The following rules are special cases of the `cut`-rule for $\Gamma = \Delta$ and $\Gamma = \Delta = \emptyset$, respectively.

$$\frac{\Gamma \vdash A, \quad \Gamma \cup \{A\} \vdash B}{\Gamma \vdash B}$$

$$\frac{\vdash A, \quad A \vdash B}{\vdash B}$$

From now on, we also denote $\Gamma \vdash A$ by the tree

$$\begin{array}{c}\Gamma\\ \mid M\\ A\end{array}$$

Proposition 1.12.2. *Let* $\Gamma \subseteq \texttt{Form}$ *and* $A, B \in \texttt{Form}$.
(i) $\Gamma \vdash (A \to B) \Rightarrow (\Gamma \vdash A \Rightarrow \Gamma \vdash B)$.
(ii) $(\Gamma \vdash A$ *or* $\Gamma \vdash B) \Rightarrow \Gamma \vdash A \vee B$.
(iii) $\Gamma \vdash (A \wedge B) \Leftrightarrow (\Gamma \vdash A$ *and* $\Gamma \vdash B)$.
(iv) $\Gamma \vdash \forall_y A \Rightarrow \Gamma \vdash A(s)$, *for every* $s \in \texttt{Term}$ *such that* $\mathrm{Free}_{s,y}(A) = 1$.
(v) *If* $s \in \texttt{Term}$ *such that* $\mathrm{Free}_{s,y}(A) = 1$ *and* $\Gamma \vdash A(s)$, *then* $\Gamma \vdash \exists_y A$.

Proof. (i) If $\Gamma \vdash (A \to B)$ and $\Gamma \vdash A$, the following is a derivation of B from Γ:

$$\dfrac{\begin{array}{c}\Gamma\\ \mid M\\ A \to B\end{array} \qquad \begin{array}{c}\Gamma\\ \mid N\\ A\end{array}}{B}\ \to^-$$

(ii) If $\Gamma \vdash A$, the following is a derivation of $A \vee B$ from Γ:

$$\dfrac{\begin{array}{c}\Gamma\\ \mid M\\ A\end{array}}{A \vee B}\ \vee_0^+$$

If $\Gamma \vdash B$, we proceed similarly.
(iii) If $\Gamma \vdash A \wedge B$, the following is a derivation of A from Γ:

$$\dfrac{\begin{array}{c}\Gamma\\ \mid M\\ A \wedge B\end{array} \qquad \dfrac{[a\colon A][v\colon B]}{A}\ 1_A}{A}\ \wedge^-\ a, v$$

Notice that in the above derivation of A we used the ext-rule. In order to show $\Gamma \vdash B$, we proceed similarly. If $\Gamma \vdash A$ and $\Gamma \vdash B$, the following is a derivation of $A \wedge B$ from Γ:

$$\dfrac{\begin{array}{c}\Gamma\\ \mid M\\ A\end{array} \qquad \begin{array}{c}\Gamma\\ \mid N\\ B\end{array}}{A \wedge B}\ \wedge^+$$

(iv) and (v) If $\Gamma \vdash \forall_y A$, the left derivation is a derivation of $A(s)$ from Γ, and if $\Gamma \vdash A(s)$, the right derivation is a derivation of $\exists_y A$ from Γ:

$$\dfrac{\begin{matrix}\Gamma \\ \mid M \\ \forall_y A\end{matrix} \qquad s \in \texttt{Term}}{A(s)}\ \forall^- \qquad \dfrac{s \in \texttt{Term} \qquad \begin{matrix}\Gamma \\ \mid M \\ A(s)\end{matrix}}{\exists_y A}\ \exists^+$$

□

Proposition 1.12.3. *Let* $\Gamma \subseteq \texttt{Form}$ *and* $A, B \in \texttt{Form}$.
(i) (Deduction theorem) $\Gamma \cup \{A\} \vdash B \Leftrightarrow \Gamma \vdash A \to B$.
(ii) *If for every* $A_1, \dots, A_n, A_{n+1} \in \texttt{Form}$, *we define*

$$\bigwedge_{i=1}^{1} A_i = A_1,$$

$$\bigwedge_{i=1}^{n+1} A_i = \left(\bigwedge_{i=1}^{n} A_i\right) \wedge A_{n+1},$$

then

$$\forall_{n \in \mathbb{N}^+}\left(\forall_{A_1,\dots,A_n,A \in \texttt{Form}}\big(\{A_1, \dots, A_n\} \vdash A \Leftrightarrow \vdash \left(\bigwedge_{i=1}^{n} A_i\right) \to A\big)\right).$$

Proof. (i) If $C_1, \dots, C_n \in \Gamma$ such that $C_1, \dots, C_n, A \vdash B$, then

$$\dfrac{\begin{matrix} u_1 \colon C_1 \dots u_n \colon C_n \ [u \colon A] \\ \mid M \\ B \end{matrix}}{A \to B}\ \to^+ u$$

is a derivation of $A \to B$ from Γ. Conversely, if $C_1, \dots, C_n \in \Gamma$ such that $C_1, \dots, C_n, \vdash A \to B$, the following is a derivation of B from $\Gamma \cup \{A\}$:

$$\dfrac{\begin{matrix} u_1 \colon C_1 \dots u_n \colon C_n \\ \mid M \\ A \to B \end{matrix} \qquad \dfrac{a \colon A}{A}\ 1_A}{B}\ \to^-$$

(ii) We use induction on $\mathbb{N}^+$. If $n = 1$, our goal-formula becomes

$$\forall_{A,B \in \texttt{Form}}\big(\{A\} \vdash B \Leftrightarrow \vdash A \to B\big),$$

which follows from (i) for $\Gamma = \emptyset$. Our inductive hypothesis is

$$\forall_{A_1,\dots,A_n,A \in \texttt{Form}}\left(\{A_1, \dots, A_n\} \vdash A \Leftrightarrow \vdash \left(\bigwedge_{i=1}^{n} A_i\right) \to A\right),$$

and we show

$$\forall_{A_1,\dots,A_n,A_{n+1},A \in \texttt{Form}}\left(\{A_1, \dots, A_n, A_{n+1}\} \vdash A \Leftrightarrow \vdash \left(\bigwedge_{i=1}^{n+1} A_i\right) \to A\right).$$

If we fix $A_1, \ldots, A_n, A_{n+1}, A$, we have that

$$\begin{aligned}\{A_1, \ldots, A_n, A_{n+1}\} \vdash A &\Leftrightarrow \{A_1, \ldots, A_n\} \cup \{A_{n+1}\} \vdash A \\ &\overset{(i)}{\Leftrightarrow} \{A_1, \ldots, A_n\} \vdash A_{n+1} \to A \\ &\overset{(*)}{\Leftrightarrow} \vdash \left(\bigwedge_{i=1}^{n} A_i\right) \to (A_{n+1} \to A) \\ &\overset{(**)}{\Leftrightarrow} \vdash \left(\bigwedge_{i=1}^{n} A_i\right) \wedge A_{n+1} \to A \\ &= \vdash \left(\bigwedge_{i=1}^{n+1} A_i\right) \to A,\end{aligned}$$

where $(*)$ follows by the inductive hypothesis on $A_1, \ldots, A_n$ and the formula $A_{n+1} \to A$, and $(**)$ follows by the derivation

$$\vdash (A \to B \to C) \leftrightarrow (A \wedge B \to C)$$

in Proposition 1.11.4(i), and the corollary of Proposition 1.12.2(i)

$$\vdash A \leftrightarrow B \Rightarrow (\vdash A \Leftrightarrow \vdash B). \qquad \square$$

1.13 The category of formulas is cartesian closed

Definition 1.13.1. *If $\boldsymbol{C}$ is a category, and $f : A \to B$ is an arrow in $\boldsymbol{C}$, f is called an iso, or an isomorphism, if there is an arrow $g : B \to A$ such that $g \circ f = \mathbf{1}_A$ and $f \circ g = \mathbf{1}_B$. In this case we say that A and B are isomorphic, and we write $A \cong B$.*

Clearly, the relation of isomorphism in a category satisfies the properties of an equivalence relation, and it is a categorical alternative to the notion of equality. In the category of formulas **Form** if $A \leq B$ and $B \leq A$ i.e., if there are derivations $M\colon A \to B$ and $N\colon B \to A$, then $A \cong B$, since by the thinness of **Form** we get $N \circ M = 1_A$ and $M \circ N = 1_B$.

Remark 1.13.2. *If $A, B \in$ `Form` such that $\vdash A$ and $\vdash B$, then $A \cong B$.*

Proof. Exercise. $\square$

Definition 1.13.3. *Let $\top$ be a fixed formula such that $\vdash \top$. We call the formula $\top$ verum i.e., true.*

Definition 1.13.4. *If $\boldsymbol{C}$ is a category, an object T of $\boldsymbol{C}$ is called terminal, if there is a unique arrow $f : A \to T$, for every object A of $\boldsymbol{C}$. Dually, an object I of $\boldsymbol{C}$ is called initial, if there is a unique arrow $g : I \to A$, for every object A of $\boldsymbol{C}$.*

Notice that the notion of an initial (terminal) object is *dual* to the notion of a terminal (initial) object i.e., we get the definition of a terminal object by reversing the arrow in the definition of an initial object, and vice versa. One could have named a terminal object a coinitial object and an initial object a coterminal one. This duality between concepts and "coconcepts" is very often in category theory.

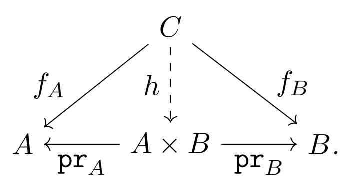
In the category of sets **Set** any singleton, like $1 = \{0\}$ is terminal, and the empty set $\emptyset$ is initial. It is straightforward to show that terminal, or initial objects in a category $\boldsymbol{C}$ are *unique up to isomorphism* i.e., any two terminal, or initial objects in a category $\boldsymbol{C}$ are isomorphic (exercise).

Proposition 1.13.5. *The formula $\top$ is a terminal object in **Form**.*

Proof. Exercise. □

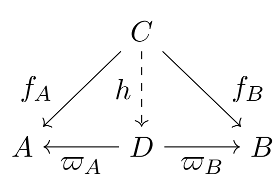
If there is a formula $I \in \texttt{Form}$, such that I is an initial element in **Form**, then this is expected to be the formula $\bot$ (can you find a reason for that?). But such a thing cannot be shown now, it has to be "postulated" (see derivations in intuitionistic logic).

Definition 1.13.6. *Let $\boldsymbol{C}$ be a category and A, B objects of $\boldsymbol{C}$. A product of A and B is an object $A \times B$ of $\boldsymbol{C}$ together with arrows $\texttt{pr}_A \colon A \times B \to A$ and $\texttt{pr}_B \colon A \times B \to B$, such that the universal property of product is satisfied i.e., if C is an object in $\boldsymbol{C}$ and $f_A \colon C \to A$ and $f_B \colon C \to B$, there is a unique arrow $h = \langle f_A, f_B \rangle \colon C \to A \times B$, such that the following inner diagrams commute i.e., $\texttt{pr}_A \circ h = f_A$ and $\texttt{pr}_B \circ h = f_B$.*

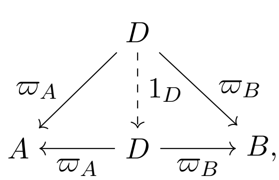


A category $\boldsymbol{C}$ has products, if for every objects A, B of $\boldsymbol{C}$, there is a product $A \times B$ in $\boldsymbol{C}$ (for simplicity we avoid to mention the corresponding projection arrows).

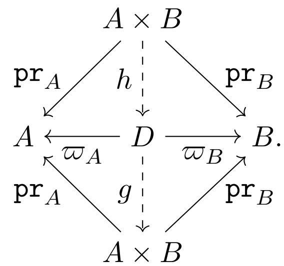
In **Set** the product of sets A, B is their cartesian product together with the two projection maps. Next we show that the product $A \times B$ in $\boldsymbol{C}$, if it exists, is unique up to isomorphism i.e., if there is some object D and arrow $\varpi_A : D \to A$ and $\varpi_B : D \to B$ such that the universal property of products is satisfied, then $D \cong A \times B$. In the universal property for D let



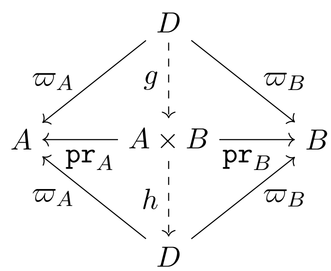
$C = D, f_A = \varpi_A$ and $f_B = \varpi_B$. Since the following inner diagrams also commute



we get $h = 1_D$, and the arrow $\langle \varpi_A, \varpi_B \rangle$ is unique, Since $A \times B$ and D both satisfy the universal property of the products, from the previous remark we get $g \circ h = 1_{A \times B}$

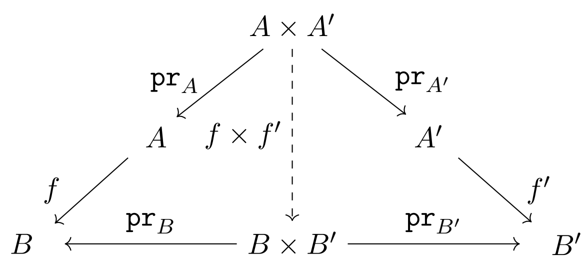


Similarly from the commutative diagrams



we get that $h \circ g = 1_D$, hence $A \times B \cong D$. The following arrow will be used in Definition 1.13.11.

Definition 1.13.7. *If a category $\mathcal{C}$ has products, $f\colon A \to B$ and $f'\colon A' \to B'$ are in C_1, then $f \times f' = \langle f \circ \mathtt{pr}_A, f' \circ \mathtt{pr}_{A'}\rangle\colon A \times A' \to B \times B'$*

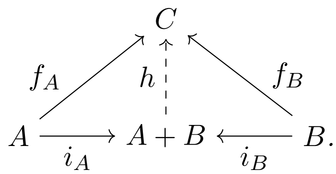


Proposition 1.13.8. *If $A, B \in \mathtt{Form}$, then $A \wedge B$ is a product of A, B in **Form**. Consequently, **Form** has products.*

Proof. Exercise. Notice that the commutativity of the corresponding diagrams is trivially satisfied, as **Form** is a thin category. ☐

Next we define the dual notion to the product of objects in a category. Notice that the arrows in the universal property of coproduct are reversed with respect to the arrows in the universal property of the product.

Definition 1.13.9. *Let $\boldsymbol{C}$ be a category and A, B objects of $\boldsymbol{C}$. A coproduct of A and B is an object $A + B$ of $\boldsymbol{C}$ together with arrows $i_A\colon A \to A + B$ and $i_B\colon B \to A + B$, such that the universal property of coproduct is satisfied i.e., if C is an object in $\boldsymbol{C}$ and $f_A\colon A \to C$ and $f_B\colon B \to C$, there is a unique arrow $h = [f_A, f_B]\colon A + B \to C$, such that the following inner diagrams commute i.e., $h \circ i_A = f_A$ and $h \circ i_B = f_B$.*



The arrows i_A, i_B are called coprojections, or injections. A category $\boldsymbol{C}$ has coproducts, if for every objects A, B of $\boldsymbol{C}$, there is a coproduct $A + B$ in $\boldsymbol{C}$ (for simplicity we avoid to mention the corresponding coprojection arrows).

In **Set** the coproduct of sets A, B is their disjoint union

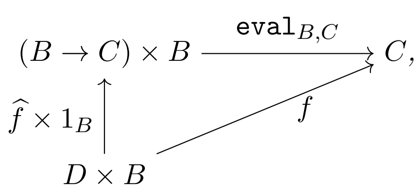
$$A + B = \{(i, x) \in \{0, 1\} \times A \cup B \mid (i = 0 \ \& \ x \in A) \text{ or } (i = 1 \ \& \ x \ \in B)\}$$

together with the injections $i_A \colon A \to A + B$, where $i_A(a) = (0, a)$, for every $a \in A$, and $i_B \colon B \to A + B$, where $i_B(b) = (1, b)$, for every $b \in B$. A coproduct $A + B$ in $\boldsymbol{C}$, if it exists, is unique up to isomorphism (the proof is dual to the proof for the product).

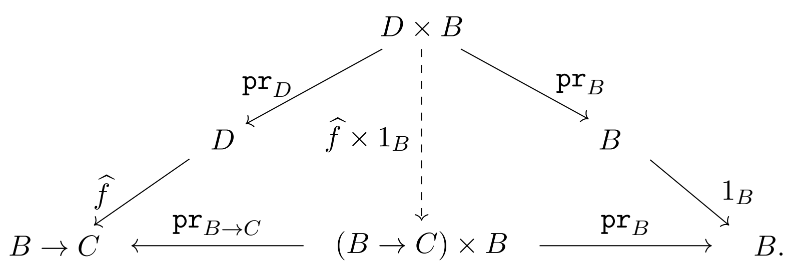
Proposition 1.13.10. *If $A, B \in \texttt{Form}$, then $A \vee B$ is a coproduct of A, B in **Form**. Consequently, **Form** has coproducts.*

Proof. Exercise. □

Definition 1.13.11. *If B, C are objects of a category $\boldsymbol{C}$ with products, an exponential of B and C is an object $B \to C$ in $\boldsymbol{C}$ together with an arrow $\texttt{eval}_{B,C} \colon (B \to C) \times B \to C$, such that for any object D in $\boldsymbol{C}$ and every arrow $f \colon D \times B \to C$ there is a unique arrow $\widehat{f} \colon D \to (B \to C)$ such that $\texttt{eval}_{B,C} \circ (\widehat{f} \times 1_B) = f$*



where the arrow $\widehat{f} \times 1_B$ is determined in Definition 1.13.7



The arrow $\widehat{f}$ is called the (exponential) transpose of f. A category has exponentials, if for every B, C in $\boldsymbol{C}$ there is an exponential $B \to C$ in $\boldsymbol{C}$.

An exponential $B \to C$ of B and C is unique up to isomorphism (exercise). In **Set** an exponential of the sets B and C is the set of all functions from B to C i.e.,

$$C^B = \{f \in \mathcal{P}(B \times C) \mid f \colon B \to C\},$$

together with the function $\texttt{eval}_{B,C} \colon C^B \times B \to C$, defined by

$$\texttt{eval}_{B,C}(f, b) = f(b); \quad f \in C^B, \ b \in B.$$

Proposition 1.13.12. *If $B, C \in \texttt{Form}$, then $B \to C$ is the exponential of B and C in **Form**. Consequently, **Form** has exponentials.*

Proof. Exercise. □

Definition 1.13.13. *A category $\boldsymbol{C}$ is called cartesian closed, if it has a terminal object, products and exponentials.*

Clearly, the category of sets **Set** is cartesian closed.

Corollary 1.13.14. *The category **Form** is cartesian closed.*

Proof. It follows from Propositions 1.13.5, 1.13.8, and 1.13.12. □

1.14 Functors associated to the main logical symbols

The concept of functor is the natural notion of "map", or "arrow", between categories.

Definition 1.14.1. *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be categories. A covariant functor, or simply a functor, from $\mathcal{C}$ to $\mathcal{D}$ is a pair $F = (F_0, F_1)$, where:*

(i) F_0 maps an object A of $\boldsymbol{C}$ to an object $F_0(A)$ of $\mathcal{D}$,

(ii) F_1 maps an arrow $f : A \to B$ of $\boldsymbol{C}$ to an arrow $F_1(f) : F_0(A) \to F_0(B)$ of $\boldsymbol{D}$, such that

(a) For every A in C_0 we have that $F_1(\mathbf{1}_A) = \mathbf{1}_{F_0(A)}$

$$F_0(A) \; \overset{\mathbf{1}_{F_0(A)}}{\underset{F_1(\mathbf{1}_A)}{\rightrightarrows}} \; F_0(A).$$

(b) If $f : A \to B$ and $g : B \to C$, then $F_1(g \circ f) = F_1(g) \circ F_1(f)$ i.e., the following diagram

$$F_0(A) \xrightarrow{F_1(f)} F_0(B) \xrightarrow{F_1(g)} F_0(C), \qquad F_0(A) \xrightarrow{F_1(g \circ f)} F_0(C)$$

commutes, where for simplicity we use the same symbol for the operation of composition in the categories $\boldsymbol{C}$ and $\boldsymbol{D}$. In this case we write[4] $F : \boldsymbol{C} \to \boldsymbol{D}$. A functor $\boldsymbol{C} \to \boldsymbol{C}$ is called an endofunctor (on $\boldsymbol{C}$). Two functors $F, G \colon \boldsymbol{C} \to \boldsymbol{D}$ are equal, if $F_0 = G_0$ and $F_1 = G_1$.

A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is a pair $F := (F_0, F_1)$, where:

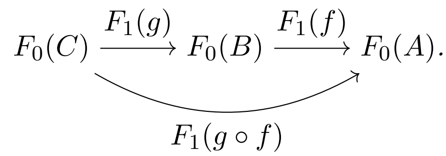
(i) F_0 maps an object A of $\boldsymbol{C}$ to an object $F_0(A)$ of $\mathcal{D}$,

(ii') F_1 maps an arrow $f : A \to B$ of $\boldsymbol{C}$ to an arrow $F_1(f) : F_0(B) \to F_0(A)$ of $\boldsymbol{D}$, such that

(a) $F_1(\mathbf{1}_A) = \mathbf{1}_{F_0(A)}$, for every A in C_0.

(b') If $f : A \to B$ and $g : B \to C$, then $F_1(g \circ f) = F_1(f) \circ F_1(g)$ i.e., the following diagram

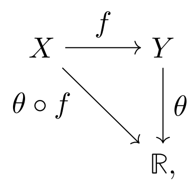
[4] In the literature it is often written $F(C)$ and $F(f)$, instead of $F_0(C)$ and $F_1(f)$.



commutes. In this case we write[5] $F : \boldsymbol{C}^{\mathrm{op}} \to \boldsymbol{D}$*, where* $\boldsymbol{C}^{\mathrm{op}}$ *is the opposite category of* $\boldsymbol{C}$ *i.e., it has the objects of* $\boldsymbol{C}$ *and an arrow* $f \colon A \to B$ *in* $\boldsymbol{C}^{\mathrm{op}}$ *is an arrow* $f \colon B \to A$ *in* $\boldsymbol{C}$*. Two contravariant functors* $F, G \colon \boldsymbol{C}^{\mathrm{op}} \to \boldsymbol{D}$ *are equal, if* $F_0 = G_0$ *and* $F_1 = G_1$*.*

Example 1.14.2. If $\boldsymbol{C}$ is a category, the *identity functor* on $\boldsymbol{C}$ is the pair $\mathrm{Id}^{\boldsymbol{C}} = (\mathrm{Id}_0^{\boldsymbol{C}}, \mathrm{Id}_1^{\boldsymbol{C}}) : \boldsymbol{C} \to \boldsymbol{C}$, where $\mathrm{Id}_0^{\boldsymbol{C}}(A) = A$, for every A in C_0, and if $f : A \to B$, then $\mathrm{Id}_1^{\boldsymbol{C}}(f) = f$.

Example 1.14.3. The pair $(G_0, G_1) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$, where $G_0(X) = \mathbb{F}(X) = \{\phi : X \to \mathbb{R}\}$, and if $f : X \to Y$, then $G_1(f) : \mathbb{F}(Y) \to \mathbb{F}(X)$ is defined by $[G_1(f)](\theta) = \theta \circ f$



for every $\theta \in \mathbb{F}(Y)$, is a contravariant functor from **Set** to **Set**. If X is a set, then

$$[G_1(\mathrm{id}_X)](\phi) = \phi \circ \mathrm{id}_X = \phi$$

and since $\phi \in \mathbb{F}(X)$ is arbitrary, we conclude that $G_1(\mathrm{id}_X) = \mathrm{id}_{\mathbb{F}(X)} = \mathrm{id}_{G_0(X)}$. If $f : X \to Y$ and $g : Y \to Z$, then $G_1(f) : \mathbb{F}(Y) \to \mathbb{F}(X)$, $G_1(g) : \mathbb{F}(Z) \to \mathbb{F}(Y)$ and $G_1(g \circ f) : \mathbb{F}(Z) \to \mathbb{F}(X)$. Moreover, if $\eta \in \mathbb{F}(Z)$, we have that

$$\begin{aligned}
[G_1(g \circ f)](\eta) &= \eta \circ (g \circ f) \\
&= (\eta \circ g) \circ f \\
&= [G_1(f)](\eta \circ g) \\
&= G_1(f)\big([G_1(g)](\eta)\big) \\
&= [G_1(f) \circ G_1(g)](\eta).
\end{aligned}$$

Example 1.14.4. If $\boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}$ are categories, $\boldsymbol{F} \colon \boldsymbol{C} \to \boldsymbol{D}$, and $\boldsymbol{G} \colon \boldsymbol{D} \to \boldsymbol{E}$ are functors, their composition $\boldsymbol{G} \circ \boldsymbol{F} \colon \boldsymbol{C} \to \boldsymbol{E}$, where $\boldsymbol{G} \circ \boldsymbol{F} = (G_0 \circ F_0, G_1 \circ F_1)$, is a functor.

Definition 1.14.5. *The collection of all categories with arrows the functors between them is a category, which is called the category of categories, and it is denoted by* ***Cat****. The unit arrow* $\mathbf{1}_{\boldsymbol{C}}$ *is the identity functor* $\mathrm{Id}^{\boldsymbol{C}}$*, while the composition of functors is defined in Example 1.14.4.*

Example 1.14.6. An endofunctor $F \colon \mathbf{Form} \to \mathbf{Form}$ is a monotone function from $(\texttt{Form}, \leq)$ to itself. The same is the case for any functor between preorders. Recall that if $(I, \leq)$ and $(J, \preceq)$ are preorders (see Definition 1.10.5), a function $f \colon I \to J$ is *monotone*, if

$$\forall_{i, i' \in I}\big(i \leq i' \Rightarrow f(i) \preceq f(i')\big).$$

Conversely, if f is a monotone function from $(\texttt{Form}, \leq)$ to itself, then f generates an endofunctor on **Form**, the 0-part of which is f.

[5]It is easy to show that a covariant functor $F : \boldsymbol{C}^{\mathrm{op}} \to \boldsymbol{D}$ is exactly a contravariant functor from $\boldsymbol{C}$ to $\boldsymbol{D}$.

Definition 1.14.7. *If $\boldsymbol{C}, \boldsymbol{D}$ are categories, the product category $\boldsymbol{C} \times \boldsymbol{D}$ has objects pairs (c, d), where $c \in C_0$ and $d \in D_0$. An arrow from (c, d) to (c', d') is a pair (f, g), where $f\colon c \to c'$ in C_1 and $g\colon d \to d'$ in D_1. If $(f, g)\colon (c, d) \to (c', d')$ and $(f', g')\colon (c', d') \to (c'', d'')$, their composition is defined by*

$$(f', g') \circ (f, g) = \big(f' \circ f, g' \circ g\big).$$

Moreover, $1_{(c,d)} = (1_c, 1_d)$. The projection functor $\mathrm{Pr}^{\boldsymbol{C}}\colon \boldsymbol{C} \times \boldsymbol{D} \to \boldsymbol{C}$ is the pair $\big(\mathrm{Pr}_0^{\boldsymbol{C}}, \mathrm{Pr}_1^{\boldsymbol{C}}\big)$, where $\mathrm{Pr}_0^{\boldsymbol{C}}(c, d) = c$, for every object (c, d) of $\boldsymbol{C} \times \boldsymbol{D}$, and $\mathrm{Pr}_1^{\boldsymbol{C}}(f, g) = f$, for every arrow (f, g) in $\boldsymbol{C} \times \boldsymbol{D}$. The projection functor $\mathrm{Pr}^{\boldsymbol{D}}\colon \boldsymbol{C} \times \boldsymbol{D} \to \boldsymbol{D}$ is defined similarly.

It is immediate to show that $\boldsymbol{C} \times \boldsymbol{D}$ is a category and $\mathrm{Pr}^{\boldsymbol{C}}$ and $\mathrm{Pr}^{\boldsymbol{D}}$ are functors. Moreover, the product category $\boldsymbol{C} \times \boldsymbol{D}$ is a product of $\boldsymbol{C}$ and $\boldsymbol{D}$ in $\mathbf{Cat}$.

Definition 1.14.8. *Let the following functors:*

(i) $\bigwedge\colon \boldsymbol{Form} \times \boldsymbol{Form} \to \boldsymbol{Form}$, *where $\bigwedge_0(A, B) = A \wedge B$, for every object (A, B) in $\boldsymbol{Form} \times \boldsymbol{Form}$, and $\bigwedge_1\big(M\colon A \to A', N\colon B \to B'\big)\colon (A \wedge B) \to (A' \wedge B')$ is the following derivation of $A' \wedge B'$ from assumption $w\colon A \wedge B$, given derivations M and N,*

$$\dfrac{\dfrac{w\colon A\wedge B}{A\wedge B}\,1_{A\wedge B} \qquad \dfrac{\begin{matrix}[u\colon A]\\ \mid M\\ A'\end{matrix} \qquad \begin{matrix}[v\colon B]\\ \mid N\\ B'\end{matrix}}{A'\wedge B'}\,\wedge^+}{A'\wedge B'}\,\wedge^- u, v$$

(ii) $\bigvee\colon \boldsymbol{Form} \times \boldsymbol{Form} \to \boldsymbol{Form}$, *where $\bigvee_0(A, B) = A \vee B$, for every object (A, B) in $\boldsymbol{Form} \times \boldsymbol{Form}$, and $\bigvee_1\big(M\colon A \to A', N\colon B \to B'\big)\colon (A \vee B) \to (A' \vee B')$ is the following derivation of $A' \vee B'$ from assumption $w\colon A \vee B$, given derivations M and N,*

$$\dfrac{\dfrac{w\colon A\vee B}{A\vee B}\,1_{A\vee B} \qquad \dfrac{\begin{matrix}[u\colon A]\\ \mid M\\ A'\end{matrix}}{A'\vee B'}\,\vee_0^+ \qquad \dfrac{\begin{matrix}[v\colon B]\\ \mid N\\ B'\end{matrix}}{A'\vee B'}\,\vee_1^+}{A'\vee B'}\,\vee^-, u, v$$

(iii) $\longrightarrow\colon \boldsymbol{Form}^{\mathrm{op}} \times \boldsymbol{Form} \to \boldsymbol{Form}$, *where $(\longrightarrow)_0(A, B) = A \to B$, for every (A, B) in $\boldsymbol{Form}^{\mathrm{op}} \times \boldsymbol{Form}$. The definition of $(\longrightarrow)_1\big((M, N)\colon (A, B) \to (A', B')\big)\colon (A \to B) \to (A' \to B')$, where $M\colon A' \to A$ and $N\colon B \to B'$, is an exercise.*

(iv) $\forall_x\colon \boldsymbol{Form} \to \boldsymbol{Form}$, *where $\big(\forall_x\big)_0(A) = \forall_x A$, for every $A \in \mathtt{Form}$, and $\big(\forall_x\big)_1\big(M\colon A \to B\big)\colon \forall_x A \to \forall_x B$ is the following derivation of $\forall_x B$ from assumption $\forall_x A$, given derivation M*

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[u\colon A]\\ \mid M\\ B\end{matrix}}{A\to B}\to^+ u \qquad \dfrac{\dfrac{w\colon \forall_x A}{\forall_x A}\,1_{\forall_x A} \qquad x\in\mathtt{Var}}{A(x)}\,\forall^-}{B}\to^-}{\forall_x B}\,\forall^+ x$$

(v) $\exists_x\colon \boldsymbol{Form} \to \boldsymbol{Form}$, *where $\big(\exists_x\big)_0(A) = \exists_x A$, for every $A \in \mathtt{Form}$, and $\big(\exists_x\big)_1\big(M\colon A \to$*

B): $\exists_x A \to \exists_x B$ is the following derivation of $\exists_x B$ from assumption $\exists_x A$, given derivation M

$$\dfrac{\dfrac{w\colon \exists_x A}{\exists_x A}\,1_{\exists_x A} \qquad \dfrac{x \in \mathtt{Var} \qquad \begin{matrix}[u\colon A]\\ \mid M\\ B\end{matrix}}{\exists_x B}\,\exists^+}{\exists_x B}\,\exists^- x, u$$

Notice that in the definition of the derivation $(\forall_x)_1(M)$, if $x \notin \mathrm{FV}(A)$, then $A(x) = A$, by Proposition 1.6.4, while if $x \in \mathrm{FV}(A)$, then $A = A(x)$, trivially. The variable condition in the application of the rule $\forall^+$ is satisfied, as in the above derivation of B the only open assumption is $w\colon \forall_x A$ and $x \notin \mathrm{FV}(\forall_x A)$. In the definition of the derivation $(\exists_x)_1(M)$ the variable condition in the application of the rule $\exists^- x, u$ is satisfied, as in the above derivation of $\exists_x B$ the variable x is not free in $\exists_x B$, while it can be free in the only open (until then) assumption $u\colon A$. The proof that these pairs are functors is immediate, as **Form** is thin.

There are more functors related to the logical symbols of a first-order language, which are variations of the functors given in Definition 1.14.8.

Definition 1.14.9. *If we fix a formula B, let the functors $\bigwedge_B, \bigvee_B, \longrightarrow_B\colon$ **Form** $\to$ **Form**, defined by the rules $A \mapsto B \wedge A$, $A \mapsto B \vee A$, and $A \mapsto B \to A$, respectively. The application of these functors on arrows is defined as in the case of the corresponding functors in Definition 1.14.8.*

The functor $\longrightarrow_B$ is a special case of the following general functor, although the corresponding proof is more involved.

Proposition 1.14.10. *Let $\boldsymbol{C}$ be a cartesian closed category and B in C_0. The rule $A \mapsto (B \to A)$, where $B \to A$ is a fixed exponential of A and B, determines an endofunctor on $\boldsymbol{C}$.*

Proof. Exercise. If $f\colon C \to D$ in C_1, you need to define an arrow $(B \to C) \to (B \to D)$ with the use of f and the universal properties of the exponentials $B \to C$ and $B \to D$. □

Definition 1.14.11. *Let the functors ${}_B\bigwedge$, ${}_B\bigvee\colon$ **Form** $\to$ **Form**, defined by the rules $A \mapsto A \wedge B$ and $A \mapsto A \vee B$, respectively.*

Clearly, the functors ${}_B\bigwedge$, $\bigwedge_B$ and ${}_B\bigvee, \bigvee_B$ are very similar, respectively. This similarity is clarified with the use of the following very important notion of "map", or arrow, between functors from a category $\boldsymbol{C}$ to a category $\boldsymbol{D}$.

1.15 Natural transformations

Definition 1.15.1. *Let $\boldsymbol{C}, \boldsymbol{D}$ be categories and $F = (F_0, F_1), G = (G_0, G_1)$ functors from $\boldsymbol{C}$ to $\boldsymbol{D}$. A natural transformation from F to G is a family of arrows in $\boldsymbol{D}$ of the form $\tau_C : F_0(C) \to G_0(C)$, such that for every C in C_0, and every $f : C \to C'$ in C_1, the following diagram commutes*

$$\begin{array}{ccc} F_0(C) & \xrightarrow{\;F_1(f)\;} & F_0(C') \\ {\scriptstyle\tau_C}\big\downarrow & & \big\downarrow{\scriptstyle\tau_{C'}} \\ G_0(C) & \xrightarrow[\;G_1(f)\;]{} & G_0(C'). \end{array}$$

We denote a natural transformation τ from F to G by $\tau : F \Rightarrow G$.

Example 1.15.2. Let $\mathrm{Id}^{\mathbf{Set}} = (\mathrm{Id}_0^{\mathbf{Set}}, \mathrm{Id}_1^{\mathbf{Set}})$ be the identity functor on **Set** (Example 1.14.2), and let the functor $H = (H_0, H_1) : \mathbf{Set} \to \mathbf{Set}$, defined by

$$H_0(X) = \mathbb{F}(\mathbb{F}(X)) = \{\Phi : \mathbb{F}(X) \to \mathbb{R}\},$$

and if $f : X \to Y$, then $H_1(f) : \mathbb{F}(\mathbb{F}(X)) \to \mathbb{F}(\mathbb{F}(Y))$ is defined by $\big[H_1(f)\big](\Phi) = \Phi \circ G_1(f)$, for every $\Phi \in \mathbb{F}(\mathbb{F}(X))$

$$\begin{array}{ccc} \mathbb{F}(Y) & \xrightarrow{\ G_1(f)\ } & \mathbb{F}(X) \\ & \searrow^{\Phi \circ G_1(f)} & \downarrow \Phi \\ & & \mathbb{R}, \end{array}$$

where G_1 is defined in the Example 1.14.3. It is straightforward to show that H is a functor (actually, one can avoid this calculation and infer immediately that $H\colon \mathbf{Set} \to \mathbf{Set}$ through the definition of H_0 and the fact that $G\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$-why?). The *Gelfand transformation* is the following family of arrows in **Set**

$$\tau = \Big(\tau_X : X \to \mathbb{F}(\mathbb{F}(X))\Big)_X$$

$$\tau_X(x) = \hat{x},$$

$$\hat{x}\colon \mathbb{F}(X) \to \mathbb{R}, \quad \hat{x}(\phi) = \phi(x); \quad \phi \in \mathbb{F}(X).$$

The Gelfand transformation τ is a natural transformation from $\mathrm{Id}_{\mathbf{Set}}$ to H, as for every $f\colon X \to Y$ the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \tau_X \downarrow & & \downarrow \tau_Y \\ \mathbb{F}(\mathbb{F}(X)) & \xrightarrow[\ H_1(f)\ ]{} & \mathbb{F}(\mathbb{F}(Y)), \end{array}$$

since, if $\theta \in \mathbb{F}(Y)$ and $x \in X$, we have that

$$\begin{aligned} \Big[H_1(f)\big(\tau_X(x)\big)\Big](\theta) &= \big[\tau_X(x) \circ G_1(f)\big](\theta) \\ &= \big[\hat{x} \circ G_1(f)\big](\theta) \\ &= \hat{x}\Big([G_1(f)](\theta)\Big) \\ &= \hat{x}\big(\theta \circ f\big) \\ &= \theta(f(x)) \\ &= \widehat{f(x)}(\theta) \\ &= \Big[(\tau_Y \circ f)(x)\Big](\theta). \end{aligned}$$

Definition 1.15.3. *If $\boldsymbol{C}, \boldsymbol{D}$ are categories the functor category $\mathtt{Fun}(\boldsymbol{C}, \boldsymbol{D})$ has objects the functors from $\boldsymbol{C}$ to $\boldsymbol{D}$, and if $F, G : \boldsymbol{C} \to \boldsymbol{D}$, an arrow from F to G is a natural transformation from F to G. The identity arrow $\mathbf{1}_F : F \Rightarrow F$ is the family of arrows $(\mathbf{1}_F)_C : F_0(C) \to F_0(C)$, where $(\mathbf{1}_F)_C = \mathbf{1}_{F_0(C)}$, and the following diagram trivially commutes*

$$\begin{array}{ccc}
F_0(C) & \xrightarrow{F_1(f)} & F_0(C') \\
\mathbf{1}_{F_0(C)} \big\downarrow & & \big\downarrow \mathbf{1}_{F_0(C')} \\
F_0(C) & \xrightarrow[F_1(f)]{} & F_0(C').
\end{array}$$

If $F, G, H : \boldsymbol{C} \to \boldsymbol{D}$, $\tau : F \Rightarrow G$ and $\sigma : G \Rightarrow H$, the composite arrow $\sigma \circ \tau$ is defined by

$$(\sigma \circ \tau)_C = \sigma_C \circ \tau_C : F_0(C) \to H_0(C),$$

for every C in C_0, and, if $f : C \to C'$ in C_1, the following outer diagram commutes

$$\begin{array}{ccccc}
 & F_0(C) & \xrightarrow{F_1(f)} & F_0(C') & \\
 & \big\downarrow \tau_C & & \tau_{C'} \big\downarrow & \\
(\sigma \circ \tau)_C \Big\downarrow & G_0(C) & \xrightarrow[G_1(f)]{} & G_0(C') & \Big\downarrow (\sigma \circ \tau)_{C'} \\
 & \big\downarrow \sigma_C & & \sigma_{C'} \big\downarrow & \\
 & H_0(C) & \xrightarrow[H_1(f)]{} & H_0(C'), &
\end{array}$$

since

$$\begin{aligned}
(\sigma \circ \tau)_{C'} \circ F_1(f) &= \big(\sigma_{C'} \circ \tau_{C'}\big) \circ F_1(f) \\
&= \sigma_{C'} \circ \big(\tau_{C'} \circ F_1(f)\big) \\
&= \sigma_{C'} \circ \big(G_1(f) \circ \tau_C\big) \\
&= \big(\sigma_{C'} \circ G_1(f)\big) \circ \tau_C \\
&= \big(H_1(f) \circ \sigma_C\big) \circ \tau_C \\
&= H_1(f) \circ \big(\sigma_C \circ \tau_C\big) \\
&= H_1(f) \circ (\sigma \circ \tau)_C.
\end{aligned}$$

Example 1.15.4. The functors ${}_B\bigwedge$, and $\bigwedge_B$ are isomorphic in $\mathtt{Fun}(\mathbf{Form}, \mathbf{Form})$, and also the functors ${}_B\bigvee$ and $\bigvee_B$, are isomorphic in the category $\mathtt{Fun}(\mathbf{Form}, \mathbf{Form})$ (exercise).

1.16 Galois connections

According to Proposition 1.11.4(i), the formula $(A \wedge B \to C) \leftrightarrow (A \to (B \to C))$ is derivable in minimal logic. This fact is rephrased as follows:

$$A \wedge B \leq C \Leftrightarrow A \leq (B \to C).$$

With the help of the functors ${}_B\bigwedge, \longrightarrow_B : \mathbf{Form} \to \mathbf{Form}$ the last equivalence is rewritten as

$$\Big({}_B\bigwedge\Big)_0(A) \le C \Leftrightarrow A \le (\longrightarrow_B)_0(C),$$

which in turn is a special case of the equivalence

$$f\colon D \times B \to C \Leftrightarrow \widehat{f}\colon D \to (B \to C)$$

that holds in a category $\boldsymbol{C}$ with exponentials. The last equivalence is understood as follows: if $f\colon D \times B \to C$, there is a unique arrow $\widehat{f}\colon D \to (B \to C)$, using the universal property of an exponential $B \to C$ of B and C in a category $\boldsymbol{C}$. Conversely, if $g\colon D \to (B \to C)$, there is a unique arrow $f\colon D \times B \to C$ such that $\widehat{f} = g$ (exercise).

Definition 1.16.1. *If $(I, \le)$ and $(J, \preceq)$ are preorders, a Galois connection, or a Galois correspondence, between them is a pair of monotone functions $(f\colon I \to J, g\colon J \to I)$, such that*

$$\forall_{i \in I}\forall_{j \in J}\Big(f(i) \preceq j \Leftrightarrow i \le g(j)\Big).$$

In this case we say that g is right adjoint to f, or f is left adjoint to g, and we write $f \dashv g$.

Clearly, we have that

$${}_B\bigwedge \dashv \longrightarrow_B.$$

Definition 1.16.2. *Let $(I, \le)$ be a preorder. If $i, i' \in I$, then $i \cong i' :\Leftrightarrow i \le i'$ & $i' \le i$, and since this is a special case of Definition 1.13.1, we say then that i and i' are isomorphic. A closure operator on I is a monotone function $\texttt{Cls}\colon I \to I$, such that $i \le \texttt{Cls}(i)$ and $\texttt{Cls}(\texttt{Cls}(i)) \le \texttt{Cls}(i)$, for every $i \in I$. An interior operator on I is a monotone function $\texttt{Int}\colon I \to I$, such that $\texttt{Int}(i) \le i$ and $\texttt{Int}(i) \le \texttt{Int}(\texttt{Int}(i))$, for every $i \in I$. An element i of I is called closed, with respect to the closure operator $\texttt{Cls}$, if $i \cong \texttt{Cls}(i)$, and it is called open, with respect to the interior operator $\texttt{Int}$, if $i \cong \texttt{Int}(i)$. We denote by $\texttt{Closed}(I)$ the set of closed elements of I, and by $\texttt{Open}(I)$ the set of open elements of I.*

As $\texttt{Cls}(i) \cong \texttt{Cls}(\texttt{Cls}(i))$ and $\texttt{Int}(i) \cong \texttt{Int}(\texttt{Int}(i))$, we get $\texttt{Cls}(i) \in \texttt{Closed}(I)$ and $\texttt{Int}(i) \in \texttt{Open}(I)$, for every $i \in I$. Notice that if $(I, \le)$ and $(J, \preceq)$ are preorders, and if $f\colon I \to J$ is monotone, then f preserves isomorphism i.e., $i \cong i' \Rightarrow f(i) \cong f(i')$, for every $i, i' \in I$, where we use the same symbol for isomorphic elements of J.

Proposition 1.16.3. *Let $(I, \le)$ and $(J, \preceq)$ be preorders and $(f\colon I \to J, g\colon J \to I)$ a Galois connection between them.*

(i) *The composition $g \circ f$ is a closure operator on I.*

(ii) *The composition $f \circ g$ is an interior operator on J.*

(iii) *The rule $i \mapsto f(i)$ determines a function from the closed elements of I with respect to $g \circ f$ to the open elements of J with respect to $f \circ g$.*

(iv) *The rule $j \mapsto g(j)$ determines a function from the open elements of J with respect to $f \circ g$ to the closed elements of I with respect to $g \circ f$.*

Proof. Exercise. $\square$

In a Galois connection the adjoints are unique up to isomorphism.

Corollary 1.16.4. *Let* $(I, \leq)$ *and* $(J, \preceq)$ *be preorders and* $(f\colon I \to J, g\colon J \to I)$ *a Galois connection between them.*
(i) *If* $(f\colon I \to J, g'\colon J \to I)$ *is a Galois connection, then* $g(j) \cong g'(j)$*, for every* $j \in J$*.*
(ii) *If* $(f'\colon I \to J, g\colon J \to I)$ *is a Galois connection, then* $f(i) \cong f'(i)$*, for every* $i \in I$*.*

Proof. Exercise. □

The quantifiers can be described as adjoints. First we give a set-interpretation of this fact.

Definition 1.16.5. *Let* X, Y *be sets. If* $u\colon X \to Y$*, let* $u^*\colon \mathcal{P}(Y) \to \mathcal{P}(X)$*, defined by*

$$u^*(B) = u^{-1}(B) = \{x \in X \mid u(x) \in B\}; \quad B \in \mathcal{P}(Y).$$

As $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \subseteq)$ are posets, it is immediate to see that u^* is a monotone function, hence, according to Example 1.14.6, a contravariant functor.

Proposition 1.16.6. *Let* X, Y *be sets, and let the preorders* $\big(\mathcal{P}(X), \subseteq\big)$ *and* $\big(\mathcal{P}(X \times Y), \subseteq\big)$*.*
(i) *The functions* $\exists_{XY}\colon \mathcal{P}(X \times Y) \to \mathcal{P}(X)$ *and* $\forall_Y\colon \mathcal{P}(X \times Y) \to \mathcal{P}(X)$*, defined by*

$$\exists_{XY}(C) = \big\{x \in X \mid \exists_{y \in Y}\big((x, y) \in C\big)\big\},$$
$$\forall_{XY}(C) = \big\{x \in X \mid \forall_{y \in Y}\big((x, y) \in C\big)\big\},$$

for every $C \subseteq X \times Y$*, respectively, are monotone.*
(ii) *If* $\pi_X\colon X \times Y \to X$ *is the projection function to* X*, then*

$$\exists_{XY} \dashv \pi_X^* \;\;\&\;\; \pi_X^* \dashv \forall_{XY},$$

where $\pi_X^*\colon \mathcal{P}(X) \to \mathcal{P}(X \times Y)$ *is defined according to Definition 1.16.5.*

Proof. Exercise. □

1.17 The quantifiers as adjoints

In this section we translate Proposition 1.16.6 into minimal logic.

Definition 1.17.1. *If* $\boldsymbol{C} = (C_0, C_1, \mathrm{dom}, \mathtt{cod}, \circ, \mathbf{1})$ *is a category, a subcategory* $\boldsymbol{D}$ *of* $\boldsymbol{C}$ *is a subcollection of objects in* $\boldsymbol{C}$ *and a subcollection of arrows in* $\boldsymbol{C}$*, which are closed under the operations* dom, $\mathtt{cod}$, $\circ$*, and* $\mathbf{1}$ *of* $\boldsymbol{C}$*. In this case we write* $\boldsymbol{D} \leq \boldsymbol{C}$*. If* $A, B \in C_0$*, let*

$$C_1(A, B) = \{f \in C_1 \mid \mathrm{dom}(f) = A \;\&\; \mathtt{cod}(f) = B\}.$$

If $\boldsymbol{D}$ *is a subcategory of* $\boldsymbol{C}$*, such that for every* $A, B \in D_0$ *we have that* $D_1(A, B) = C_1(A, B)$*, then* $\boldsymbol{D}$ *is called a full subcategory of* $\boldsymbol{C}$*. A category* $\boldsymbol{C}$ *is called small, if the collections* C_0 *and* C_1 *are both sets. If one of them is a proper class i.e., a class that is not a set, then* $\boldsymbol{C}$ *is called large*[6]*. If for every* $A, B \in C_0$ *the collection* $C_1(A, B)$ *is a set, then* $\boldsymbol{C}$ *is called locally small.*

[6] In Zermelo-Fraenkel set theory a class is either a set or a proper class. The collection of all sets, or the universe, $\mathbb{V}$ is a proper class. That can be shown via the so-celled Russell's paradox: if $\mathbb{V}$ was a set, then we can define with the scheme of separation the set $R = \{x \in \mathbb{V} \mid x \notin x\}$, and then we reach the contradiction $R \in R \Leftrightarrow R \notin R$.

Example 1.17.2. The category $\mathbf{Set}_{\text{fin}}$ of all finite sets and functions between them is a full subcategory of **Set**. The category **Set** is large, as the collection $\mathbb{V}$ of all sets is a proper class, but it is locally small, since the collection of all functions between two sets is a set. The category **Form** is small.

Example 1.17.3. If $x_1, \ldots, x_n \in \mathtt{Var}$, the category $\mathbf{Form}(x_1, \ldots, x_n)$ with objects the set $\mathtt{Form}(x_1, \ldots, x_n)$ of all formulas A, such that $\mathrm{FV}(A) \subseteq \{x_1, \ldots, x_n\}$, together with the usual derivations between them as arrows, is a full subcategory of **Form**.

Definition 1.17.4. *If $x, y \in \mathtt{Var}$, let the functors $\exists(x, y), \forall(x, y) \colon \boldsymbol{Form}(x, y) \to \boldsymbol{Form}(x)$, defined by the rules*

$$\Big(\exists_{xy}\Big)_0(A) = \exists_y A \quad \& \quad \Big(\forall_{xy}\Big)_0(A) = \forall_y A; \qquad A \in \mathtt{Form}(x, y).$$

Let also $W(x, y) \colon \boldsymbol{Form}(x) \to \boldsymbol{Form}(x, y)$, where $\big(W(x, y)\big)_0(A) = A$, for every $A \in \mathtt{Form}$.

Next follows the immediate translation of Proposition 1.16.6 into minimal logic.

Theorem 1.17.5. *The following adjunctions hold:*
(i) *$\exists(x, y) \dashv W(x, y)$.*
(ii) *$W(x, y) \dashv \forall(x, y)$.*

Proof. Exercise. □

Example 1.17.6. The category $\mathbf{Form}_x$ of all formulas A, such that $x \notin \mathrm{FV}(A)$, together with the usual derivations between them as arrows, is a full subcategory of **Form**.

Definition 1.17.7. *The functors $\exists_x \colon \boldsymbol{Form} \to \boldsymbol{Form}$ and $\forall_x \colon \boldsymbol{Form} \to \boldsymbol{Form}$, defined in Definition 1.14.8, can be written as functors of the form $\exists_x \colon \boldsymbol{Form} \to \boldsymbol{Form}_x$ and $\forall_x \colon \boldsymbol{Form} \to \boldsymbol{Form}_x$, since $x \notin \mathrm{FV}(\exists_x A)$ and $x \notin \mathrm{FV}(\forall_x A)$. Let the functor $W_x \colon \boldsymbol{Form}_x \to \boldsymbol{Form}$, defined by $\big(W_x\big)_0(A) = A$, for every $A \in \mathtt{Form}$.*

Theorem 1.17.8. *The following adjunctions hold:*
(i) *$\exists_x \dashv W_x$.*
(ii) *$W_x \dashv \forall_x$.*

Proof. Let $A \in \mathtt{Form}$ such that $x \notin \mathrm{FV}(A)$, and $C \in \mathtt{Form}$.
(i) We show that $\big(\exists_x\big)_0(C) \leq A \Leftrightarrow C \leq \big(W_x)_0(A)$ i.e., there is an arrow $M \colon \exists_x C \to A$ if and only if there is an arrow $N \colon C \to A$. Suppose first that $M \colon \exists_x C \to A$. We find a derivation N of A from assumption C, as follows:

$$\dfrac{\dfrac{\begin{matrix}[v\colon \exists_x C]\\ \mid M\\ A\end{matrix}}{\exists_x C \to A}\,{\to^+}v \qquad \dfrac{x \in \mathtt{Term} \qquad \dfrac{w\colon C}{C}\,1_C}{\exists_x C}\,\exists^+}{A}\,{\to^-}.$$

For the converse, we suppose that there is a derivation N of A with assumption C, and we find a derivation M of A with assumption $\exists_x C$, as follows:

$$\dfrac{\dfrac{w\colon \exists_x C}{\exists_x C}\,1_{\exists_x C} \qquad \begin{matrix}[u\colon C]\\ \mid N\\ A\end{matrix}}{A}\,\exists^- x, u.$$

The variable condition in $\exists^- x, u$ is satisfied: as A is in $\mathbf{Form}_x$, $x \notin \mathrm{FV}(A)$, and the only open assumption in the above derivation N of A is $u\colon C$, and x can be free in C).
(ii) We show that $\big(W_x\big)_0(A) \leq C \Leftrightarrow A \leq \big(\forall_x\big)_0(C)$ i.e., there is an arrow $M\colon A \to C$ if and only if there is an arrow $N\colon A \to \forall_x C$. Suppose first that $M\colon A \wedge B(x) \to C$. We find a derivation N of $\forall_x C$ from assumption A, as follows:

$$\cfrac{\cfrac{\cfrac{\begin{matrix}[v\colon A]\\ \mid M\\ C\end{matrix}}{A \to C}\,{\to^+}v \qquad \cfrac{a\colon A}{A}\,1_A}{C}\,{\to^-}}{\forall_x C}\,\forall^+ x$$

The variable condition in $\forall^+ x$ is satisfied: the only open assumption in the derivation of C is A, and by our hypothesis $x \notin \mathrm{FV}(A)$. For the converse, let N be a derivation of $\forall_x C$ with assumption A. We find a derivation M of C with assumption A, as follows:

$$\cfrac{\cfrac{\cfrac{\begin{matrix}[u\colon A]\\ \mid N\\ \forall_x C\end{matrix}}{A \to \forall_x C}\,{\to^+}u \qquad \cfrac{a\colon A}{A}\,1_A}{\forall_x C}\,{\to^-} \qquad x \in \mathtt{Term}}{C}\,\forall^- \qquad \square$$

Next we give one more variation of the previous theorem.

Definition 1.17.9. *Let $\mathcal{L}$ be a first-order language, x is a fixed variable. Moreover, we suppose that there is a formula B of $\mathcal{L}$, such that $x \in \mathrm{FV}(B)$ and a fixed derivation K of B in minimal logic. E.g., if $R \in \mathtt{Rel}^{(1)}$, we can take B to be the formula $R(x) \to R(x)$. Let the functor $W_x^B\colon \boldsymbol{Form}_x \to \boldsymbol{Form}$, defined by $\big(W_x^B\big)_0(A) = A \wedge B(x)$, for every $A \in \mathtt{Form}$.*

Theorem 1.17.10. *The following adjunctions hold:*
(i) $\exists_x \dashv W_x^B$.
(ii) $W_x^B \dashv \forall_x$.

Proof. We proceed exactly as in the proof of Theorem 1.17.8. $\square$

In accordance to Corollary 1.16.4, if $\mathcal{L}$ is a first-order language as described in Theorem 1.17.10, we have that $W_x(A) \cong W_x^B(A)$, for every $A \in \mathtt{Form}$.

Chapter 2

Derivations in Intuitionistic and Classical Logic

In this chapter we study derivations in intuitionistic and classical logic. We also explore the relation between minimal, intuitionisitc and classical logic.

2.1 Derivations in intuitionistic logic

The intuitionistic derivations are the minimal derivations extended with the rule "ex-falso-quodlibet" (from falsity everything follows).

Definition 2.1.1. *We define inductively the set $\mathfrak{D}^i_V(A)$ of intuitionistic derivations of a formula A with assumption variables in V, where V is a finite subset of* `Aform` *(see Definition 1.9.1). If $V = \emptyset$, we write $\mathfrak{D}^i(A)$. The introduction-rules for $\mathfrak{D}^i_V(A)$ are the introduction-rules for $\mathfrak{D}_V(A)$, given in Definition 1.9.2, together with the following rule:*

(0_A) *The following tree 0_A*

$$\frac{o\colon \bot}{A}\, 0_A$$

is an element of $\mathfrak{D}^i_{\{o\colon \bot\}}(A)$, for every $A \in \mathtt{Form} \setminus \{\bot\}$.

Unless otherwise stated, a derivation in $\mathfrak{D}^i_V(A)$ is denoted by M_i. If $V = \{u\colon A\}, W = \{v\colon A\}$, $M_i \in \mathfrak{D}^i_V(B)$ and $N_i \in \mathfrak{D}^i_W(B)$, we define $M_i = N_i$. A formula A is derivable in intuitionistic logic, written $\vdash_i A$, if there is an intuitionistic derivation of A without free assumptions i.e.,

$$\vdash_i A :\Leftrightarrow \exists_{M_i}(M_i \in \mathfrak{D}^i(A)).$$

A formula A is intuitionistically derivable from assumptions $A_1, \ldots, A_n$, written $\{A_1, \ldots, A_n\} \vdash_i A$, or $A_1, \ldots, A_n \vdash_i A$, if there is an intuitionistic derivation of A with free assumptions among $A_1, \ldots, A_n$ i.e.,

$$A_1, \ldots, A_n \vdash_i A :\Leftrightarrow \exists_{V \subseteq^{fin} \mathtt{Aform}}\Big(\mathtt{Form}(V) \subseteq \{A_1, \ldots, A_n\} \ \&\ \exists_{M_i}\big(M_i \in \mathfrak{D}^i_V(A)\big)\Big).$$

If $\Gamma \subseteq$ `Form`, a formula A is called intuitionistically derivable from Γ, written $\Gamma \vdash_i A$, if A is intuitionistically derivable from finitely many assumptions $A_1, \ldots, A_n \in \Gamma$. The category of *intuitionistic formulas* $\boldsymbol{Form}_i$ *has objects the formulas in* `Form` *and an arrow from* A *to* B *is an intuitionistic derivation of* B *from an assumption of the form* $u\colon A$*, and we write*

$$M_i\colon A \to B :\Leftrightarrow M_i \in \mathfrak{D}^i_{\{u\colon A\}}(B).$$

The induced preorder and isomorphism of the thin category $\boldsymbol{Form}_i$ *are given by*

$$A \leq_i B :\Leftrightarrow \exists_{M_i}\big(M_i\colon A \to B\big),$$

$$A \cong_i B :\Leftrightarrow A \leq_i B \ \& \ B \leq_i A.$$

If $A = \bot$, then $1_\bot$ is already a minimal derivation of $\bot$ from $\bot$. This is why we exclude the derivation $0_\bot$ from the rule (0_A) in Definition 2.1.1. We use the following notation.

Definition 2.1.2. *Let* $0_\bot = 1_\bot$.

If $A \in \texttt{Form} \setminus \{\bot\}$, then $\vdash_i \bot \to A$, as it is shown by the following tree:

$$\cfrac{\cfrac{[o\colon \bot]}{A}\,0_A}{\bot \to A}\to^+ o.$$

The addition of the rule (0_A) in the inductive definition of $\mathfrak{D}^i_V(A)$ has an immediate consequence to the category of intuitionistic formulas $\mathbf{Form}_i$ and to the preorder $\leq_i$.

Proposition 2.1.3. *The category of intuitionistic formulas* $\boldsymbol{Form}_i$ *has an initial element, and the preorder* $\leq_i$ *has a minimal element.*

Proof. If $A \in \texttt{Form} \setminus \{\bot\}$, then $0_A \in \mathfrak{D}^i_{\{o\colon \bot\}}(A)$ i.e., $0_A\colon \bot \to A$. If $A = \bot$, then $1_\bot\colon \bot \to \bot$. The uniqueness of these arrows follows immediately by the thinness of $\mathbf{Form}_i$. By Definition 2.1.2, $0_A\colon \bot \to A \Leftrightarrow \bot \leq_i A$, for every $A \in \texttt{Form}$ i.e., $\bot$ is $\leq_i$-minimal. □

Proposition 2.1.4. *If* $A \in \texttt{Form}$ *and* $V \subseteq^{fin} \texttt{Aform}$*, then* $\mathfrak{D}_V(A) \subseteq \mathfrak{D}^i_V(A)$.

Proof. We use induction on $\mathfrak{D}_V(A)$. Let $P(M) :\Leftrightarrow M \in \mathfrak{D}^i_V(A)$, a formula of our metatheory $\mathcal{M}$ on $\mathfrak{D}_V(A)$. The cumbersome to write induction principle $\mathrm{Ind}_{\mathfrak{D}_V(A)}$ gives us that

$$\forall_{M \in \mathfrak{D}_V(A)}\big(M \in \mathfrak{D}^i_V(A)\big).$$

E.g., according to the clause of $\mathrm{Ind}_{\mathfrak{D}_V(A)}$ with respect to the rule $(\to^+)$, if $M \in \mathfrak{D}_{\{u\colon A\}}(B)$ such that $M \in \mathfrak{D}^i_{\{u\colon A\}}(B)$, then $\frac{M}{A\to B}\to^+ u \in \mathfrak{D}^i(A \to B)$, as we apply the rule $(\to^+ u)$ of $\mathfrak{D}^i_V(A)$ on M. For the rest rules of $\mathfrak{D}_V(A)$ we work similarly. □

Corollary 2.1.5. *Let* $A, B \in \texttt{Form}$ *and* $\Gamma \subseteq \texttt{Form}$.

(i) *If* $\Gamma \vdash A$*, then* $\Gamma \vdash_i A$.

(ii) *The category* $\boldsymbol{Form}$ *is a subcategory of* $\boldsymbol{Form}_i$ *(is it full?).*

(iii) *The identity rules* $A \mapsto A$ *and* $(M\colon A \to B) \mapsto (M\colon A \to B)$ *determine the functor* $\mathrm{Id}^{mi}\colon \boldsymbol{Form} \to \boldsymbol{Form}_i$.

(iv) *If* $A \leq B$*, then* $A \leq_i B$.

(v) *If* $A \cong B$*, then* $A \cong_i B$.

Proof. All cases follow immediately from Proposition 2.1.3. □

The category $\mathbf{Form}_i$, as the category $\mathbf{Form}$, has a terminal object $\top$, which is a $\leq$-maximal, and hence by Corollary 2.1.5, it is also $\leq_i$-maximal. As there are many $\leq_i$-maximal elements, there are many $\leq_i$-minimal elements, although isomorphic to each other. E.g., $A \wedge \neg A \cong \bot$, for every $A \in \mathtt{Form}$. The inequality $\bot \leq_i A \wedge \neg A$ follows from $0_{A\wedge\neg A}$, while the inequality $A \wedge \neg A \leq_i \bot$ follows immediately (i.e., by Corollary 2.1.5(iv)) from the minimal inequality $A \wedge \neg A \leq \bot$, as the following tree is a derivation of $\bot$ from $A \wedge \neg A$ in minimal logic:

$$\dfrac{\dfrac{w\colon A\wedge\neg A}{A\wedge\neg A}\,1_{A\wedge\neg A} \qquad \dfrac{\dfrac{[a\colon A]}{A}\,1_A \qquad \dfrac{[v\colon \neg A]}{\neg A}\,1_{\neg A}}{\bot}\,\to^-}{\bot}\,\wedge^- a,v.$$

One could have used a weaker notion of intuitionistic derivability, by not accepting all instances of the ex-falso-quodlibet. One could have defined $\vdash_i A :\Leftrightarrow \mathrm{Efq} \vdash A$, where Efq is the set of formulas defined next.

Definition 2.1.6. *Let* Efq *be the following set of formulas:*

$$\begin{aligned}\mathrm{Efq} = \{&\forall_{x_1,\ldots,x_n}(\bot \to R(x_1,\ldots,x_n)) \mid n \in \mathbb{N}^+, R \in \mathtt{Rel}^{(n)}, x_1,\ldots,x_n \in \mathtt{Var}\}\\ &\cup \{\bot \to R \mid R \in \mathtt{Rel}^{(0)} \setminus \{\bot\}\}.\end{aligned}$$

Theorem 2.1.7. $\forall_{A\in\mathtt{Form}}\big(\mathrm{Efq} \vdash (\bot \to A)\big)$.

Proof. If $A = R(t_1,\ldots,t_n)$, where $n \in \mathbb{N}^+$, $R \in \mathtt{Rel}^{(n)}$ and $t_1,\ldots,t_n \in \mathtt{Term}$, the following is an intuitionistic derivation of $\bot \to R(t_1,\ldots,t_n)$:

$$\dfrac{\dfrac{\forall_{x_1,\ldots,x_n}(\bot \to R(x_1,\ldots,x_n)) \qquad t_1 \in \mathtt{Term}}{\forall_{x_2,\ldots,x_n}(\bot \to R(t_1,x_2\ldots,x_n))}\,\forall^- \qquad t_2 \in \mathtt{Term}}{\forall_{x_3,\ldots,x_n}(\bot \to R(t_1,t_2,x_3\ldots,x_n))}\,\forall^-$$

$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$

$$\dfrac{\forall_{x_n}(\bot \to R(t_1,\ldots,t_{n-1},x_n)) \qquad t_n \in \mathtt{Term}}{\bot \to R(t_1,t_2\ldots,t_n)}\,\forall^-$$

If we suppose that $\mathrm{Efq} \vdash (\bot \to A)$ and $\mathrm{Efq} \vdash (\bot \to B)$ i.e., that there are minimal derivations M, N of $\bot \to A$ and $\bot \to B$ from Efq, respectively, the following are minimal derivations of $\bot \to A \to B, \bot \to A \vee B, \bot \to A \wedge B, \bot \to \forall_x A$ and $\bot \to \exists_x A$ from Efq, respectively:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathrm{Efq}\\ \mid N\\ \bot \to B\end{matrix} \qquad [v\colon \bot]}{B}\,\to^-}{A \to B}\,\to^+ u\colon A}{\bot \to (A \to B)}\,\to^+ v$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathrm{Efq}\\ \mid M\\ \bot \to A\end{matrix} \qquad [u\colon \bot]}{A}\,\to^- \qquad \dfrac{\begin{matrix}\mathrm{Efq}\\ \mid N\\ \bot \to B\end{matrix} \qquad [u\colon \bot]}{B}\,\to^-}{A \wedge B}\,\wedge^+}{\bot \to (A \wedge B)}\,\to^+ u$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathrm{Efq}\\ \mid M\\ \bot\to A\end{array}\qquad [u\colon \bot]}{A}\,\to^-}{A\vee B}\,\vee_0^+}{\bot\to(A\vee B)}\,\to^+u
\qquad\qquad
\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathrm{Efq}\\ \mid M\\ \bot\to A\end{array}\qquad [u\colon \bot]}{A}\,\to^-}{\forall_x A}\,\forall^+x}{\bot\to\forall_x A}\,\to^+u$$

$$\dfrac{\dfrac{x\in\mathtt{Var}\qquad \dfrac{\begin{array}{c}\mathrm{Efq}\\ \mid M\\ \bot\to A\end{array}\qquad [u\colon \bot]}{A}\,\to^-}{\exists_x A}\,\exists^+}{\bot\to\exists_x A}\,\to^+u$$

In the above use of the $\forall^+x$-rule the variable condition is satisfied, as $x\notin \mathrm{FV}(\bot)=\mathrm{FV}(S)=\emptyset$, for every $S\in \mathrm{Efq}$. $\square$

Proposition 2.1.8. *Let the functor* $\mathrm{EFQ}\colon \boldsymbol{Form}\to\boldsymbol{Form}$, *where* $\mathrm{EFQ}_0(A)=\bot\to A$, *for every* $A\in\mathtt{Form}$ (*of which already established functor is* EFQ *a special case?*).

(i) EFQ *preserves products i.e.,* $\mathrm{EFQ}_0(A\wedge B)\cong \mathrm{EFQ}_0(A)\wedge\mathrm{EFQ}_0(B)$, *for every* $A,B\in\mathtt{Form}$.

(ii) $\mathrm{EFQ}_0(A\vee B)\geq \mathrm{EFQ}_0(A)\vee\mathrm{EFQ}_0(B)$, *for every* $A,B\in\mathtt{Form}$.

(iii) EFQ *preserves the terminal object* $\top$ *i.e.,* $\mathrm{EFQ}_0(\top)\cong\top$.

(iv) *If* $\mathrm{EFQ}^{ii}\colon \boldsymbol{Form}_i\to\boldsymbol{Form}_i$ *is also defined by* $\mathrm{EFQ}^{ii}_0(A)=\bot\to A$, *for every* $A\in\mathtt{Form}$, *then* EFQ^{ii} *does not preserve the initial element* $\bot$ *i.e., it is not the case that* $\mathrm{EFQ}^{ii}_0(\bot)\cong_i\bot$.

Proof. Exercise. $\square$

Given that there is no minimal derivation of $\bot\to A$, for every $A\in\mathtt{Form}$, is the rule $\mathrm{EFQ}^{im}\colon \mathbf{Form}_i\to\mathbf{Form}$, defined as above, a functor (exercise)? Note that the extension-rule, the cut-rule and the deduction theorem for minimal logic (see section 1.12) are easily extended to intuitionistic logic. Next we describe the functors associated to negation.

Proposition 2.1.9. *Let* $\mathrm{Id}^{\boldsymbol{Form}}$ *be the identity functor on* $\boldsymbol{Form}$ (*see Example 1.14.2*) *and let* $\neg\colon \boldsymbol{Form}\to\boldsymbol{Form}$, *defined by* $\neg_0(A)=\neg A$, *for every* $A\in\mathtt{Form}$. *For every* $n\in\mathbb{N}$ *we define*

$$\neg^n=\begin{cases}\mathrm{Id}^{\boldsymbol{Form}} & ,\ n=0\\ \neg & ,\ n=1\\ \neg^{n-1}\circ\neg & ,\ n>1.\end{cases}$$

(i) $\neg^{2n+1}$ *is a contravariant endofunctor, for every* $n\in\mathbb{N}$.

(ii) $\neg^{2n}$ *is a covariant endofunctor, for every* $n\in\mathbb{N}$.

(iii) *The endofunctor* $\neg^{2n+1}$ *is isomorphic to* $\neg$ *in* $\mathtt{Fun}(\boldsymbol{Form},\boldsymbol{Form})$, *for every* $n\in\mathbb{N}^+$.

Proof. Exercise. $\square$

If $\neg^n_i\colon \mathbf{Form}_i\to\mathbf{Form}_i$ is defined similarly, for every $n\in\mathbb{N}$, then it also satisfies Proposition 2.1.9(i)-(iii). The corresponding negation endofunctor $\neg^n_c$ on the category of classical formulas, defined in the next section, satisfies extra properties. E.g., the endofunctor $\neg^{2n}_c$ is isomorphic to $\neg^{2n-2}_c$, and hence it is isomorphic to $\mathrm{Id}^{\mathbf{Form}}$, for every $n\geq 2$.

2.2 Derivations in classical logic

The classical derivations are the minimal derivations extended with the rule of "double-negation-elimination".

Definition 2.2.1. *We define inductively the set $\mathfrak{D}^c_V(A)$ of classical derivations of a formula A with assumption variables in V, where V is a finite subset of* $\texttt{Aform}$ *(see Definition 1.9.1). If $V = \emptyset$, we write $\mathfrak{D}^c(A)$. The introduction-rules for $\mathfrak{D}^c_V(A)$ are the introduction-rules for $\mathfrak{D}_V(A)$, given in Definition 1.9.2, together with the following rule*[1]*:*

(DNE_A) *The following tree* DNE_A

$$\frac{u\colon \neg\neg A}{A}\,\mathrm{DNE}_A$$

is an element of $\mathfrak{D}^c_{\{u\colon \neg\neg A\}}(A)$, for every $A \in \texttt{Form} \setminus \{\bot\}$.

Unless otherwise stated, a derivation in $\mathfrak{D}^c_V(A)$ is denoted by M_c. If $V = \{u\colon A\}, W = \{v\colon A\}$, $M_c \in \mathfrak{D}^c_V(B)$ and $N_c \in \mathfrak{D}^c_W(B)$, we define $M_c = N_c$. A formula A is derivable in classical logic, written $\vdash_c A$, if there is a classical derivation of A without free assumptions i.e.,

$$\vdash_c A :\Leftrightarrow \exists_{M_c}(M_c \in \mathfrak{D}^c(A)).$$

A formula A is classically derivable from assumptions $A_1, \ldots, A_n$, written $\{A_1, \ldots, A_n\} \vdash_c A$, or $A_1, \ldots, A_n \vdash_c A$, if there is a classical derivation of A with free assumptions among $A_1, \ldots, A_n$ i.e.,

$$A_1, \ldots, A_n \vdash_c A :\Leftrightarrow \exists_{V \subseteq^{fin} \texttt{Aform}}\Big(\texttt{Form}(V) \subseteq \{A_1, \ldots, A_n\} \ \&\ \exists_{M_c}\big(M_c \in \mathfrak{D}^c_V(A)\big)\Big).$$

If $\Gamma \subseteq \texttt{Form}$, a formula A is called classically derivable from Γ, written $\Gamma \vdash_c A$, if A is classically derivable from finitely many assumptions $A_1, \ldots, A_n \in \Gamma$. The category of classical formulas $\boldsymbol{Form}_c$ *has objects the formulas in* $\texttt{Form}$ *and an arrow from A to B is a classical derivation of B from an assumption of the form $u\colon A$, and we write*

$$M_c\colon A \to B :\Leftrightarrow M_c \in \mathfrak{D}^c_{\{u\colon A\}}(B).$$

The induced preorder and isomorphism of the thin category $\boldsymbol{Form}_c$ *are given by*

$$A \leq_c B :\Leftrightarrow \exists_{M_c}\big(M_c\colon A \to B\big),$$

$$A \cong_c B :\Leftrightarrow A \leq_c B \ \&\ B \leq_c A.$$

The derivation $\neg\neg\bot \to \bot$ is not considered in the rule (DNE_A), as a derivation of $\bot$ from $\neg\neg\bot$ already exists in minimal logic:

$$\cfrac{\cfrac{[v\colon (\bot \to \bot) \to \bot] \qquad \cfrac{\cfrac{[o\colon \bot]}{\bot}\,1_\bot}{\bot \to \bot}\,\to^+ o}{\bot}\,\to^-}{((\bot \to \bot) \to \bot) \to \bot}\,\to^+ v$$

[1]For simplicity we use the same notation for the tree DNE_A and for the formula $\mathrm{DNE}_A = \neg\neg A \to A$. It will always be clear from the context where the notation DNE_A refers to.

Definition 2.2.2. *We denote the above minimal derivation of* $\bot$ *from* $\neg\neg\bot$ *by* $\mathrm{DNE}_\bot$.

If $A \in \texttt{Form} \setminus \{\bot\}$, then $\vdash_c \neg\neg A \to A$, as it is shown by the following tree:

$$\dfrac{\dfrac{[v\colon \neg\neg A]}{A}\,\mathrm{DNE}_A}{\neg\neg A \to A}\,\to^{+} v.$$

Proposition 2.2.3. *If* $A \in \texttt{Form}$ *and* $V \subseteq^{fin} \texttt{Aform}$, *there is a unique, canonical embedding* ${}^c\colon \mathfrak{D}^i_V(A) \to \mathfrak{D}^c_V(A)$.

Proof. We use recursion on $\mathfrak{D}^i_V(A)$. As the introduction rules of $\mathfrak{D}^i_V(A)$ differ from the introduction rules of $\mathfrak{D}^c_V(A)$ only with respect to the rule $(0_A) \in \mathfrak{D}^i_{\{o\colon \bot\}}(A)$, it suffices to describe the rule $(0_A)^c \in \mathfrak{D}^c_{\{o\colon \bot\}}(A)$. If $A \neq \bot$, let the following derivation

$$\dfrac{\dfrac{\dfrac{[u\colon \neg\neg A]}{A}\,\mathrm{DNE}_A}{\neg\neg A \to A}\,\to^{+}u \qquad \dfrac{\dfrac{o\colon \bot \quad [w\colon \neg A]}{\bot}\,1_\bot}{\neg\neg A}\,\to^{+}w}{A}\,\to^{-}.$$

be the derivation $(0_A)^c$, which is clearly in $\mathfrak{D}^c_{\{o\colon \bot\}}(A)$. For all the rest introduction rules of $\mathfrak{D}^i_V(A)$ the embedding $M_i \mapsto M_i^c$ is defined by the identity rule. □

Corollary 2.2.4. *Let* $A, B \in \texttt{Form}$, $\Gamma \subseteq \texttt{Form}$, *and* $V \subseteq^{fin} \texttt{Aform}$.
(i) *If* $\Gamma \vdash_i A$, *then* $\Gamma \vdash_c A$.
(ii) *The rules* $A \mapsto A$ *and* $(M_i\colon A \to B) \mapsto (M_i^c\colon A \to B)$ *determine the functor* $\mathrm{Id}^{ic}\colon \boldsymbol{Form}_i \to \boldsymbol{Form}_c$.
(iii) $\mathfrak{D}_V(A) \subseteq \mathfrak{D}^c_V(A)$.
(iv) *If* $A \leq_i B$, *then* $A \leq_c B$.
(v) *If* $A \cong_i B$, *then* $A \cong_c B$.

Proof. All cases follow immediately from Proposition 2.2.3. Notice that for the proof of (iii) the preservation of the unit arrow 1_A follows immediately from the definition of the canonical embedding ^{c}. As we use the identity rule in its definition for all introduction rules of $\mathfrak{D}^i_V(A)$, other than (0_A), we get $(1_A)^c = 1_A$. □

Combining Corollaries 2.1.5 and 2.2.4, we get $\Gamma \vdash A \Rightarrow \Gamma \vdash_i A \Rightarrow \Gamma \vdash_c A$,

$$\begin{array}{ccc} & \mathfrak{D}_V(A) & \\ {\scriptstyle \mathrm{Id}}\swarrow & & \searrow{\scriptstyle \mathrm{Id}} \\ \mathfrak{D}^i_V(A) & \xrightarrow{\;\;c\;\;} & \mathfrak{D}^c_V(A), \end{array}$$

$$\begin{array}{ccc} & \textbf{Form} & \\ {\scriptstyle \mathrm{Id}^{mi}}\swarrow & & \searrow{\scriptstyle \mathrm{Id}^{mc}} \\ \textbf{Form}_i & \xrightarrow{\;\;\mathrm{Id}^{ic}\;\;} & \textbf{Form}_c, \end{array}$$

where $\mathrm{Id}^{mc} = \mathrm{Id}^{ic} \circ \mathrm{Id}^{mi} \colon \mathbf{Form} \to \mathbf{Form}_c$, is given by the identity rules, and

$$A \leq B \Rightarrow A \leq_i B \Rightarrow A \leq_c B,$$

$$A \cong B \Rightarrow A \cong_i B \Rightarrow A \cong_c B.$$

Proposition 2.2.5. *If* $A \in \mathtt{Form}$*, let* $\mathrm{PEM}_A = A \vee \neg A$.
(i) $\vdash \neg\neg\mathrm{PEM}_A$.
(ii) $\vdash_i \mathrm{PEM}_A \to \mathrm{DNE}_A$.
(iii) $\vdash \mathrm{DNE}_{\mathrm{PEM}_A} \to \mathrm{PEM}_A$*, hence* $\vdash_c \mathrm{PEM}_A$.

Proof. Exercise. □

The addition of the rule (DNE_A) in the inductive definition of $\mathfrak{D}^c_V(A)$ has the following consequence to the preorder $\leq_c$. Note that because of the above implications between the various preorders and congruences, we usually use the subscript i (or none), if an intuitionistic (or minimal) preorder or congruence holds in $\mathbf{Form}_c$.

Corollary 2.2.6. *If* $A \in \mathtt{Form}$*, then* A *is* $\leq_c$*-pseudo-complemented i.e., there is a unique, up to* $\cong$*-isomorphism,* $B \in \mathtt{Form}$ *such that* $A \wedge B \cong_c \bot$ *and* $A \vee B \cong_c \top$.

Proof. Let $B = \neg A$. Then $A \wedge \neg A \leq \bot$ and $\bot \leq_i A \wedge \neg A$. Moreover, $A \vee \neg A \leq \top$ and we show that $\top \leq_c A \vee \neg A$. Actually, by Proposition 2.2.5(iii) we have that $\vdash_c A \vee \neg A$. If we suppose that $B \in \mathtt{Form}$, such that $A \wedge B \cong_c \bot$, we can show (exercise) that $B \cong \neg A$. □

In contrast to intuitionistic derivability, one gets a weaker notion of classical derivability, if the corresponding fewer instances of the double-negation-elimination-principle are considered.

Definition 2.2.7. *Let* Dne *be the following set of formulas:*

$$\mathrm{Dne} = \{\forall_{x_1,\ldots,x_n}(\neg\neg R(x_1,\ldots,x_n) \to R(x_1,\ldots,x_n)) \mid n \in \mathbb{N}^+, R \in \mathtt{Rel}^{(n)},$$
$$x_1,\ldots,x_n \in \mathtt{Var}\} \ \cup \ \{\neg\neg R \to R \mid R \in \mathtt{Rel}^{(0)} \setminus \{\bot\}\}.$$

Let $\vdash^*_c A \Leftrightarrow \mathrm{Dne} \vdash A$ *and* $\Gamma \vdash^*_c A \Leftrightarrow \Gamma \cup \mathrm{Dne} \vdash A$*. We denote a derivation* $\Gamma \vdash^*_c A$ *by* M^*_c.

Clearly, $\vdash^*_c A \Rightarrow \vdash_c A$, but not conversely. Next we see which part of the rule (DNE_A) is captured by the weaker classical derivability $\vdash^*_c$. For that we need a lemma and a definition.

Lemma 2.2.8. *Let* $A, B \in \mathtt{Form}$.
(i) $\vdash (\neg\neg A \to A) \to (\neg\neg B \to B) \to \neg\neg(A \wedge B) \to A \wedge B$.
(ii) $\vdash (\neg\neg B \to B) \to \neg\neg(A \to B) \to A \to B$.
(iii) $\vdash (\neg\neg A \to A) \to \neg\neg\forall_x A \to A$.

Proof. Exercise. □

Definition 2.2.9. *The formulas* $\mathtt{Form}^*$ *without* $\vee, \exists$ *are defined inductively by the rules:*

$$\frac{P \in \mathtt{Prime}}{P \in \mathtt{Form}^*}, \qquad \frac{A, B \in \mathtt{Form}^*}{(A \to B),\ (A \wedge B) \in \mathtt{Form}^*}, \qquad \frac{A \in \mathtt{Form}^*, \quad x \in \mathtt{Var}}{\forall_x A \in \mathtt{Form}^*}.$$

An induction principle and a recursion theorem correspond to this definition of $\mathtt{Form}^*$.

Theorem 2.2.10. $\forall_{A\in\texttt{Form}^*}(\vdash_c^* \neg\neg A \to A)$.

Proof. We use induction on $\texttt{Form}^*$. If A is atomic we work exactly as in the corresponding case of the proof of Theorem 2.1.7 i.e.,

$$\dfrac{\dfrac{\forall_{x_1,\dots,x_n}(\neg\neg R(x_1,\dots,x_n)\to R(x_1,\dots,x_n)) \qquad t_1\in\texttt{Term}}{\forall_{x_2,\dots,x_n}(\neg\neg R(t_1,x_2\dots,x_n)\to R(t_1,x_2\dots,x_n))}\,\forall^- \qquad t_2\in\texttt{Term}}{\forall_{x_3,\dots,x_n}(\neg\neg R(t_1,t_2,x_3\dots,x_n)\to R(t_1,t_2,x_3\dots,x_n))}\,\forall^-$$

$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$

$$\dfrac{\forall_{x_n}(\neg\neg R(t_1,\dots,t_{n-1},x_n)\to R(t_1,\dots,t_{n-1},x_n)) \qquad t_n\in\texttt{Term}}{\neg\neg R(t_1,t_2,\dots,t_n)\to R(t_1,t_2\dots,t_n)}\,\forall^-$$

Next we suppose that there are classical derivations of $\vdash_c^* \neg\neg A \to A, \vdash_c^* \neg\neg B \to B$ and we find classical derivations of $\vdash_c^* \neg\neg(A\to B)\to A\to B, \vdash_c^* \neg\neg(A\wedge B)\to A\wedge B$ and $\vdash_c^* \neg\neg\forall_x A \to \forall_x A$. By Lemma 2.2.8(ii) there is a derivation M of $(\neg\neg B\to B)\to\neg\neg(A\to B)\to A\to B$, and the required classical derivation is

$$\dfrac{\begin{matrix}\mid M \\ (\neg\neg B\to B)\to\neg\neg(A\to B)\to A\to B\end{matrix} \qquad \begin{matrix}\mid M_c^* \\ \neg\neg B\to B\end{matrix}}{\neg\neg(A\to B)\to A\to B}\,\to^-$$

By Lemma 2.2.8(i) there is a derivation N of $C = (\neg\neg A\to A)\to(\neg\neg B\to B)\to\neg\neg(A\wedge B)\to A\wedge B$, and the required classical derivation is

$$\dfrac{\dfrac{\begin{matrix}\mid N \\ C\end{matrix} \qquad \begin{matrix}\mid N_c^* \\ \neg\neg A\to A\end{matrix}}{(\neg\neg B\to B)\to\neg\neg(A\wedge B)\to A\wedge B}\,\to^- \qquad \begin{matrix}\mid M_c^* \\ \neg\neg B\to B\end{matrix}}{\neg\neg(A\wedge B)\to A\wedge B}\,\to^-$$

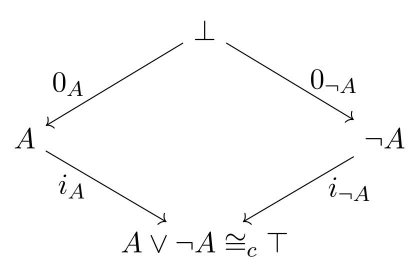
By Lemma 2.2.8(iii) there is a derivation K of $D = (\neg\neg A\to A)\to\neg\neg\forall_x A\to A$, and the required classical derivation, where the variable condition is easy to see that it is satisfied, is

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mid K \\ D\end{matrix} \qquad \begin{matrix}\mid N_c^* \\ \neg\neg A\to A\end{matrix}}{\neg\neg\forall_x A\to A}\,\to^- \qquad [u\colon \neg\neg\forall_x A]}{A}\,\to^-}{\forall_x A}\,\forall^+ x}{\neg\neg\forall_x A\to\forall_x A}\,\to^+ u \qquad \square$$

The extension-rule, the cut-rule and the deduction theorem for minimal logic (see section 1.12) are easily extended to classical logic.

2.3 Monos, epis and subobjects

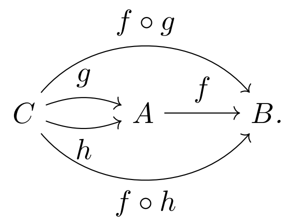
Categorically speaking, the obvious commutativity of the following diagram in $\mathbf{Form}_c$



expresses that in $\mathbf{Form}_c$ the objects A and $\neg A$ are *complemented subobjects* of $\top$.

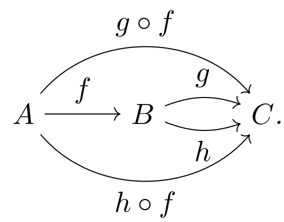
Definition 2.3.1. *Let $\boldsymbol{C}$ be a category and $f\colon A \to B$ in C_1. The arrow f is called a monic arrow, or a mono(morphism), and we write $f\colon A \hookrightarrow B$, if*

$$\forall_{C \in C_0} \forall_{g,h \in C_1(C,A)} \big(f \circ g = f \circ h \Rightarrow g = h\big)$$

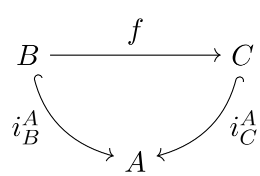


The arrow f is called an epi, or an epi(morphism), and we write $f\colon A \twoheadrightarrow B$, if

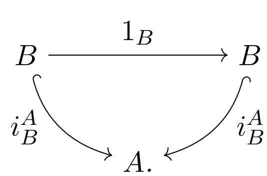
$$\forall_{C \in C_0} \forall_{g,h \in C_1(B,C)} \big(g \circ f = h \circ f \Rightarrow g = h\big)$$



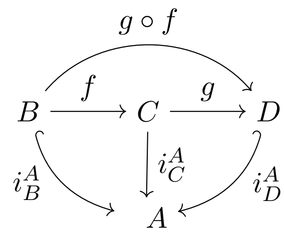
If $A \in C_0$, a subobject of A in $\boldsymbol{C}$ is a pair (B, i_B^A), where $B \in C_0$ and $i_B^A\colon B \hookrightarrow A$. The category $\mathrm{Sub}_{\boldsymbol{C}}(A)$ of subobjects of A in $\boldsymbol{C}$ has objects the subobjects of A in $\boldsymbol{C}$ and an arrow $f\colon (B, i_B^A) \to (C, i_C^A)$ is an arrow $f\colon B \to C$ in C_1 such that the following diagram commutes:



If (B, i_B^A) is an object in $\mathrm{Sub}_{\boldsymbol{C}}(A)$, its unit arrow is the unit 1_B in C_1, since the following diagram is trivially commutative



If $f\colon (B, i_B^A) \to (C, i_C^A)$ and $g\colon (C, i_C^A) \to (D, i_D^A)$ in $\mathrm{Sub}_{\boldsymbol{C}}(A)$, their composition in $\mathrm{Sub}_{\boldsymbol{C}}(A)$ is the composition $g \circ f$ in $\boldsymbol{C}$, as the commutativity of the following inner diagrams implies the commutativity of the following outer diagram



$$i^A_D \circ (g \circ f) = (i^A_D \circ g) \circ f = i^A_C \circ f = i^A_B.$$

Notice that in the above definition of the abstract injectivity (surjectivity) of arrows is expressed without reference to the membership relation $\in$ of sets. Moreover, the notion of a subobject is the abstract, categorical version of the notion of subset, and the category of subobjects $\mathrm{Sub}_{\boldsymbol{C}}(A)$ of A in $\boldsymbol{C}$ is the abstract, categorical version of the set of subsets of a set.

Proposition 2.3.2. *Let $\boldsymbol{C}$ be a category, $A \in C_0$ and $f \in C_1$.*

(i) *If f is an iso, then f is a mono and an epi.*

(ii) *Every arrow in **Form** (or in **Form**$_i$, **Form**$_c$) is both a mono and an epi.*

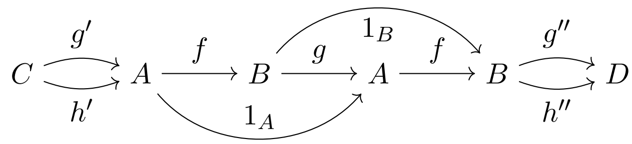
(iii) *The converse to* (i) *does not hold, in general.*

(iv) *If $f\colon (B, i^A_B) \to (C, i^A_C)$ in $\mathrm{Sub}_{\boldsymbol{C}}(A)$, then f is a mono.*

(v) *There is at most one arrow $f\colon (B, i^A_B) \to (C, i^A_C)$ in $\mathrm{Sub}_{\boldsymbol{C}}(A)$ i.e., $\mathrm{Sub}_{\boldsymbol{C}}(A)$ is thin.*

(vi) *In the category of sets **Set** a function $f\colon A \to B$ is a mono if and only if f is an injection, and f is an epi if and only if f is a surjection.*

Proof. (i) Let $g\colon B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. If $g', h'\colon C \to A$, such that $f \circ g' = f \circ h'$, then

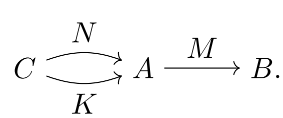


$$\begin{aligned} g \circ (f \circ g') = g \circ (f \circ h') &\Rightarrow (g \circ f) \circ g' = (g \circ f) \circ h' \\ &\Rightarrow 1_A \circ g' = 1_A \circ h' \\ &\Rightarrow g' = h'. \end{aligned}$$

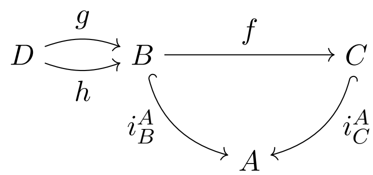
If $g'', h''\colon B \to C$, such that $g'' \circ f = h'' \circ f$, then

$$\begin{aligned} (g'' \circ f) \circ g = (h'' \circ f) \circ g &\Rightarrow g'' \circ (f \circ g) = h'' \circ (f \circ g) \\ &\Rightarrow g'' \circ 1_B = h'' \circ 1_B \\ &\Rightarrow g'' = h''. \end{aligned}$$

(ii) If $M\colon A \to B$ and $K, N\colon C \to A$ in **Form** such that $M \circ N = M \circ K$

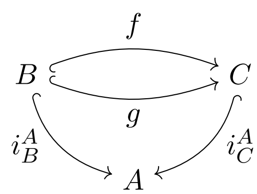


the equality $N = K$ follows from the thinness of **Form**. Similarly, M is an epi.
(iii) By (ii) all arrows in the category **Form** are monos and epis, but not all arrows in **Form** are (going to be) isos (can we show that now?).
(iv) Let $g, h\colon D \to B$ such that $f \circ g = f \circ h$. Then, since i^A_B is a mono, we get



$$\begin{aligned} i^A_C \circ (f \circ g) = i^C_A \circ (f \circ h) &\Rightarrow (i^A_C \circ f) \circ g = (i^A_C \circ f) \circ h \\ &\Rightarrow i^A_B \circ g = i^A_B \circ h \\ &\Rightarrow g = h. \end{aligned}$$

(v) Let $f, g\colon B \to C$ such that the following two diagrams formed by the arrows i^A_B, i^A_C



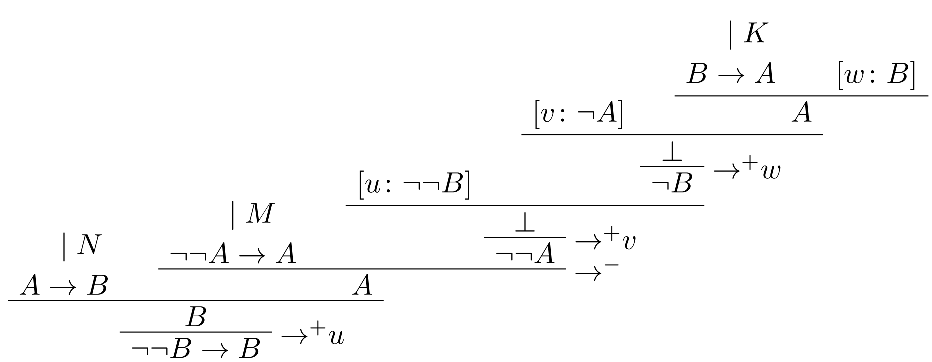
commute. Then the third diagram, formed by the arrows f, g, also commutes, since $i^A_B = i^A_C \circ g = i^A_C \circ f \Rightarrow g = f$. Case (vi) is an exercise. ☐

2.4 The groupoid category of formulas

The rule $A \mapsto (\neg\neg A \to A)$ does not define an endofunctor on **Form** (or on $\mathbf{Form}_i$, or on $\mathbf{Form}_c$). If $M\colon A \to B$, in order to get a derivation $M'\colon (\neg\neg A \to A) \to (\neg\neg B \to B)$ one also needs, in general, a derivation $N\colon B \to A$. The situation is similar for the rule $A \mapsto A \vee \neg A$.

Proposition 2.4.1. *Let* $A, B \in \texttt{Form}$ *such that* $\vdash A \leftrightarrow B$. *Then* $\vdash (\neg\neg A \to A) \Leftrightarrow \vdash (\neg\neg B \to B)$, *and* $\vdash (A \vee \neg A) \Leftrightarrow \vdash (B \vee \neg B)$.

Proof. We prove only the first equivalence, while the second is an exercise. If M is a derivation of $\neg\neg A \to A$, N a derivation of $A \to B$, and K a derivation of $B \to A$, then the following is a derivation of $\neg\neg B \to B$:



The converse implication follows similarly. □

The rule $A \mapsto (\neg\neg A \to A)$ defines an endofunctor on the various categories of formulas, if we consider an arrow $A \to B$ to be a derivation of the equivalence $A \leftrightarrow B$.

Definition 2.4.2. *The groupoid category* $\boldsymbol{Form}^{\mathrm{grp}}$ *of formulas has objects the formulas and an arrow* $A \to B$ *is a pair* $(M : A \to B, N : B \to A)$ *i.e.,* (M, N) *is a derivation of* $A \leftrightarrow B$*, or of* $A \cong B$*. In this case we write* $(M, N) : A \to B$*. If* $(M', N') : B \to C$*, their composition* $(M', N') \circ (M, N) : A \to C$ *is the pair* $(M' \circ M, N \circ N')$

$$A \underset{N}{\overset{M}{\rightleftarrows}} B \underset{N'}{\overset{M'}{\rightleftarrows}} C.$$

Moreover, $1^{\mathrm{grp}}_A = (1_A, 1_A)$*, and* $(M, N) = (K, L) \Leftrightarrow M = K \ \&\ N = L$*. Similarly we define the groupoid categories* $\boldsymbol{Form}^{\mathrm{grp}}_i$ *and* $\boldsymbol{Form}^{\mathrm{grp}}_c$*.*

It is immediate to show that $\mathbf{Form}^{\mathrm{grp}}$ is a (small) category. Moreover, $\mathbf{Form}^{\mathrm{grp}}$ is thin; if $(M, N), (M', N') : A \to B$, then $M, M' : A \to B$ and $N, N' : B \to A$, hence by the thinness of **Form** we have that $M = M'$ and $N = N'$, and then $(M, N) = (M', N')$. Notice that in $\mathbf{Form}^{\mathrm{grp}}$ the object $\top$ is no longer a terminal object, since if it was there would be an arrow $(M, N) : \bot \to \top$, hence a derivation $N : \top \to \bot$. For the same reason in $\mathbf{Form}^{\mathrm{grp}}_i$ the object $\bot$ is no longer an initial object. The above construction of the groupoid category $\mathbf{Form}^{\mathrm{grp}}$ from the preorder category **Form** can be generalised to an arbitrary preorder category.

Definition 2.4.3. *A small category* $\boldsymbol{C}$ *is a groupoid, if every arrow in* C_1 *is an isomorphism.*

Corollary 2.4.4. (i) *The groupoid category* $\boldsymbol{Form}^{\mathrm{grp}}$ *is a groupoid.*
(ii) *If* $A, B \in \mathtt{Form}$*, then* $A \cong B \Leftrightarrow A \cong^{\mathrm{grp}} B$*.*
(iii) *If* $F : \boldsymbol{Form} \to \boldsymbol{Form}$*, the induced endofunctor* $F^{\mathrm{grp}} : \boldsymbol{Form}^{\mathrm{grp}} \to \boldsymbol{Form}^{\mathrm{grp}}$ *is defined by*

$$F^{\mathrm{grp}}_0(A) = F_0(A),$$

$$F^{\mathrm{grp}}_1\big(M : A \to B, N : B \to A\big) : F_0(A) \to F_0(B),$$

$$F^{\mathrm{grp}}_1(M, N) = \big(F_1(M) : F_0(A) \to F_0(B), F_1(N) : F_0(B) \to F_0(A)\big),$$

for every $A, B \in \mathtt{Form}$ *and every arrow* $(M, N) : A \to B$ *in* $\boldsymbol{Form}^{\mathrm{grp}}$*.*

Proof. (i) Clearly, $\mathbf{Form}^{\mathrm{grp}}$ is small (see Definition 1.17.1). If $(M, N) : A \to B$, then $(N, M) : B \to A$, and hence $(N, M) \circ (M, N) = (N \circ M, N \circ M) = (1_A, 1_A) = 1^{\mathrm{grp}}_A$

$$A \underset{N}{\overset{M}{\rightleftarrows}} B \underset{M}{\overset{N}{\rightleftarrows}} A \underset{N}{\overset{M}{\rightleftarrows}} B.$$

Similarly, $(M, N) \circ (N, M) = (M \circ N, M \circ N) = (1_B, 1_B) = 1^{\mathrm{grp}}_B$.
(ii) If $A \cong B$ i.e., if there are $M : A \to B$ and $N : B \to A$, then $(M, N) : A \to B$ in $\mathbf{Form}^{\mathrm{grp}}$, and by (i) $A \cong^{\mathrm{grp}} B$. If $A \cong^{\mathrm{grp}} B$, there is $(M, N) : A \to B$ in $\mathbf{Form}^{\mathrm{grp}}$ i.e., $A \cong B$.
(iii) By definition of F^{grp} we get

$$F^{\mathrm{grp}}_1\big(1^{\mathrm{grp}}_A\big) = F^{\mathrm{grp}}_1(1_A, 1_A) = (F_1(1_A), F_1(1_A)) = (1_{F_0(A)}, 1_{F_0(A)}) = 1^{\mathrm{grp}}_{F_0(A)},$$

$$\begin{aligned}F_1^{\text{grp}}\big((M',N')\circ(M,N)\big) &= F_1^{\text{grp}}(M'\circ M, N\circ N')\\ &= \big(F_1(M'\circ M), F_1(N\circ N')\big)\\ &= \big(F_1(M')\circ F_1(M), F_1(N)\circ F_1(N')\big)\\ &= \big(F_1(M'), F_1(N')\big)\circ\big(F_1(M), F_1(N)\big)\\ &= F_1^{\text{grp}}(M',N')\circ F_1^{\text{grp}}(M,N).\end{aligned}$$ □

Corollary 2.4.5. *Let* DNE: $\boldsymbol{Form}^{\text{grp}} \to \boldsymbol{Form}^{\text{grp}}$ *be defined by*

$$\text{DNE}_0(A) = \neg\neg A \to A,$$

$$\text{DNE}_1\big(M\colon A \to B, N\colon B \to A\big) = (M', N'),$$

$$M'\colon (\neg\neg A \to A) \to (\neg\neg B \to B) \ \ \& \ \ N'\colon (\neg\neg B \to B) \to (\neg\neg A \to A),$$

where the derivations M' and N' are determined in Proposition 2.4.1.
(i) DNE *is an endofunctor on* $\boldsymbol{Form}^{\text{grp}}$.
(ii) *If* $A, B \in \texttt{Form}$, *then* $\text{DNE}_0(A \wedge B) \geq \text{DNE}_0(A) \wedge \text{DNE}_0(B)$.

Proof. Exercise. □

Corollary 2.4.6. *Let* PEM: $\boldsymbol{Form}^{\text{grp}} \to \boldsymbol{Form}^{\text{grp}}$ *be defined by*

$$\text{PEM}_0(A) = A \vee \neg A,$$

$$\text{PEM}_1\big(M\colon A \to B, N\colon B \to A\big) = (M', N'),$$

$$M'\colon (A \vee \neg A) \to (B \vee \neg B) \ \ \& \ \ N'\colon (B \vee \neg B) \to (A \vee \neg A),$$

where the derivations M' and N' are determined in Proposition 2.4.1.
(i) PEM *is an endofunctor on* $\boldsymbol{Form}^{\text{grp}}$.
(ii) *If* $A, B \in \texttt{Form}$, *then* $\text{PEM}_0(A \vee B) \leq \text{PEM}_0(A) \vee \text{PEM}_0(B)$.
(ii) *If* $A, B \in \texttt{Form}$, *then* $\text{PEM}_0(A) \wedge \text{PEM}_0(B) \leq \text{PEM}_0(A \wedge B)$.

Proof. Exercise. □

2.5 The negative fragment

The question answered in this section is the following: are there formulas A, other than $\bot$, for which it is possible to show that $\text{DNE}_A = \neg\neg A \to A$ is derivable in minimal logic?

Definition 2.5.1. *The negative formulas* $\texttt{Form}^-$ *of* $\texttt{Form}$, *or the negative fragment of* $\texttt{Form}$, *is defined by the following inductive rules:*

$$\frac{}{\bot \in \texttt{Form}^-}, \quad \frac{P \in \texttt{Prime}}{P \to \bot \in \texttt{Form}^-}, \quad \frac{A, B \in \texttt{Form}^-}{(A \circ B) \in \texttt{Form}^-}, \quad \frac{A \in \texttt{Form}^-, \quad x \in \texttt{Var}}{\forall_x A \in \texttt{Form}^-},$$

where $\circ \in \{\to, \wedge\}$. To the definition of $\texttt{Form}^-$ *corresponds the obvious induction principle. The negative fragment* $\boldsymbol{Form}^-$ *of* $\boldsymbol{Form}$ *is the corresponding full subcategory of negative formulas. The negative fragments* $\boldsymbol{Form}_i^-$ *and* $\boldsymbol{Form}_c^-$ *are defined similarly.*

It is immediate to show inductively that $\neg A \in {}^{-}$, if $A \in {}^{-}$.

Proposition 2.5.2. (i) $\forall_{A \in \mathtt{Form}^-}\big(A \in \mathtt{Form}^*\big)$.
(ii) Let $R \in \mathbb{R}^{(n)}$. If $n > 1$ and $t_1, \ldots, t_n \in \mathtt{Term}$, then $R(t_1, \ldots, t_n) \in \mathtt{Form}^* \setminus \mathtt{Form}^-$. If $n = 0$ and $R \neq \bot$, then $R \in \mathtt{Form}^* \setminus \mathtt{Form}^-$.

Proof. (i) By induction on $\mathtt{Form}$ (exercise). (ii) Since $R \in \mathrm{Prim}$, we get $R \in {}^*$. If $R \in {}^-$, then R is either $\bot$, or of the form $P \to \bot$, for some $P \in \mathrm{Prim}$, or of the form $A \circ B$, or of the form $\forall_x A$, for some $A, B \in \mathtt{Form}^-$. In all these cases we get a contradiction. ☐

Proposition 2.5.3. $\forall_{A \in \mathtt{Form}^-}\big(\vdash \neg\neg A \to A\big)$.

Proof. By induction on $\mathtt{Form}^-$. If $A = \bot$, we use $\vdash \neg\neg\bot \to \bot$. If $A = \neg R\vec{t}$ with R distinct from $\bot$, we must show $\neg\neg\neg R\vec{t} \to \neg R\vec{t}$, which is a special case of $\vdash \neg\neg\neg B \to \neg B$, Proposition 1.11.1(iii). Next we suppose that $\vdash \neg\neg A \to A$, $\vdash \neg\neg B \to B$ and we show $\vdash \neg\neg(A \to B) \to (A \to B)$. If $C = (\neg\neg B \to B) \to \neg\neg(A \to B) \to A \to B$, we use Lemma 2.2.8(ii) as follows:

$$\dfrac{\begin{array}{c} \mid M \\ C \end{array} \qquad \begin{array}{c} \mid N \\ \neg\neg B \to B \end{array}}{\neg\neg(A \to B) \to A \to B}$$

For the derivation of $\vdash \neg\neg(A \wedge B) \to (A \wedge B)$ we use Lemma 2.2.8(i) in a similar manner. If

$$D = (\neg\neg A \to A) \to \neg\neg\forall_x A \to A,$$

for the derivation of $\vdash \neg\neg\forall_x A \to \forall_x A$ we use Lemma 2.2.8(iii) as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c} \mid M \\ D \end{array} \qquad \begin{array}{c} \mid K \\ \neg\neg A \to A \end{array}}{\neg\neg\forall_x A \to A} \qquad u\colon \neg\neg\forall_x A}{A}}{\forall_x A}\,\forall^+ x}{\neg\neg\forall_x A \to \forall_x A}\,\to^+ u$$

The variable condition is trivially satisfied in the previous use of the rule $\forall^+ x$. ☐

2.6 Weak disjunction and weak existence

One reason for restricting the classical derivation $\vdash_c$ to the weak classical derivation $\vdash_c^*$ is that one can replace existential formulas and disjunctions with weak existential formulas and weak disjunctions, respectively. In this way Theorem 2.2.10 is "enough" for our needs, as through it we get the double-negation-elimination of $\mathtt{Form}^*$ i.e., of all formulas without $\exists$ and $\vee$. Here we distinguish between two kinds of "exists" and two kinds of "or": the "weak, or classical ones, and the "strong" or non-classical ones, with constructive content. In the present context both kinds occur together and hence we must mark the distinction; we do so by writing a tilde above the weak disjunction and existence symbols thus $\tilde{\exists}, \tilde{\vee}$.

Definition 2.6.1. *If $A, B \in \mathtt{Form}$, let*

$$A \mathbin{\tilde{\vee}} B = \neg A \to \neg B \to \bot \quad \& \quad \tilde{\exists}_x A = \neg\forall_x \neg A.$$

Proposition 2.6.2. *Let* $A, B \in \texttt{Form}$.
(i) *If* $A, B \in \texttt{Form}^*$, *then* $A \tilde{\vee} B \in \texttt{Form}^*$ *and* $\tilde{\exists}_x A \in \texttt{Form}^*$.
(ii) *If* $A, B \in \texttt{Form}^-$, *then* $A \tilde{\vee} B \in \texttt{Form}^-$ *and* $\tilde{\exists}_x A \in \texttt{Form}^-$.
(iii) $\vdash A \vee B \to A \tilde{\vee} B$.
(iv) $\vdash \exists_x A \to \tilde{\exists}_x A$.
(v) $\vdash A \tilde{\vee} B \leftrightarrow \neg\neg(A \vee B)$.
(vi) $\vdash \tilde{\exists}_x A \leftrightarrow \neg\neg\exists_x A$.
(vii) $\vdash A \tilde{\vee} B \leftrightarrow \neg(\neg A \wedge \neg B)$.
(viii) $\vdash \neg\neg(A \tilde{\vee} B) \to A \tilde{\vee} B$.
(ix) $\vdash \neg\neg(\tilde{\exists}_x A) \to \tilde{\exists}_x A$.

Proof. Exercise. For the proof of (x) and (xi) we only mention the following. By (i) Theorem 2.2.10 implies the classical derivability of the double-negation-elimination of $A \tilde{\vee} B$ and $\tilde{\exists}_x A$, if $A, B \in \texttt{Form}^*$. By (ii) Proposition 2.5.3 implies the derivability of the double-negation-elimination of $A \tilde{\vee} B$ and $\tilde{\exists}_x A$, if $A, B \in \texttt{Form}^-$. Using Brouwer's double-negation-elimination of a negated formula (Proposition 1.11.1(iii)), we derive these eliminations in minimal logic. $\square$

Proposition 2.6.3. *The following formulas are derivable.*

$$(i) \qquad (\tilde{\exists}_x A \to B) \to \forall_x (A \to B), \quad \text{if } x \notin \mathrm{FV}(B).$$

$$(ii) \qquad (\neg\neg B \to B) \to \forall_x (A \to B) \to \tilde{\exists}_x A \to B, \quad \text{if } x \notin \mathrm{FV}(B).$$

$$(iii) \qquad (\bot \to B(c)) \to (A \to \tilde{\exists}_x B) \to \tilde{\exists}_x (A \to B), \quad \text{if } x \notin \mathrm{FV}(A).$$

$$(iv) \qquad \tilde{\exists}_x (A \to B) \to A \to \tilde{\exists}_x B, \quad \text{if } x \notin \mathrm{FV}(A).$$

Proof. The following is a derivation of (i):

$$\dfrac{\dfrac{\dfrac{\tilde{\exists}_x A \to B \qquad \dfrac{\dfrac{\dfrac{[u_1 \colon \forall_x \neg A] \qquad x}{\neg A} \qquad [w \colon A]}{\bot}}{\neg\forall_x \neg A} \to^+ u_1}{B}}{A \to B} \to^+ w}{\forall_x (A \to B)} \forall^+ x$$

The following is a derivation of (ii) without the last $\to^+$-rules:

$$\dfrac{\neg\neg B \to B \qquad \dfrac{\dfrac{\neg\forall_x \neg A \qquad \dfrac{\dfrac{\dfrac{[u_2 \colon \neg B] \qquad \dfrac{\dfrac{\forall_x (A \to B) \qquad x}{A \to B} \qquad [u_1 \colon A]}{B}}{\bot}}{\neg A} \to^+ u_1}{\forall_x \neg A}}{\bot}}{\neg\neg B} \to^+ u_2}{B}$$

The following is a derivation of (iii) without the last $\to^+$-rules:

$$\dfrac{\dfrac{\forall_x\neg(A\to B)\quad c}{\neg(A\to B(c))} \qquad \dfrac{\dfrac{\bot\to B(c) \qquad \dfrac{\dfrac{A\to\tilde{\exists}_x B \quad u_2\colon A}{\tilde{\exists}_x B} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\forall_x\neg(A\to B)\quad x}{\neg(A\to B)} \qquad \dfrac{[u_1\colon B]}{A\to B}}{\bot}}{\neg B}\,\to^+ u_1}{\forall_x\neg B}}{\bot}}{B(c)}}{A\to B(c)}\,\to^+ u_2}{\bot}$$

Note that above we used the fact that if $x\notin \mathrm{FV}(A)$, then $A(c)=A$ (Proposition 1.6.4). The following is a derivation of (iv) without the last $\to^+$-rules:

$$\dfrac{\tilde{\exists}_x(A\to B) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\forall_x\neg B\quad x}{\neg B} \qquad \dfrac{[u_1\colon A\to B]\quad A}{B}}{\bot}}{\neg(A\to B)}\,\to^+ u_1}{\forall_x\neg(A\to B)}\,\forall^+ x}{\bot}\,\to^-$$

□

Proposition 2.6.4. *The following formulas are derivable.*

(i) $$\forall_x(\bot\to A)\to(\forall_x A\to B)\to\forall_x\neg(A\to B)\to\neg\neg A.$$

(ii) $$\forall_x(\neg\neg A\to A)\to(\forall_x A\to B)\to\tilde{\exists}_x(A\to B)\quad \textit{if } x\notin\mathrm{FV}(B).$$

Proof. If Ax, Ay stand for $A(x), A(y)$, respectively, we get the following derivation M of (i):

$$\dfrac{\dfrac{\dfrac{\forall_x\neg(Ax\to B)\quad x}{\neg(Ax\to B)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\forall_x Ax\to B \qquad \dfrac{\dfrac{\forall_y Ay\quad x}{Ax}}{\forall_x Ax}\,\forall^+ x}{B}\,\to^-}{\forall_y Ay\to B} \qquad \dfrac{\dfrac{\dfrac{\forall_x(\bot\to Ax)\quad y}{\bot\to Ay} \qquad \dfrac{u_1\colon\neg Ax\quad u_2\colon Ax}{\bot}}{Ay}}{\forall_y Ay}\,\forall^+ y}{B}}{Ax\to B}\,\to^+ u_2}{\bot}}{\neg\neg Ax}\,\to^+ u_1,$$

where the last $\to^+$-rules are not included. Using this derivation M we obtain

$$\dfrac{\dfrac{\forall_x\neg(Ax\to B)\quad x}{\neg(Ax\to B)} \qquad \dfrac{\dfrac{\forall_x Ax\to B \qquad \dfrac{\dfrac{\dfrac{\forall_x(\neg\neg Ax\to Ax)\quad x}{\neg\neg Ax\to Ax} \qquad \begin{array}{c}\mid M\\ \neg\neg Ax\end{array}}{Ax}}{\forall_x Ax}}{B}}{Ax\to B}}{\bot}\,\to^-.$$

Note that the assumption $\forall_x(\neg\neg A\to A)$ in (ii) is used to derive the assumption $\forall_x(\bot\to A)$ in (i), since $\vdash(\neg\neg A\to A)\to\bot\to A$ (see the proof of Proposition 2.2.3). □

Corollary 2.6.5. *If $R \in \mathtt{Rel}^{(1)}$, then $\vdash_c^* \tilde{\exists}_x(R(x) \to \forall_x R(x))$.*

Proof. let $A = R(x)$ and $B = \forall_x R(x)$ in Proposition 2.6.4(ii). □

The formula $\tilde{\exists}_x(R(x) \to \forall_x R(x))$ is known as the *drinker formula*, and can be read as "in every non-empty bar there is a person such that, if this person drinks, then everybody drinks". The next proposition on weak disjunction is similar to Proposition 2.6.3.

Proposition 2.6.6. *The following formulas are derivable.*

$$\begin{aligned}
& && (A \tilde{\vee} B \to C) \to (A \to C) \wedge (B \to C), \\
&(\neg\neg C \to C) \to && (A \to C) \to (B \to C) \to A \tilde{\vee} B \to C, \\
&(\bot \to B) \to && (A \to B \tilde{\vee} C) \to (A \to B) \tilde{\vee} (A \to C), \\
& && (A \to B) \tilde{\vee} (A \to C) \to A \to B \tilde{\vee} C, \\
&(\neg\neg C \to C) \to && (A \to C) \tilde{\vee} (B \to C) \to A \to B \to C, \\
&(\bot \to C) \to && (A \to B \to C) \to (A \to C) \tilde{\vee} (B \to C).
\end{aligned}$$

Proof. The derivations of the second and the final formula are

$$\dfrac{\neg\neg C \to C \qquad \dfrac{\dfrac{\dfrac{\neg A \to \neg B \to \bot \qquad \dfrac{\dfrac{u_1\colon \neg C \qquad \dfrac{A \to C \quad u_2\colon A}{C}}{\bot}}{\neg A}\to^+ u_2}{\neg B \to \bot} \qquad \dfrac{\dfrac{u_1\colon \neg C \qquad \dfrac{B \to C \quad u_3\colon B}{C}}{\bot}}{\neg B}\to^+ u_3}{\bot}}{\neg\neg C}\to^+ u_1}{C}$$

$$\dfrac{\neg(B \to C) \qquad \dfrac{\dfrac{\bot \to C \qquad \dfrac{\neg(A \to C) \qquad \dfrac{\dfrac{\dfrac{A \to B \to C \quad u_1\colon A}{B \to C} \qquad u_2\colon B}{C}}{A \to C}\to^+ u_1}{\bot}}{C}}{B \to C}\to^+ u_2}{\bot}\to^-$$

□

The weak disjunction and the weak existential quantifier satisfy the same axioms as the strong variants, if one restricts the conclusion of the elimination axioms to formulas in $\mathtt{Form}^*$.

Proposition 2.6.7. *The following formulas are derivable.*

$$\begin{aligned}
&\vdash A \to A \tilde{\vee} B, \\
&\vdash B \to A \tilde{\vee} B, \\
&\vdash_c^* A \tilde{\vee} B \to (A \to C) \to (B \to C) \to C \quad (C \in \mathtt{Form}^*), \\
&\vdash A \to \tilde{\exists}_x A, \\
&\vdash_c^* \tilde{\exists}_x A \to \forall_x (A \to B) \to B \quad (x \notin \mathrm{FV}(B),\ B \in \mathtt{Form}^*).
\end{aligned}$$

Proof. The derivations of the last formula is

$$\dfrac{\neg\neg B \to B \qquad \dfrac{\dfrac{\neg\forall_x\neg A \qquad \dfrac{\dfrac{\dfrac{u_1\colon \neg B \qquad \dfrac{\dfrac{\forall_x(A \to B) \quad x}{A \to B} \qquad u_2\colon A}{B}}{\bot}}{\neg A}\,\to^+ u_2}{\forall_x\neg A}}{\bot}}{\neg\neg B}\,\to^+ u_1}{B}\,\to^- .$$

□

2.7 Logical operations on functors

Next we generalise the composition of functors defined in Example 1.14.4.

Definition 2.7.1. *Let $\boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}$ be categories. If $F\colon \boldsymbol{C} \to \boldsymbol{D}$ and $G\colon \boldsymbol{D} \to \boldsymbol{E}$ are covariant (contravariant) functors their composition $G \circ F$ is the pair $(G_0 \circ F_0, G_1 \circ F_1)$.*

Proposition 2.7.2. *Let $\boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}$ be categories.*
(i) *If $F\colon \boldsymbol{C} \to \boldsymbol{D}$ and $G\colon \boldsymbol{D}^{\mathrm{op}} \to \boldsymbol{E}$, then $G \circ F\colon \boldsymbol{C}^{\mathrm{op}} \to \boldsymbol{E}$.*
(ii) *If $F\colon \boldsymbol{C}^{\mathrm{op}} \to \boldsymbol{D}$ and $G\colon \boldsymbol{D} \to \boldsymbol{E}$, then $G \circ F\colon \boldsymbol{C}^{\mathrm{op}} \to \boldsymbol{E}$.*
(iii) *If $F\colon \boldsymbol{C}^{\mathrm{op}} \to \boldsymbol{D}$ and $G\colon \boldsymbol{D}^{\mathrm{op}} \to \boldsymbol{E}$, then $G \circ F\colon \boldsymbol{C} \to \boldsymbol{E}$.*
(iii) *If $F\colon \boldsymbol{C} \to \boldsymbol{D}$ and $G\colon \boldsymbol{D} \to \boldsymbol{E}$, then $G \circ F\colon \boldsymbol{C} \to \boldsymbol{E}$.*

Proof. Exercise. □

Corollary 2.7.3. *Let $F\colon \boldsymbol{Form} \to \boldsymbol{Form}$, and $\neg\colon \boldsymbol{Form}^{\mathrm{op}} \to \boldsymbol{Form}$, defined in Propositions 2.1.9. We define the following endofunctors on $\boldsymbol{Form}$:*

$$\neg^n F = \begin{cases} F & , n = 0 \\ \neg \circ F & , n = 1 \\ \neg \circ \big(\neg^{n-1} F\big) & , n > 1. \end{cases}$$

(i) *$\neg^{2n+1} F$ is a contravariant endofunctor, for every $n \in \mathbb{N}$.*
(ii) *$\neg^{2n} F$ is a covariant endofunctor, for every $n \in \mathbb{N}$.*
(iii) *The endofunctor $\neg^{2n+1} F$ is isomorphic to $\neg F$ in $\mathtt{Fun}(\boldsymbol{Form}, \boldsymbol{Form})$, for every $n \in \mathbb{N}$.*
(iv) *If $F\colon \boldsymbol{Form}_c \to \boldsymbol{Form}_c$, then $\neg^{2n} F$ is isomorphic to F in $\mathtt{Fun}(\boldsymbol{Form}_c, \boldsymbol{Form}_c)$, for every $n \in \mathbb{N}$.*

Proof. It follows immediately from Propositions 2.1.9 and 2.7.2. □

Proposition 2.7.4. *If $F, G\colon \boldsymbol{Form} \to \boldsymbol{Form}$ and $B \in \mathtt{Form}$, the following are functors.*
(i) *$F \times G\colon \boldsymbol{Form} \to \boldsymbol{Form} \times \boldsymbol{Form}$, where*

$$(F \times G)_0(A) = \big(F_0(A), G_0(A)\big); \qquad A \in \mathtt{Form},$$

$$(F \times G)_1(M\colon A \to B)\colon \big(F_0(A), G_0(A)\big) \to \big(F_0(B), G_0(B)\big); \qquad M\colon A \to B,$$

$$(F \times G)_1(M) = \big(F_1(M) \colon F_0(A) \to F_0(B), G_1(M) \colon G_0(A) \to G_0(B)\big).$$

(ii) $F \wedge G \colon \textbf{\textit{Form}} \to \textbf{\textit{Form}}$, *where* $(F \wedge G)_0(A) = F_0(A) \wedge G_0(A)$, *for every* $A \in \mathtt{Form}$.

(iii) $F \vee G \colon \textbf{\textit{Form}} \to \textbf{\textit{Form}}$, *where* $(F \vee G)_0(A) = F_0(A) \wedge G_0(A)$, *for every* $A \in \mathtt{Form}$.

(iv) $B \to F \colon \textbf{\textit{Form}} \to \textbf{\textit{Form}}$, *where* $(B \to F)_0(A) = B \to F_0(A)$, *for every* $A \in \mathtt{Form}$.

(v) $\exists_x F \colon \textbf{\textit{Form}} \to \textbf{\textit{Form}}$, *where* $(\exists_x F)_0(A) = \exists_x F_0(A)$, *for every* $A \in \mathtt{Form}$.

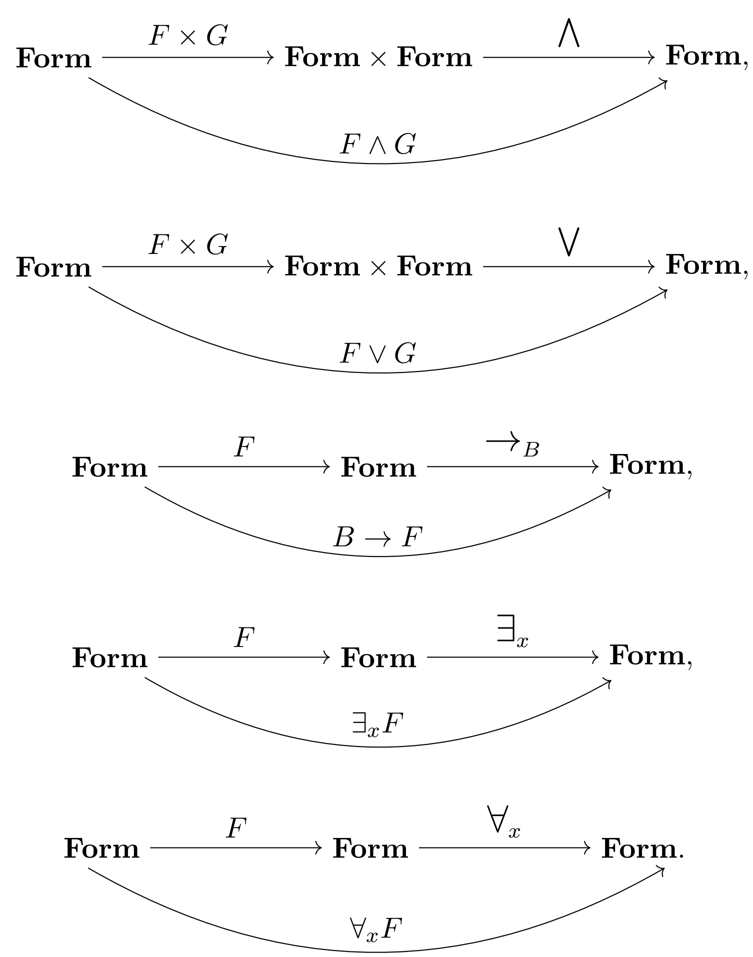
(vi) $\forall_x F \colon \textbf{\textit{Form}} \to \textbf{\textit{Form}}$, *where* $(\forall_x F)_0(A) = \forall_x F_0(A)$, *for every* $A \in \mathtt{Form}$.

Proof. (i) If $A, B, C \in \mathtt{Form}, M \colon A \to B$ and $N \colon B \to C$, then by the definition of the unit arrow and composition in the product category **Form** × **Form** we get

$$(F \times G)_1(1_A) = \big(F_1(1_A), G_1(1_A)\big) = \big(1_{F_0(A)}, 1_{G_0(A)}\big) = 1_{(F_0(A), G_0(A))} = 1_{(F \times G)_0(A)},$$

$$\begin{aligned}(F \times G)_1(N \circ M) &= \big(F_1(N \circ M), G_1(N \circ M)\big)\\ &= \big(F_1(N) \circ F_1(M), G_1(N) \circ G_(M)\big)\\ &= \big(F_1(N), G_1(N)\big) \circ \big(F_1(M), G_1(M)\big)\\ &= (F \times G)_1(N) \circ (F \times G)_1(M).\end{aligned}$$

(ii)-(vi) These are functors as composition of the functors in Definitions 1.14.8 and 1.14.9 with $F \times G$ or F, respectively, as it is shown in the following commutative diagrams



□

If $F, G \colon \mathbf{Form}^{\mathrm{op}} \to \mathbf{Form}$ are contravariant functors, all results in Proposition 2.7.4 are extended accordingly. If $F, G \colon \mathbf{Form} \times \mathbf{Form} \to \mathbf{Form}$, or more generally, $F, G \colon \mathbf{Form}^n \to \mathbf{Form}$, where $n > 1$, the corresponding functors $F \wedge G$ and $F \vee G$ are defined similarly.

Proposition 2.7.5. *Let $F \colon \boldsymbol{Form} \times \boldsymbol{Form} \to \boldsymbol{Form}$ a functor and let a function $G_0 \colon \mathtt{Form} \times \mathtt{Form} \to \mathtt{Form}$, such that $\vdash F_0(A, B) \leftrightarrow G_0(A, B)$, for every $A, B \in \mathtt{Form}$. Then G_0 generates a functor $G \colon \boldsymbol{Form} \times \boldsymbol{Form} \to \boldsymbol{Form}$.*

Proof. If $(M, N) \colon (A, B) \to (A', B')$ in $\mathbf{Form} \times \mathbf{Form}$, then we define $G_1(M, N) \colon G_0(A, B) \to G_0(A', B')$ the arrow $L_{A',B'} \circ F_1(M, N) \circ K_{A,B}$

$$\begin{array}{ccc} F_0(A,B) & \xrightarrow{\;F_1(M,N)\;} & F_0(A',B') \\ K_{A,B} \Big\uparrow & & \Big\downarrow L_{A',B'} \\ G_0(A,B) & \xrightarrow[\;G_1(M,N)\;]{} & G_0(A',B'), \end{array}$$

where $K_{A,B} \colon G_0(A, B) \to F_0(A, B)$ and $L_{A',B'} \colon F_0(A', B') \to G_0(A', B')$ are found by the hypotheses $\vdash F_0(A, B) \leftrightarrow G_0(A, B)$ and $\vdash F_0(A', B') \leftrightarrow G_0(A', B')$. □

A similar result holds, if $F \colon \mathbf{Form}^n \to \mathbf{Form}$ and $G_0 \colon \mathtt{Form}^n \to \mathtt{Form}$, for every $n > 0$. All results of this section are extended naturally to functors $F, G \colon \boldsymbol{C} \to \mathbf{Form}$ (in Proposition 2.7.4) and $F \colon \boldsymbol{C} \times \boldsymbol{D} \to \mathbf{Form}$ (in Proposition 2.7.5), where $\boldsymbol{C}$ and $\boldsymbol{D}$ are categories.

2.8 Functors on functors

To the functors on formulas associated to the logical symbols in Definition 1.14.8 correspond functors on endofunctors on **Form**. For simplicity we use the same symbols for them.

Definition 2.8.1. *Let the following functors:*

(i) $\bigwedge \colon \mathtt{Fun}(\boldsymbol{Form}, \boldsymbol{Form}) \times \mathtt{Fun}(\boldsymbol{Form}, \boldsymbol{Form}) \to \mathtt{Fun}(\boldsymbol{Form}, \boldsymbol{Form})$, *defined by*

$$\bigwedge_0 (F, G) = F \wedge G,$$

$$\bigwedge_1 \big((\eta, \tau) \colon (F, G) \to (F', G')\big) \colon F \wedge G \Rightarrow F' \wedge G',$$

$$\Big[\bigwedge_1 (\eta, \tau)\Big]_A = \bigwedge_1 (\eta_A, \tau_A) \colon F_0(A) \wedge G_0(A) \to {F_0}'(A) \wedge {G_0}'(A); \qquad A \in \mathtt{Form},$$

$\eta_A \colon F_0(A) \to {F_0}'(A), \tau_A \colon G_0(A) \to {G_0}'(A)$, and $\bigwedge_1(\eta_A, \tau_A)$ is defined in Definition 1.14.8(i).

(ii) $\bigvee \colon \mathtt{Fun}(\boldsymbol{Form}, \boldsymbol{Form}) \times \mathtt{Fun}(\boldsymbol{Form}, \boldsymbol{Form}) \to \mathtt{Fun}(\boldsymbol{Form}, \boldsymbol{Form})$, *defined by*

$$\bigvee_0 (F, G) = F \vee G,$$

$$\bigvee_1 \big((\eta, \tau) \colon (F, G) \to (F', G')\big) \colon F \vee G \Rightarrow F' \vee G',$$

$$\Big[\bigvee_1(\eta,\tau)\Big]_A = \bigvee_1(\eta_A,\tau_A)\colon F_0(A)\vee G_0(A)\to {F_0}'(A)\vee {G_0}'(A); \qquad A\in \mathtt{Form},$$

$\eta_A\colon F_0(A)\to {F_0}'(A), \tau_A\colon G_0(A)\to {G_0}'(A)$, and $\bigvee_1(\eta_A,\tau_A)$ is defined in Definition 1.14.8(ii).

(iii) *If $B\in\mathtt{Form}$, let $\longrightarrow_B\colon \mathtt{Fun}(\boldsymbol{Form},\boldsymbol{Form})\to\mathtt{Fun}(\boldsymbol{Form},\boldsymbol{Form})$, defined by*

$$(\longrightarrow_B)_0(F) = B\to F,$$

$$(\longrightarrow_B)_1\big(\eta\colon F\Rightarrow G)\big)\colon (B\to F)\Rightarrow(B\to G),$$

$$\big[(\longrightarrow_B)_1(\eta)\big]_A = (\longrightarrow_B)_1(\eta_A)\colon (B\to F_0(A))\to(B\to G_0(A)); \qquad A\in\mathtt{Form},$$

$\eta_A\colon F_0(A)\to G_0(A)$, and $(\longrightarrow_B)_1(\eta_A)$ is defined in Definition 1.14.9.

(iv) *$\forall_x\colon \mathtt{Fun}(\boldsymbol{Form},\boldsymbol{Form})\to\mathtt{Fun}(\boldsymbol{Form},\boldsymbol{Form})$, defined by*

$$\Big(\forall_x\Big)_0(F) = \forall_x F,$$

$$\Big(\forall_x\Big)_1\big(\eta\colon F\Rightarrow G)\big)\colon \forall_x F\Rightarrow\forall_x G,$$

$$\Big[\Big(\forall_x\Big)_1(\eta)\Big]_A = \Big(\forall_x\Big)_1(\eta_A)\colon \forall_x F_0(A)\to\forall_x G_0(A); \qquad A\in\mathtt{Form},$$

where $\eta_A\colon F_0(A)\to G_0(A)$, and $\big(\forall_x\big)_1(\eta_A)$ is defined in Definition 1.14.8(iv).

(iv) *$\exists_x\colon \mathtt{Fun}(\boldsymbol{Form},\boldsymbol{Form})\to\mathtt{Fun}(\boldsymbol{Form},\boldsymbol{Form})$, defined by*

$$\Big(\exists_x\Big)_0(F) = \exists_x F,$$

$$\Big(\exists_x\Big)_1\big(\eta\colon F\Rightarrow G)\big)\colon \exists_x F\Rightarrow\exists_x G,$$

$$\Big[\Big(\exists_x\Big)_1(\eta)\Big]_A = \Big(\exists_x\Big)_1(\eta_A)\colon \exists_x F_0(A)\to\exists_x G_0(A); \qquad A\in\mathtt{Form},$$

where $\eta_A\colon F_0(A)\to G_0(A)$, and $\big(\exists_x\big)_1(\eta_A)$ is defined in Definition 1.14.8(v).

Other functors on formulas, like $\bigwedge_B$ and ${}_B\bigwedge$, induce the corresponding functors on functors on formulas. The preorder on **Form** also induces a preorder on $\mathtt{Fun}(\mathbf{Form},\mathbf{Form})$.

Definition 2.8.2. *If $F,G\in\mathtt{Fun}(\boldsymbol{Form},\boldsymbol{Form})$ and $\eta,\tau\colon F\Rightarrow G$, let*

$$F\leq G\Leftrightarrow \forall_{A\in\mathtt{Form}}\big(F_0(A)\leq G_0(A)\big),$$

$$\eta=\tau\Leftrightarrow\forall_{A\in\mathtt{Form}}\big(\eta_A=\tau_A\big).$$

A witness μ of $F\leq G$, in symbols $\mu\colon F\leq G$, is a family $(\mu_A\colon F_0(A)\to G_0(A))_{A\in\mathtt{Form}}$.

If $\mu\colon F\leq G$, by the thinness of **Form** we get $\mu\colon F\Rightarrow G$, and if $\tau\colon F\Rightarrow G$, then $\tau\colon F\leq G$. By the definition of equality between natural transformations $F\Rightarrow G$ we have that the thinness of **Form** implies the thinness of $\mathtt{Fun}(\mathbf{Form},\mathbf{Form})$. Moreover, the adjunctions of sections 1.16 and 1.17 are extended to functors on functors on formulas.

Proposition 2.8.3. *If* $B \in \texttt{Form}$ *and* $F, G \in \texttt{Fun}(\boldsymbol{Form}, \boldsymbol{Form})$, *then*

$$_B\bigwedge \circ F \leq G \Leftrightarrow F \leq (\longrightarrow_B \circ\, G).$$

Proof. By Definition 2.8.2 we have that

$$\begin{aligned}
_B\bigwedge \circ F \leq G &\Leftrightarrow \forall_{A \in \texttt{Form}}\Big(\Big(_B\bigwedge \circ F\Big)_0(A) \leq G_0(A)\Big)\\
&\Leftrightarrow \forall_{A \in \texttt{Form}}\Big(_B\bigwedge\nolimits_0(F_0(A)) \leq G_0(A)\Big)\\
&\Leftrightarrow \forall_{A \in \texttt{Form}}\big(F_0(A) \wedge B \leq G_0(A)\big)\\
&\Leftrightarrow \forall_{A \in \texttt{Form}}\big(F_0(A) \leq (B \to G_0(A))\big)\\
&\Leftrightarrow \forall_{A \in \texttt{Form}}\big(F_0(A) \leq (\longrightarrow_B)_0(G_0(A))\big)\\
&\Leftrightarrow \forall_{A \in \texttt{Form}}\big(F_0(A) \leq (\longrightarrow_B \circ\, G)_0(A)\big)\\
&\Leftrightarrow F \leq (\longrightarrow_B \circ\, G).
\end{aligned}$$

□

Definition 2.8.4. *The functors* $\forall_x$ *and* $\exists_x$ *in Definition 2.8.1 can be seen as functors*

$$\forall_x, \exists_x \colon \texttt{Fun}(\boldsymbol{Form}, \boldsymbol{Form}) \to \texttt{Fun}(\boldsymbol{Form}, \boldsymbol{Form}_x),$$

as e.g., $(\exists_x F)_0(A) = (\exists_x \circ F)_0(A) = \exists_x F_0(A) \in \texttt{Form}_x$ *(see Example 1.17.6). Let the functor* $W_x \colon \texttt{Fun}(\boldsymbol{Form}, \boldsymbol{Form}_x) \to \texttt{Fun}(\boldsymbol{Form}, \boldsymbol{Form})$ *be defined by the identity rule* $F \mapsto F$, *for every* $F \in \texttt{Fun}(\boldsymbol{Form}, \boldsymbol{Form}_x)$.

Proposition 2.8.5. *The following adjunctions hold:* $\exists_x \dashv W_x$ *and* $W_x \dashv \forall_x$.

Proof. Exercise. □

2.9 Functors associated to weak "or" and weak "exists"

Definition 2.9.1. (i) *Let the functor* $\widetilde{\bigvee} \colon \boldsymbol{Form} \times \boldsymbol{Form} \to \boldsymbol{Form}$, *defined by*

$$\widetilde{\bigvee}_0(A, B) = A \mathbin{\tilde{\vee}} B; \qquad A, B \in \texttt{Form},$$

$$\widetilde{\bigvee}_1\big(M \colon A \to A', N \colon B \to B'\big) \colon A \mathbin{\tilde{\vee}} B \to A' \mathbin{\tilde{\vee}} B',$$

is the following derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{u\colon \neg A \to \neg B \to \bot \qquad \dfrac{\dfrac{\begin{array}{c}[w\colon A]\\ \mid M\\ A'\end{array} \qquad [u'\colon \neg A']}{\bot}\to^-}{A \to \bot}\to^+ w}{\neg B \to \bot}\to^+ \qquad \dfrac{\dfrac{\begin{array}{c}[v\colon B]\\ \mid N\\ B'\end{array} \qquad [v'\colon \neg B']}{\bot}\to^-}{B \to \bot}\to^+ v}{\bot}\to^+}{\neg B \to \bot}\to^+ v'}{\neg A' \to \neg B' \to \bot}\to^+ u'.$$

(ii) *Let* $\widetilde{\mathrm{PEM}} \colon \boldsymbol{Form} \to \boldsymbol{Form}$, *defined by* $\widetilde{\mathrm{PEM}}_0(A) = A \mathbin{\tilde{\vee}} \neg A$, *for every* $A \in \texttt{Form}$.

We can show that $\widetilde{\bigvee}$ is a functor using also Propositions 2.7.2 and 2.7.5 (exercise). The fact that the rule $\widetilde{\mathrm{PEM}}_A = A \mathbin{\tilde{\vee}} \neg A$ defines an endofunctor on **Form** follows from the trivial fact that $\vdash A \mathbin{\tilde{\vee}} \neg A$, for every $A \in \texttt{Form}$.

Definition 2.9.2. *Let the functor* $\tilde{\exists}_x \colon \boldsymbol{Form} \to \boldsymbol{Form}$, *defined by*

$$\Big(\tilde{\exists}\Big)_0(A) = \tilde{\exists}_x A; \quad \& \quad \Big(\tilde{\exists}\Big)_1\big(M \colon A \to B\big) \colon \tilde{\exists}_x A \to \tilde{\exists}_x B,$$

is the following derivation

$$\begin{prooftree}
\AxiomC{$u \colon \neg\forall_x\neg A$}
\AxiomC{$[w \colon A]$}
\noLine
\UnaryInfC{$\mid M$}
\noLine
\UnaryInfC{B}
\RightLabel{$\to^+ w$}
\UnaryInfC{$A \to B$}
\UnaryInfC{$\neg B \to \neg A$}
\AxiomC{$[v \colon \forall_x\neg B]$}
\AxiomC{x}
\RightLabel{$\forall^-$}
\BinaryInfC{$\neg B$}
\RightLabel{$\to^-$}
\BinaryInfC{$\neg A$}
\RightLabel{$\forall^+ x$}
\UnaryInfC{$\forall_x\neg A$}
\RightLabel{$\to^-$}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ v$}
\UnaryInfC{$\neg\forall_x\neg B$}
\end{prooftree}$$

The variable condition in the above derivation of $\neg A$ is satisfied, as the only open assumption is the formula $\forall_x\neg B$. In the derivation tree above we omit to mention the derivation $\vdash (A \to B) \to (\neg B \to \neg A)$ and the use of $(\to^-)$ in order to derive $\neg B \to \neg A$. We can show that $\tilde{\exists}_x$ is a functor using also Proposition 2.7.2 (exercise). Next follows the weak analogue to Theorem 1.17.8.

Proposition 2.9.3. *The functors* $\tilde{\exists}_x \colon \boldsymbol{Form} \to \boldsymbol{Form}$ *can be written as a functor of the form* $\tilde{\exists}_x \colon \boldsymbol{Form} \to \boldsymbol{Form}_x$, *where* $\boldsymbol{Form}_x$ *is the subcategory of formulas* A *with* $x \notin \mathrm{FV}(A)$. *Let again the functor* $W_x \colon \boldsymbol{Form}_x \to \boldsymbol{Form}$, *defined by* $\big(W_x\big)_0(A) = A$, *for every* $A \in \texttt{Form}$. *If* $A \in \texttt{Form}_x$ *and* $C \in \texttt{Form}$, *the following hold:*

(i) $\Big(\tilde{\exists}_x\Big)_0(C) \leq A \Rightarrow C \leq \big(W_x\big)_0(A)$.

(ii) *If* $\vdash \neg\neg A \to A$, *then* $C \leq \big(W_x\big)_0(A) \Rightarrow \Big(\tilde{\exists}_x\Big)_0(C) \leq A$.

Proof. (i) If $N \colon \neg\forall_x\neg C \to A$, and since there is a derivation $M \colon \exists_x C \to \tilde{\exists}_x C$, we get the the composition $N \circ M \colon \exists_x C \to A$, hence by the proof of Theorem 1.17.8(i) there is $K \colon C \to A$. (ii) If $K \colon C \to A$, we get the following derivation of $(\neg\forall_x\neg C) \to \neg\neg A$:

$$\begin{prooftree}
\AxiomC{$u \colon \neg\forall_x\neg C$}
\AxiomC{$[w \colon C]$}
\noLine
\UnaryInfC{$\mid K$}
\noLine
\UnaryInfC{A}
\RightLabel{$\to^+ w$}
\UnaryInfC{$C \to A$}
\UnaryInfC{$\neg A \to \neg C$}
\AxiomC{$[v \colon \neg A]$}
\RightLabel{$\to^-$}
\BinaryInfC{$\neg C$}
\RightLabel{$\forall^+ x$}
\UnaryInfC{$\forall_x\neg C$}
\RightLabel{$\to^-$}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ v.$}
\UnaryInfC{$\neg\neg A$}
\end{prooftree}$$

The variable condition in the above derivation of $\neg C$ is satisfied, as $x \notin \mathrm{FV}(\neg A) = \mathrm{FV}(A)$, where $\neg A$ is the only open assumption. $\square$

2.10 The Gödel-Gentzen translation

Definition 2.10.1. *The Gödel-Gentzen translation is the unique function*

$$^g : \mathtt{Form} \to \mathtt{Form}$$

$$A \mapsto A^g; \qquad A \in \mathtt{Form}$$

defined by recursion on Form *from the following clauses:*

$$\begin{aligned}
\bot^g &= \bot,\\
R^g &= \neg\neg R, \quad R \in \mathtt{Rel}^{(0)} \setminus \{\bot\},\\
\big(R(t_1,\ldots,t_n)\big)^g &= \neg\neg R(t_1,\ldots,t_n), \quad R \in \mathtt{Rel}^{(n)}, n \in \mathbb{N}^+, t_1,\ldots,t_n \in \mathtt{Term},\\
(A \circ B)^g &= A^g \circ B^g, \quad \circ \in \{\to, \wedge\},\\
(\forall_x A)^g &= \forall_x A^g,\\
(A \vee B)^g &= A^g \mathbin{\tilde{\vee}} B^g = \neg A^g \to \neg B^g \to \bot,\\
(\exists_x A)^g &= \tilde{\exists}_x A^g = \neg\forall_x \neg A^g.
\end{aligned}$$

If $\Gamma \subseteq \mathtt{Form}$*, let* $\Gamma^g = \{C^g \mid C \in \Gamma\}$.

Corollary 2.10.2. *Let* $n > 1$ *and* $A, A_1, \ldots, A_n \in \mathtt{Form}$.
(i) $(\neg A)^g = \neg A^g$.
(ii) $\big(A_1 \to A_2 \to \ldots \to A_{n-1} \to A_n\big)^g = A_1^g \to A_2^g \to \ldots \to A_{n-1}^g \to A_n^g$, *for every* $n > 0$.
(iii) $(\mathrm{EFQ}_A)^g = \mathrm{EFQ}_{A^g}$.
(iv) $(\mathrm{DNE}_A)^g = \mathrm{DNE}_{A^g}$.
(v) $\big(\widetilde{\mathrm{PEM}}_A\big)^g = \widetilde{\mathrm{PEM}}_{A^g}$.

Proof. The proofs of all cases are immediate. □

Proposition 2.10.3. (i) $\forall_{A \in \mathtt{Form}}\big(A^g \in \mathtt{Form}^-\big)$.
(ii) *Let* $R \in \mathbb{R}^{(n)}$. *If* $n > 1$ *and* $t_1, \ldots, t_n \in \mathtt{Term}$, *then* $R(t_1,\ldots,t_n) \to \bot \in \mathtt{Form}^- \setminus \mathtt{Form}^g$. *If* $n = 0$ *and* $R \neq \bot$, *then* $R \to \bot \in \mathtt{Form}^- \setminus \mathtt{Form}^g$.
(iii) *The Gödel-Gentzen translation is not an injection.*

Proof. Exercise. □

Combining Propositions 2.5.2(ii) and 2.10.3(ii) we get $\mathtt{Form}^g \subsetneq \mathtt{Form}^- \subsetneq \mathtt{Form}^*$.

Proposition 2.10.4. *Let* $x \in \mathtt{Var}$ *and* $s \in \mathtt{Term}$.
(i) $\forall_{A \in \mathtt{Form}}\big(A^g \in \mathtt{Form}^*\big)$.
(ii) $\forall_{A \in \mathtt{Form}}\big(\mathrm{FV}(A) = \mathrm{FV}(A^g)\big)$.
(iii) $\forall_{A \in \mathtt{Form}}\big(\mathrm{Free}_{s,x}(A) = \mathrm{Free}_{s,x}(A^g)\big)$.
(iv) $\forall_{A \in \mathtt{Form}}\big((A[x := s])^g = A^g[x := s]\big)$.

Proof. (i) It follows immediately from Propositions 2.10.3(i) and 2.5.2(i).
(ii) We use induction on $\texttt{Form}$. Let $A \in \texttt{Prime}$. If $A = \bot$, then $\mathrm{FV}(\bot^g) = \mathrm{FV}(\bot) = \emptyset$. If $A = R \in \texttt{Rel}^{(0)} \setminus \{\bot\}$, then $\mathrm{FV}(R^g) = \mathrm{FV}((R \to \bot) \to \bot) = \emptyset = \mathrm{FV}(R)$. If $A = R(t_1, \ldots, t_n)$,

$$\mathrm{FV}\big(R(t_1, \ldots, t_n)^g\big) = \mathrm{FV}\big((R(t_1, \ldots, t_n) \to \bot) \to \bot\big) = \mathrm{FV}\big(R(t_1, \ldots, t_n)\big).$$

If $\circ \in \{\to, \wedge\}$ and $A, B \in \texttt{Form}$, by the inductive hypotheses we get

$$\mathrm{FV}\big((A \circ B)^g\big) = \mathrm{FV}\big(A^g \circ B^g\big) = \mathrm{FV}(A^g) \cup \mathrm{FV}(B^g) = \mathrm{FV}(A) \cup \mathrm{FV}(B) = \mathrm{FV}(A \circ B),$$

$$\mathrm{FV}\big(\forall_x A)^g\big) = \mathrm{FV}\big(\forall_x A^g\big) = \mathrm{FV}(A^g) \setminus \{x\} = \mathrm{FV}(A) \setminus \{x\} = \mathrm{FV}(\forall_x A),$$

$$\mathrm{FV}\big(A \vee B)^g\big) = \mathrm{FV}(\neg A^g \to \neg B^g \to \bot) = \mathrm{FV}(A^g) \cup \mathrm{FV}(B^g) = \mathrm{FV}(A) \cup \mathrm{FV}(B) = \mathrm{FV}(A \vee B),$$

$$\mathrm{FV}\big(\exists_x A)^g\big) = \mathrm{FV}\big([\forall_x (A^g \to \bot)] \to \bot\big) = \mathrm{FV}(A^g) \setminus \{x\} = \mathrm{FV}(A) \setminus \{x\} = \mathrm{FV}(\exists_x A).$$

(iii) By Definition 1.6.1 we get the following equalities. Let $A \in \texttt{Prime}$. If $A = \bot$, then $\mathrm{Free}_{s,x}(\bot^g) = \mathrm{Free}_{s,x}(\bot) = 1$. If $A = R \in \texttt{Rel}^{(0)} \setminus \{\bot\}$, or if $A = R(t_1, \ldots, t_n)$, then

$$\mathrm{Free}_{s,x}(A^g) = \mathrm{Free}_{s,x}((A \to \bot) \to \bot) = \mathrm{Free}_{s,x}(A) \cdot \mathrm{Free}_{s,x}(\bot) \cdot \mathrm{Free}_{s,x}(\bot) = \mathrm{Free}_{s,x}(A).$$

If $\circ \in \{\to, \wedge\}$ and $A, B \in \texttt{Form}$, by the inductive hypotheses we get

$$\begin{aligned}
\mathrm{Free}_{s,x}\big((A \circ B)^g\big) &= \mathrm{Free}_{s,x}\big(A^g \circ B^g\big) \\
&= \mathrm{Free}_{s,x}(A^g) \cdot \mathrm{Free}_{s,x}(B^g) \\
&= \mathrm{Free}_{s,x}(A) \cdot \mathrm{Free}_{s,x}(B) \\
&= \mathrm{Free}_{s,x}(A \circ B),
\end{aligned}$$

$$\begin{aligned}
\mathrm{Free}_{s,x}\big(A \vee B)^g\big) &= \mathrm{Free}_{s,x}(\neg A^g \to \neg B^g \to \bot) \\
&= \mathrm{Free}_{s,x}(\neg A) \cdot \mathrm{Free}_{s,x}(\neg B) \\
&= \mathrm{Free}_{s,x}(A) \cdot \mathrm{Free}_{s,x}(B) \\
&= \mathrm{Free}_{s,x}(A \vee B),
\end{aligned}$$

$$\begin{aligned}
\mathrm{Free}_{s,x}\big((\forall_y A)^g\big) &= \mathrm{Free}_{s,x}\big(\forall_y A^g\big) \\
&= \begin{cases} 0 & , x = y \ \vee \ [x \neq y \ \& \ y \in \{y_1, \ldots, y_m\}] \\ 1, & , x \neq y \ \& \ x \notin \mathrm{FV}(A^g) \setminus \{y\} \\ \mathrm{Free}_{s,x}(A^g) & , x \neq y \ \& \ y \notin \{y_1, \ldots, y_m\} \ \& \ x \in \mathrm{FV}(A^g) \end{cases} \\
&= \begin{cases} 0 & , x = y \ \vee \ [x \neq y \ \& \ y \in \{y_1, \ldots, y_m\}] \\ 1, & , x \neq y \ \& \ x \notin \mathrm{FV}(A) \setminus \{y\} \\ \mathrm{Free}_{s,x}(A) & , x \neq y \ \& \ y \notin \{y_1, \ldots, y_m\} \ \& \ x \in \mathrm{FV}(A) \end{cases} \\
&= \mathrm{Free}_{s,x}\big(\forall_y A\big).
\end{aligned}$$

$$\begin{aligned}
\mathrm{Free}_{s,x}\big((\exists_y A)^g\big) &= \mathrm{Free}_{s,x}\big(\neg\forall_y\neg A^g\big)\\
&= \mathrm{Free}_{s,x}\big(\forall_y\neg A^g\big)\\
&= \begin{cases} 0 & , x = y \ \vee\ [x \neq y \ \&\ y \in \{y_1,\ldots,y_m\}]\\ 1, & , x \neq y \ \&\ x \notin \mathrm{FV}(\neg A^g)\setminus\{y\}\\ \mathrm{Free}_{s,x}(\neg A^g) & , x \neq y \ \&\ y \notin \{y_1,\ldots,y_m\} \ \&\ x \in \mathrm{FV}(\neg A^g)\end{cases}\\
&= \begin{cases} 0 & , x = y \ \vee\ [x \neq y \ \&\ y \in \{y_1,\ldots,y_m\}]\\ 1, & , x \neq y \ \&\ x \notin \mathrm{FV}(A^g)\setminus\{y\}\\ \mathrm{Free}_{s,x}(A^g) & , x \neq y \ \&\ y \notin \{y_1,\ldots,y_m\} \ \&\ x \in \mathrm{FV}(A^g)\end{cases}\\
&= \begin{cases} 0 & , x = y \ \vee\ [x \neq y \ \&\ y \in \{y_1,\ldots,y_m\}]\\ 1, & , x \neq y \ \&\ x \notin \mathrm{FV}(A)\setminus\{y\}\\ \mathrm{Free}_{s,x}(A) & , x \neq y \ \&\ y \notin \{y_1,\ldots,y_m\} \ \&\ x \in \mathrm{FV}(A)\end{cases}\\
&= \mathrm{Free}_{s,x}\big(\exists_y A\big).
\end{aligned}$$

(iv) Let $A \in \texttt{Prime}$. If $A = \bot$, then $(\bot[x := s])^g = \bot^g = \bot = \bot^g[x := s]$. If $A = R \in \texttt{Rel}^{(0)} \setminus \{\bot\}$ and if $A = R(t_1,\ldots,t_n)$, then we we get, respectively,

$$(R[x := s])^g = R^g = \neg\neg R = \neg\neg R(x := s] = (\neg\neg R)[x := s] = R^g[x := s].$$

$$\begin{aligned}
\big(R(t_1,\ldots,t_n)[x := s]\big)^g &= \big(R(t_1[x := s],\ldots,t_n[x := s])\big)^g\\
&= \neg\neg R(t_1[x := s],\ldots,t_n[x := s])\\
&= \big(\neg\neg R(t_1,\ldots,t_n)\big)[x := s]\\
&= \big(R(t_1,\ldots,t_n)\big)^g[x := s].
\end{aligned}$$

If $\mathrm{Free}_{s,x}(A) = 0$ or $\mathrm{Free}_{s,x}(B) = 0$, then $\mathrm{Free}_{s,x}(A \circ B) = 0$, and hence

$$\big((A \circ B)[x := s]\big)^g = (A \circ B)^g = A^g \circ B^g = (A^g \circ B^g)[x := s] = (A \circ B)^g[x := s],$$

since by (iii) we get $\mathrm{Free}_{s,x}\big((A \circ B)^g\big) = \mathrm{Free}_{s,x}(A \circ B) = 0$. Suppose next that $\mathrm{Free}_{s,x}(A) = 1 = \mathrm{Free}_{s,x}(B)$, hence $\mathrm{Free}_{s,x}(A^g) = 1 = \mathrm{Free}_{s,x}(B^g)$. By the inductive hypotheses we get

$$\begin{aligned}
\big((A \circ B)[x := s]\big)^g &= \big(A[x := s] \circ B[x := s]\big)^g\\
&= \big(A[x := s]\big)^g \circ \big(B[x := s]\big)^g\\
&= A^g[x := s] \circ B^g[x := s]\\
&= (A^g \circ B^g)[x := s]\\
&= (A \circ B)^g[x := s].
\end{aligned}$$

If $\mathrm{Free}_{s,x}(A) = 0$ or $\mathrm{Free}_{s,x}(B) = 0$, then $\mathrm{Free}_{s,x}(A \vee B) = 0$, and hence

$$\big((A \vee B)[x := s]\big)^g = (A \vee B)^g = (A \vee B)^g[x := s],$$

since by (iii) we get $\mathrm{Free}_{s,x}\big((A \vee B)^g\big) = \mathrm{Free}_{s,x}(A \vee B) = 0$. Suppose next that $\mathrm{Free}_{s,x}(A) = 1 = \mathrm{Free}_{s,x}(B)$, hence $\mathrm{Free}_{s,x}(A^g) = 1 = \mathrm{Free}_{s,x}(B^g)$. By the inductive hypotheses we get

$$\begin{aligned}
\big((A \vee B)[x := s]\big)^g &= \big(A[x := s] \vee B[x := s]\big)^g\\
&= \neg\big(A[x := s]\big)^g \to \neg\big(B[x := s]\big)^g \to \bot\\
&= \neg A^g[x := s] \to \neg B^g[x := s] \to \bot\\
&= (A^g \mathbin{\tilde{\vee}} B^g)[x := s]\\
&= (A \vee B)^g[x := s].
\end{aligned}$$

If $\text{Free}_{s,x}(\forall_y A) = 0 = \text{Free}_{s,x}\big((\forall_y A)^g\big) = \text{Free}_{s,x}(\forall_y A^g)$, then

$$\big((\forall_y A)[x := s]\big)^g = \big(\forall_y A\big)^g = \big(\forall_y A\big)^g[x := s].$$

If $\text{Free}_{s,x}(\forall_y A) = 1 = \text{Free}_{s,x}\big((\forall_y A)^g\big) = \text{Free}_{s,x}(\forall_y A^g)$, then

$$\begin{aligned}\big((\forall_y A)[x := s]\big)^g &= \big(\forall_y A[x := s]\big)^g \\ &= \forall_y (A[x := s])^g \\ &= \forall_y A^g[x := s] \\ &= \big(\forall_y A^g\big)[x := s] \\ &= (\forall_y A)^g[x := s].\end{aligned}$$

If $\text{Free}_{s,x}(\exists_y A) = 0 = \text{Free}_{s,x}\big((\exists_y A)^g\big)$, then

$$\big((\exists_y A)[x := s]\big)^g = \big(\exists_y A\big)^g = \big(\exists_y A\big)^g[x := s].$$

If $\text{Free}_{s,x}(\exists_y A) = 1 = \text{Free}_{s,x}\big((\exists_y A)^g\big) = \text{Free}_{s,x}(\neg\forall_y \neg A^g)$, then

$$\begin{aligned}\big((\exists_y A)[x := s]\big)^g &= \big(\exists_y A[x := s]\big)^g \\ &= \neg\forall_y \neg(A[x := s])^g \\ &= \neg\forall_y \neg A^g[x := s] \\ &= \neg\forall_y (\neg A^g)[x := s] \\ &= \big(\neg\forall_y \neg A^g\big)[x := s]. \\ &= (\exists_y A)^g[x := s].\end{aligned}$$

□

Because of Proposition 2.5.2(i), the Gödel-Gentzen translation is also called the *negative* translation. Since $A^g \in \texttt{Form}^*$, by Theorem 2.2.10 we get that $\vdash_c^* \neg\neg A^g \to A^g$. For the formulas in $\texttt{Form}^* \cap \texttt{Form}^g$ one gets though, the minimal derivability of their double-negation-elimination.

Corollary 2.10.5. $\forall_{A \in \texttt{Form}}\big(\vdash \neg\neg A^g \to A^g\big)$.

Proof. Immediately from Propositions 2.5.2(i) and 2.5.3. □

Proposition 2.10.6. (i) $\forall_{A \in \texttt{Form}}\big(\vdash_c A \leftrightarrow A^g\big)$.
(ii) $\forall_{A \in \texttt{Form}^*}\big(\vdash_c^* A \leftrightarrow A^g\big)$.

Proof. Exercise. □

2.11 The Gödel-Gentzen functor

In this section we show that the Gödel-Gentzen translation generates a functor $\textbf{Form}_c \to \textbf{Form}$ i.e., not only formulas are translated into formulas in the negative fragment, but also classical derivations are translated into minimal derivations. The first step in the proof of the functorial character of the Gödel-Gentzen translation is the existence of the following mapping.

Proposition 2.11.1. *There is a function* $\text{dne}\colon \texttt{Form}^- \to \mathfrak{D}_V(A)$ *such that* $\text{dne}(A)\colon \neg\neg A \to A$, *for every* $A \in \texttt{Form}^-$.

Proof. We use recursion on $\mathtt{Form}^-$ and we rewrite accordingly the proof of Proposition 2.5.3 (the details is an exercise). □

For simplicity we use the same symbol to the Gödel-Gentzen translation for the function that translates classical derivations into minimal ones.

Theorem 2.11.2. *There is a function* $^g \colon \mathfrak{D}^c_V(A) \to \mathfrak{D}_{V^g}(A^g)$*, where*

$$\mathfrak{D}^c_V(A) \ni M_c \mapsto M^g_c \in \mathfrak{D}_{V^g}(A^g),$$

$$V^g = \{u^g \colon C^g \mid u \colon C \in V\}.$$

Proof. By recursion[2] on $\mathfrak{D}^c_V(A)$ we map a classical derivation M_c in $\mathfrak{D}^c_V(A)$ to a minimal derivation M^g_c in $\mathfrak{D}_{V^g}(A^g)$ by mapping each introduction-rule of $\mathfrak{D}^c_V(A)$ to an element of $\mathfrak{D}_{V^g}(A^g)$.

(DNE_A)

$$\frac{u \colon \neg\neg A}{A}\,\mathrm{DNE}_A \qquad \longmapsto \qquad \mathrm{dne}(A^g) \colon \neg\neg A^g \to A^g,$$

where as $A^g \in \mathtt{Form}^-$ and $\mathrm{dne} \colon \mathtt{Form}^- \to \mathfrak{D}_V(A)$, we get $\mathrm{dne}(A^g) \in \mathfrak{D}_{\{u^g \colon \neg\neg A^g = (\neg\neg A)^g\}}(A^g)$.

(1_A)

$$\frac{a \colon A}{A}\,1_A \qquad \longmapsto \qquad \frac{a^g \colon A^g}{A^g}\,1_{A^g}$$

$(\to^+)$ If we consider the following left, classical derivation M_c and if we suppose that N^g_c is already defined i.e.,

$$\begin{array}{c} [u \colon A]\ u_1 \colon C_1 \ldots u_n \colon C_n \\ \mid N_c \\ \dfrac{B}{A \to B}\,\to^+ u \end{array} \qquad \text{and} \qquad \begin{array}{c} u^g \colon A^g\ u_1{}^g \colon C_1{}^g \ldots u_n{}^g \colon C_n{}^g \\ \mid N^g_c \\ B^g \end{array}$$

we define as M^g_c the following minimal derivation

$$\begin{array}{c} [u^g \colon A^g]\ u_1{}^g \colon C_1{}^g \ldots u_n{}^g \colon C_n{}^g \\ \mid N^g_c \\ \dfrac{B^g}{A^g \to B^g}\,\to^+ u^g \end{array}$$

as $A^g \to B^g = (A \to B)^g$.

$(\to^-)$ If we consider the following classical derivation M_c

$$\dfrac{\begin{array}{c} u_1 \colon C_1 \ldots u_n \colon C_n \\ \mid N_c \\ A \to B \end{array} \qquad \begin{array}{c} v_1 \colon D_1 \ldots v_m \colon D_m \\ \mid K_c \\ A \end{array}}{B}\,\to^-$$

[2]Actually, what we need do here, as in the proof of Proposition 2.11.1 is the following: first we define by recursion a function in the set of trees of formulas, and then by induction we prove that the value of this function is a minimal derivation of A^g from assumptions V^g. For simplicity, here we perform the two steps simultaneously.

and if we suppose that that N_c^g and K_c^g have been defined i.e.,

$$\begin{array}{c} {u_1}^g\colon {C_1}^g \ldots {u_n}^g\colon {C_n}^g \\ \mid N_c^g \\ (A \to B)^g \end{array} \qquad \text{and} \qquad \begin{array}{c} {v_1}^g\colon {D_1}^g \ldots {v_m}^g\colon {D_m}^g \\ \mid K_c^g \\ A^g \end{array}$$

we define M_c^g to be the following minimal derivation

$$\dfrac{\begin{array}{c} {u_1}^g\colon {C_1}^g \ldots {u_n}^g\colon {C_n}^g \\ \mid N_c^g \\ A^g \to B^g \end{array} \qquad \begin{array}{c} {v_1}^g\colon {D_1}^g \ldots {v_m}^g\colon {D_m}^g \\ \mid K_c^g \\ A^g \end{array}}{B^g} \to^-$$

($\forall^+$) If we consider the following left, classical derivation M_c, and if we suppose that the minimal derivation N_c^g is already defined i.e.,

$$\begin{array}{c} u_1\colon C_1 \ldots u_n\colon C_n \\ \mid N_c \\ \dfrac{A}{\forall_x A}\,\forall^+ x \end{array} \qquad \text{and} \qquad \begin{array}{c} {u_1}^g\colon {C_1}^g \ldots {u_n}^g\colon {C_n}^g \\ \mid N_c^g \\ A^g \end{array}$$

we define M_c^g to be the following minimal derivation

$$\begin{array}{c} {u_1}^g\colon {C_1}^g \ldots {u_n}^g\colon {C_n}^g \\ \mid N_c^g \\ \dfrac{A^g}{\forall_x A^g}\,\forall^+ x \end{array}$$

where the variable condition $x \notin \mathrm{FV}(C_1^g)$ & ... & $\notin \mathrm{FV}(C_n^g)$, is satisfied, since by Proposition 2.10.4(ii) $\mathrm{FV}(C_i) = \mathrm{FV}({C_i}^g)$, for every $i \in \{1, \ldots, n\}$, and the variable condition $x \notin \mathrm{FV}(C_1)$ & ... & $x \notin \mathrm{FV}(C_n)$ is satisfied in N_c.

($\forall^-$) If we consider the following left, classical derivation M_c, and if we suppose that N_c^g is already defined i.e.,

$$\dfrac{\begin{array}{c} u_1\colon C_1 \ldots u_n\colon C_n \\ \mid N_c \\ \forall_x A \end{array} \qquad r \in \mathtt{Term}}{A(r)} \forall^- \qquad \text{and} \qquad \begin{array}{c} {u_1}^g\colon {C_1}^g \ldots {u_n}^g\colon {C_n}^g \\ \mid N_c^g \\ (\forall_x A)^g = \forall_x A^g \end{array}$$

we define M_c^g to be the following classical derivation

$$\dfrac{\begin{array}{c} {u_1}^g\colon {C_1}^g \ldots {u_n}^g\colon {C_n}^g \\ \mid N_c^g \\ \forall_x A^g \end{array} \qquad r \in \mathtt{Term}}{A^g(r) = A(r)^g} \forall^-$$

where by Proposition 2.10.4(iv) we get the required equality $A^g(r) = A(r)^g$.

($\wedge^+$) If we consider the following classical derivation derivation M_c

$$\dfrac{\begin{array}{c} u_1\colon C_1 \ldots u_n\colon C_n \\ \mid N_c \\ A \end{array} \qquad \begin{array}{c} v_1\colon D_1 \ldots v_m\colon D_m \\ \mid K_c \\ AB \end{array}}{A \wedge B} \wedge^+$$

and if we suppose that the following minimal derivations N_c^g and K_c^g are already defined i.e.,

$$\begin{array}{c} {u_1}^g\colon {C_1}^g \ldots {u_n}^g\colon {C_n}^g \\ \mid N_c^g \\ A^g \end{array} \quad \text{and} \quad \begin{array}{c} {v_1}^g\colon {D_1}^g \ldots {v_m}^g\colon {D_m}^g \\ \mid K_c^g \\ B^g \end{array}$$

we define M_c^g to be the following minimal derivation

$$\dfrac{\begin{array}{c} {u_1}^g\colon {C_1}^g \ldots {u_n}^g\colon {C_n}^g \\ \mid N_c^g \\ A^g \end{array} \qquad \begin{array}{c} {v_1}^g\colon {D_1}^g \ldots {v_m}^g\colon {D_m}^g \\ \mid K_c^g \\ B^g \end{array}}{A^g \wedge B^g = (A \wedge B)^g}\ \wedge^+$$

($\wedge^-$) If we consider the following classical derivation M_c

$$\dfrac{\begin{array}{c} u_1\colon C_1 \ldots u_n\colon C_n \\ \mid N_c \\ A \wedge B \end{array} \qquad \begin{array}{c} [u\colon A]\ [v\colon B] \\ \mid K_c \\ C \end{array}}{C}\ \wedge^- u, v$$

and if we suppose that the minimal derivations N_c^g and K_c^g are already defined i.e.,

$$\begin{array}{c} {u_1}^g\colon {C_1}^g \ldots {u_n}^g\colon {C_n}^g \\ \mid N_c^g \\ (A \wedge B)^g \end{array} \quad \text{and} \quad \begin{array}{c} u^g\colon A^g\ v^g\colon B^g \\ \mid K_c^g \\ C^g \end{array}$$

we define M_c^g to be the following minimal derivation

$$\dfrac{\begin{array}{c} {u_1}^g\colon {C_1}^g \ldots {u_n}^g\colon {C_n}^g \\ \mid N_c^g \\ A^g \wedge B^g \end{array} \qquad \begin{array}{c} [u^g\colon A^g]\ [v^g\colon B^g] \\ \mid K_c^g \\ C^g \end{array}}{C^g}\ \wedge^- u^g, v^g$$

($\vee_0^+$) If we consider the following, left classical derivation M_c, and if we suppose that the minimal derivation N_c^g is already defined i.e.,

$$\begin{array}{c} u_1\colon C_1 \ldots u_n\colon C_n \\ \mid N_c \\ \dfrac{A}{A \vee B}\ \vee_0^+ \end{array} \quad \text{and} \quad \begin{array}{c} {u_1}^g\colon {C_1}^g \ldots {u_n}^g\colon {C_n}^g \\ \mid N_c^g \\ A^g \end{array}$$

we define M_c^g to be the following minimal derivation of $(A \vee B)^g = \neg A^g \to \neg B^g \to \bot$

$$\dfrac{\dfrac{\dfrac{u\colon A^g \to \bot \qquad \begin{array}{c} {u_1}^g\colon {C_1}^g \ldots {u_n}^g\colon {C_n}^g \\ \mid N_c^g \\ A^g \end{array}}{\bot}\ \to^-}{\neg B^g \to \bot}\ \to^+ v\colon \neg B^g}{\neg A^g \to \neg B^g \to \bot}\ \to^+ u$$

For the rule $\vee_1^+$ we proceed similarly.

($\vee^-$) For simplicity we use the notations $w\colon \Gamma$ for $w_1\colon C_1 \ldots w_n\colon C_n$, and $w'\colon \Delta$ for ${w_1}'\colon D_1 \ldots {w_m}'\colon D_m$, and $w''\colon E$ for ${w_1}''\colon E_1 \ldots {w_k}''\colon E_m$. Let the following classical derivation M_c

$$\dfrac{\begin{array}{c} w\colon \Gamma \\ \mid N_c \\ A \vee B \end{array} \qquad \begin{array}{c} [u\colon A]\ w'\colon \Delta \\ \mid K_c \\ C \end{array} \qquad \begin{array}{c} [v\colon B]\ w''\colon E \\ \mid L_c \\ C \end{array}}{C}\ \vee^- u, v.$$

We suppose that the minimal derivations N_c^g, K_c^g and L_c^g are already defined i.e.,

$$\begin{array}{c} w^g\colon \Gamma^g \\ \mid N_c^g \\ (A \vee B)^g \end{array} \qquad \begin{array}{c} u^g\colon A^g\ w'^g\colon \Delta^g \\ \mid K_c^g \\ C^g \end{array} \qquad \begin{array}{c} v^g\colon B^g\ w''^g\colon E^g \\ \mid L_c^g \\ C^g. \end{array}$$

By Proposition 2.6.6 there is a minimal derivation of the formula

$$D(A, B, C) = (\neg\neg C \to C) \to (A \to C) \to (B \to C) \to A \mathbin{\tilde{\vee}} B \to C.$$

Hence we can define a function $f\colon \mathtt{Form}^3 \to \mathfrak{D}_V(A)$ with

$$f(A, B, C) \in \mathfrak{D}(D(A, B, C)); \qquad (A, B, C) \in \mathtt{Form}^3.$$

By Proposition 2.11.1 there is a minimal derivation $\mathrm{dne}(C^g)\colon \neg\neg C^g \to C^g$. If

$$D'(A^g, B^g, C^g) = (A^g \to C^g) \to (B^g \to C^g) \to A^g \mathbin{\tilde{\vee}} B^g \to C^g,$$

$$D''(A^g, B^g, C^g) = (B^g \to C^g) \to A^g \mathbin{\tilde{\vee}} B^g \to C^g,$$

we define M_c^g to be the following derivation of C^g from assumptions Γ^g, Δ^g and E^g

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c} \mid f(A^g,B^g,C^g) \\ D(A^g,B^g,C^g) \end{array} \qquad \dfrac{\begin{array}{c} u'^g\colon\ \neg\neg C^g \\ \mid \mathrm{dne}(C^g) \\ (C^g \end{array}}{\neg\neg C^g \to C^g}\ \to^+ u'^g}{D'(A^g, B^g, C^g)} \qquad \dfrac{\begin{array}{c} [u^g\colon A^g] w'^g \in \Delta^g \\ \mid K_c^g \\ C^g \end{array}}{A^g \to C^g}}{D''(A^g, B^g, C^g)} \qquad \dfrac{\begin{array}{c} [v^g\colon B^g] w''^g \in E^g \\ \mid L_c^g \\ C^g \end{array}}{B^g \to C^g}}{A^g \mathbin{\tilde{\vee}} B^g \to C^g} \qquad \begin{array}{c} w^g\colon \Gamma^g \\ \mid N_c^g \\ A^g \mathbin{\tilde{\vee}} B^g \end{array}}{C^g}$$

In the above definition of M_c^g we write all intermediate derivations as values of appropriate functions, in order to be compatible to the formulation of the recursion theorem for $\mathfrak{D}_V^c(A)$ that we employ in the proof.

($\exists^+$) If we consider the following left, classical derivation M_c, and if we suppose that the minimal derivation N_c^g is already defined i.e.,

$$\dfrac{r \in \mathtt{Term} \qquad \begin{array}{c} u_1\colon C_1 \ldots u_n\colon C_n \\ \mid N_c \\ A(r) \end{array}}{\exists_x A}\ \exists^+ \qquad \text{and} \qquad \begin{array}{c} {u_1}^g\colon {C_1}^g \ldots {u_n}^g\colon {C_n}^g \\ \mid N_c^g \\ A(r)^g = A^g(r) \end{array}$$

we define M_c^g to be the following minimal derivation of $(\exists_x A)^g = \forall_x(A^g \to \bot) \to \bot$

$$\dfrac{\dfrac{\dfrac{[u\colon \forall_x(A^g \to \bot)] \qquad r \in \texttt{Term}}{A^g(r) \to \bot}\,\forall^- \qquad \begin{array}{c} u_1{}^g\colon C_1{}^g \dots u_n{}^g\colon C_n{}^g \\ \mid N_c^g \\ A^g(r) \end{array}}{\bot}\,\to^-}{\forall_x(A^g \to \bot) \to \bot}\,\to^+ u.$$

$(\exists^-)$ Again for simplicity we use the notate $w\colon \Gamma$ for $w_1\colon C_1 \dots w_n\colon C_n$, and $w'\colon \Delta$ for $w_1{}'\colon D_1 \dots w_m{}'\colon D_m$. Let the following classical derivation M_c

$$\dfrac{\begin{array}{c} w\colon \Gamma \\ \mid N_c \\ \exists_x A \end{array} \qquad \begin{array}{c} [u\colon A]\ w'\colon \Delta \\ \mid K_c \\ B \end{array}}{B}\,\exists^- x, u$$

with $x \notin \mathrm{FV}(\Delta)$ and $x \notin \mathrm{FV}(B)$. Suppose that the derivations N_c^g and K_c^g are defined i.e.,

$$\begin{array}{c} w^g\colon \Gamma^g \\ \mid N_c^g \\ \tilde{\exists}_x A^g \end{array} \quad \text{and} \quad \begin{array}{c} u^g\colon A^g\ w'^g\colon \Delta^g \\ \mid K_c^g \\ B^g \end{array}$$

As $x \notin \mathrm{FV}(B^g) = \mathrm{FV}(B)$, by Proposition 2.6.3(ii) there is a derivation Λ of

$$E_x(A^g, B^g) = (\neg\neg B^g \to B^g) \to \tilde{\exists}_x A^g \to \forall_x(A^g \to B^g) \to B^g.$$

Hence we can define a function $g_x\colon \texttt{Form} \to \texttt{Form}_x \to \mathfrak{D}(A)$ with

$$g_x(A, B) \in \mathfrak{D}(E_x(A, B)); \qquad A \in \texttt{Form} \ \& \ B \in \texttt{Form}_x,$$

as x must not be free in B. By Proposition 2.11.1 let the derivation $\mathrm{dne}(B^g)\colon \neg\neg B^g \to B^g$. If

$$E'_x(A^g, B^g) = \tilde{\exists}_x A^g \to \forall_x(A^g \to B^g) \to B^g,$$

$$E''_x(A^g, B^g) = \forall_x(A^g \to B^g) \to B^g,$$

we define M_c^g to be the following minimal derivation of B^g

$$\dfrac{\dfrac{\dfrac{\begin{array}{c} \mid g_x(A^g, B^g) \\ E_x(A^g, B^g) \end{array} \qquad \dfrac{\begin{array}{c} u'^g\colon \neg\neg B^g \\ \mid \mathrm{dne}(B^g) \\ (B^g \end{array}}{\neg\neg B^g \to B^g}\,\to^+ u'^g}{E'_x(A^g, B^g)} \qquad \begin{array}{c} w\colon \Gamma^g \\ \mid N_c^g \\ \tilde{\exists}_x A^g \end{array}}{E''_x(A^g, B^g)}\,\to^- \qquad \dfrac{\dfrac{\begin{array}{c} [u^g\colon A^g]\ w'^g\colon \Delta^g \\ \mid K_c^g \\ B^g \end{array}}{A^g \to B^g}\,\to^+ u^g}{\forall_x(A^g \to B^g)}\,\forall^+ x}{B^g}\,\to^-$$

from assumptions Γ^g and Δ^g. Note that the variable condition is satisfied in the above use of $\forall^+ x$, since $x \notin \mathrm{FV}(\Delta^g) = \mathrm{FV}(\Delta)$. □

Definition 2.11.3. *The Gödel-Gentzen functor $GG\colon \boldsymbol{Form}_c \to \boldsymbol{Form}$ is defined by $GG_0(A) = A^g$, for every $A \in \texttt{Form}$, and for every arrow $M_c\colon A \to B$ in $\boldsymbol{Form}_c$ let*

$$GG_1(M_c\colon A \to B) = M_c^g\colon A^g \to B^g.$$

The fact that GG is a functor follows from the equalities

$$GG_1(1_A) = 1_A^g = 1_{A^g} = 1_{GG_0(A)},$$

$$GG_1(N \circ M) = (N \circ M)^g = N^g \circ M^g = GG_1(N) \circ GG_2(M),$$

where the equality $(N \circ M)^g = N^g \circ M^g$ follows immediately by the thinness of **Form**.

2.12 Applications of the Gödel-Gentzen functor

Corollary 2.12.1. *Let* $\Gamma \subseteq \mathtt{Form}$ *and* $A \in \mathtt{Form}$.
(i) $\Gamma \vdash_c A \Rightarrow \Gamma^g \vdash A^g$.
(ii) $\Gamma \vdash A \Rightarrow \Gamma^g \vdash A^g$.

Proof. (i)Let $C_1, \ldots, C_n \in G$ such that $C_1, \ldots, C_n \vdash_c A$ i.e., there is a classical derivation M_c in $\mathfrak{D}^c_{\{u:\, C_1,\ldots,u_n:\, C_n\}}(A)$. By Theorem 2.11.2 the derivation M_c^g is in $\mathfrak{D}_{\{u^g:\, C_1^g,\ldots,u_n^g:\, C_n^g\}}(A^g)$ i.e., $C_1^g, \ldots, C_n^g \vdash A^g$, hence $\Gamma^g \vdash A^g$.
(ii) It follows immediately from (i) and the fact that $\Gamma \vdash A \Rightarrow \Gamma \vdash_c A$. □

Proposition 2.12.2. (i) $GG_0(A \wedge B) \cong GG_0(A) \wedge GG_0(B)$.
(ii) $GG_0(\top) \cong \top$.
(iii) *The Gödel-Gentzen translation defines a functor* $GG^{\mathrm{grp}} \colon \boldsymbol{Form}_c^{\mathrm{grp}} \to \boldsymbol{Form}^{\mathrm{grp}}$ *such that* $\forall_{A \in \mathtt{Form}} \exists_{B \in \mathtt{Form}} \big(B^g = GG_0^{\mathrm{grp}}(B) \cong_c A\big)$.

Proof. Exercise. □

Definition 2.12.3. (i) *A logic, minimal, intuitionistic, or classical, is consistent, if there is no derivation of* $\bot$ *within it.*
(ii) *A logic, minimal, intuitionistic, or classical, is inconsistent, if there is a derivation of* $\bot$ *within it.*
(iii) *A pair of logics* $(\vdash, \vdash_c)$, $(\vdash, \vdash_i)$, *or* $(\vdash_c, \vdash_i)$, *is a pair of equiconsistent logics, if the consistency of one logic of the pair is equivalent to the consistency of the other.*

At the moment, we cannot prove the consistency of the logics studied. What we can show though, is that all possible pairs of logics studied here are pairs of equiconsistent logics.

Corollary 2.12.4. (i) *If minimal logic is consistent, then classical logic and intuitionistic logic are consistent.*
(ii) *If classical logic is consistent, then minimal logic and intuitionistic logic are consistent.*
(iii) *If intuitionistic logic is consistent, then minimal logic and classical logic are consistent.*
(iv) *The pairs* $(\vdash, \vdash_c)$, $(\vdash, \vdash_i)$, *and* $(\vdash_c, \vdash_i)$ *are pairs of equiconsistent logics.*

Proof. (i) If in Corollary 2.12.1 we set $\Gamma = \emptyset$ and $A = \bot$, we get

$$(*) \qquad \vdash_c \bot \Rightarrow \vdash \bot^g = \bot.$$

Suppose that there is a derivation $\vdash_c \bot$. Then there is a derivation $\vdash \bot$, which contradicts our hypothesis. Hence, there is no $\vdash_c \bot$. We have already shown the implications

$$(**) \qquad \vdash \bot \Rightarrow \vdash_i \bot \Rightarrow \vdash_c \bot.$$

By $(*)$ $\vdash_i \bot \Rightarrow \vdash_c \bot \Rightarrow \vdash \bot$, hence, if there is a derivation $\vdash_i \bot$, there is a derivation $\vdash \bot$.
(ii) It follows immediately from $(**)$.
(iii) The consistency of minimal logic follows from $(**)$, and the consistency of classical logic follows from $(*)$ and $(**)$. (iv) follows immediately from (i)-(iii). ☐

Definition 2.12.5. *The height $|M|$ of a derivation M is the maximum length of a branch in M, where if B is a branch of M, then its length is the number of its nodes minus 1. On can define (exercise) accordingly the functions $|.|\colon \mathfrak{D}_V(A) \to \mathbb{N}$, $|.|\colon \mathfrak{D}^i_V(A) \to \mathbb{N}$ and $|.|\colon \mathfrak{D}^c_V(A) \to \mathbb{N}$, where for simplicity we use the same symbol for all of them.*

E.g., for the following derivation tree M

$$\dfrac{\dfrac{\dfrac{\forall_x\neg(Ax \to B) \quad x}{\neg(Ax \to B)} \qquad \dfrac{\dfrac{\forall_x Ax \to B \qquad \dfrac{\dfrac{\dfrac{\dfrac{\forall_y(\bot \to Ay) \quad y}{\bot \to Ay} \qquad \dfrac{u_1\colon \neg Ax \quad u_2\colon Ax}{\bot}}{Ay}}{\forall_y Ay}}{B}}{Ax \to B}\to^+ u_2}{\bot}}{\neg\neg Ax}\to^+ u_1$$

we have that $|M| = 7$, since the length of its longest branch

$$\{\neg\neg Ax, \bot, Ax \to B, B, \forall_y Ay, Ay, \bot, Ax\}$$

is $8 - 1 = 7$. Clearly, $|M_A| = 1$, and $|M| \geq 2$, for all other elements M of $\mathcal{D}$.

Corollary 2.12.6. $\forall_{M_c \in \mathfrak{D}^c_V(A)}(|M^g_c| \geq |M_c|)$.

Proof. By induction on $\mathfrak{D}^c_V(A)$ and inspection of the proof of Theorem 2.11.2. ☐

Proposition 2.12.7. *There are functions $g^c_0, g^c_1\colon \mathtt{Form} \to \mathfrak{D}^c_V(A)$ such that*

$$g^c_0(A)\colon A^g \to A \;\;\&\;\; g^c_1\colon A \to A^g; \qquad A \in \mathtt{Form}.$$

Proof. By recursion on `Form` and by inspection of the proof of Proposition 2.10.6(i). ☐

The next theorem is the converse to Theorem 2.11.2.

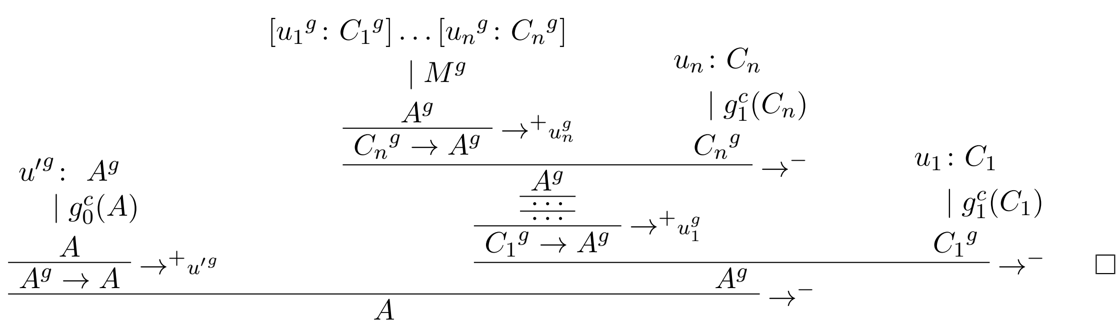
Theorem 2.12.8. *There is a function $_c\colon \mathfrak{D}_{V^g}(A^g) \to \mathfrak{D}^c_V(A)$, where*

$$\mathfrak{D}_{V^g}(A^g) \ni M^g \mapsto (M^g)_c \in \mathfrak{D}^c_V(A).$$

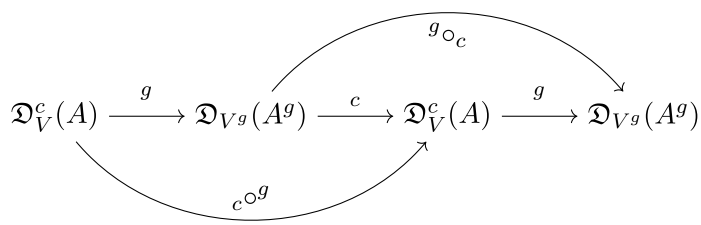
Proof. We map each minimal derivation M^g of A^g from assumptions $C^g_1, \ldots, C^g_n$

$$\begin{array}{c} {u_1}^g\colon {C_1}^g \ldots {u_n}^g\colon {C_n}^g \\ \mid M^g \\ A^g \end{array}$$

to the following classical derivation $(M^g)_c$ of A from assumptions $C_1, \ldots, C_n$:



Notice that the above function is not defined by recursion (why this is not possible?). Moreover, if $M^g\colon A^g \to B^g$, then $(M^g)_c\colon A \to B$. Despite this "functorial" behaviour of the function ${}_c$, we cannot define a functor $\mathbf{Form} \to \mathbf{Form}_c$, having ${}_c$ as its 1-part (why?). Consequently, the following compositions are defined



$$M_c \overset{g}{\longmapsto} (M_c)^g \overset{c}{\longmapsto} [(M_c)^g]_c,$$
$$M^g \overset{c}{\longmapsto} (M^g)_c \overset{c}{\longmapsto} [(M^g)_c]^g.$$

Corollary 2.12.9. *If* $\Gamma \subseteq \texttt{Form}$ *and* $A \in \texttt{Form}$, *then* $\Gamma^g \vdash A^g \Rightarrow \Gamma \vdash_c A$.

Proof. We proceed as in the proof of Corollary 2.12.1. $\square$

The following translation is a variation of the Gödel-Gentzen translation.

Definition 2.12.10. *The Kolmogorov translation is the unique mapping*

$$^k : \texttt{Form} \to \texttt{Form}$$

$$A \mapsto A^k$$

defined by recursion on Form *through the following clauses:*

$$\begin{aligned}
\bot^k &= \bot,\\
R^k &= \neg\neg R, \quad R \in \texttt{Rel}^{(0)} \setminus \{\bot\},\\
(R(t_1, \ldots, t_n) &= \neg\neg R(t_1, \ldots, t_n), \quad R \in \texttt{Rel}^{(n)}, n \in \mathbb{N}^+, t_1, \ldots, t_n \in \texttt{Term},\\
(A \Box B)^k &= \neg\neg(A^k \Box B^k), \quad \Box \in \{\to, \wedge, \vee\},\\
(\triangle_x A)^k &= \neg\neg(\triangle_x A^k), \quad \triangle \in \{\forall, \exists\}.
\end{aligned}$$

If $\Gamma \subseteq \texttt{Form}$, *let* $\Gamma^k = \{C^k \mid C \in \Gamma\}$.

Proposition 2.12.11. (i) $\texttt{Form}^k \nsubseteq \texttt{Form}^*$.
(ii) $\forall_{A\in\texttt{Form}}\big(\vdash (A^g \leftrightarrow A^k)\big)$.
(iii) $\forall_{A\in\texttt{Form}}\big(\vdash \neg\neg A^k \to A^k)\big)$.
(iv) *The set of formulas for which there is a minimal derivation of their double-negation-elimination is not included in* $\texttt{Form}^-$.

Proof. (i) Clearly $(A \vee B)^k, (\exists_x A)^k \notin \texttt{Form}^*$.
(ii) and (iii) are exercises.
(iv) It follows from (i) and (iii) and the fact that $\texttt{Form}^- \subset \texttt{Form}^*$. □

Corollary 2.12.12. (i) *The Kolmogorov translation defines a functor* $K\colon \boldsymbol{Form}_c \to \boldsymbol{Form}$

$$K_0(A) = A^k,$$

$$K_1(M_c\colon A \to B)\colon A^k \to B^k.$$

(ii) *If* $\Gamma \subseteq \texttt{Form}$ *and* $A \in \texttt{Form}$*, then* $\Gamma \vdash_c A \Rightarrow \Gamma^k \vdash A^k$.
(iii) *If* $\Gamma \subseteq \texttt{Form}$ *and* $A \in \texttt{Form}$*, then* $\Gamma \vdash A \Rightarrow \Gamma^k \vdash A^k$.

Proof. Exercise. □

The Gödel-Gentzen translation was introduced from Gödel in [10], and independently from Gentzen in [8]. The Kolmogorov translation was introduced even earlier in [13], but it was not known neither to Gödel nor to Gentzen.

2.13 The Gödel-Gentzen translation as a continuous function

Definition 2.13.1. *If* $A \in \texttt{Form}$*, we define the set*

$$O_A = \{C \in \texttt{Form} \mid A \vdash C\} = \{C \in \texttt{Form} \mid \vdash A \to C\}.$$

Lemma 2.13.2. *Let* $A, C \in \texttt{Form}$.
(i) $A \in O_A$.
(ii) $C \in O_A \Leftrightarrow O_C \subseteq O_A$.
(iii) $\vdash A \leftrightarrow C \Leftrightarrow O_C = O_A$.

Proof. (i) Since $\vdash A \to A$, we get $A \in O_A$.
(ii) Let $D \in O_C$ i.e., $\vdash C \to D$. Since by hypothesis we also have that $\vdash A \to C$, then by the $\texttt{cut}$-rule of Proposition 1.12.1 for $\Gamma = \Delta = \{A\}$

$$\frac{\{A\} \vdash C, \quad \{A\} \cup \{C\} \vdash D}{\{A\} \vdash D}\,\texttt{cut}$$

we get $\vdash A \to D$ i.e., $D \in O_A$. If $O_C \subseteq O_A$, then by (i) we get $C \in O_C$, hence $C \in O_A$.
(iii) By (ii) $O_C = O_A$ is equivalent to $C \in O_A$ and $A \in O_C$, hence to $\vdash A \leftrightarrow C$. □

Proposition 2.13.3. *The collection of sets*

$$\mathcal{B} = \{O_A \mid A \in \texttt{Form}\} \cup \{\emptyset, \texttt{Form}\}$$

is a basis for a topology $\mathcal{T}(\mathcal{B})$ *on* $\texttt{Form}$.

Proof. For this it suffices to show[3] that if $A, B, C \in \texttt{Form}$ such that $C \in O_A \cap O_B$, there is some $D \in \texttt{Form}$ such that

$$C \in O_D \subseteq O_A \cap O_B.$$

The hypothesis $C \in O_A \cap O_B$ implies that $A \vdash C$ and $B \vdash C$ i.e., $C \in O_A$ and $C \in O_B$, hence by Lemma 2.13.2(ii) we get $O_C \subseteq O_A$ and $O_C \subseteq O_B$. Hence $C \in O_C \subseteq O_A \cap O_B$. □

We denote the resulting topological space as $\mathcal{F} = (\texttt{Form}, \mathcal{T}(\mathcal{B}))$. It is easy to see that this space does not behave well with respect to the separation properties. E.g., it is not T_1, since $A \wedge A$ is in the complement $\texttt{Form} \setminus \{A\}$ of $\{A\}$, which is not open; if there was some $C \in \texttt{Form}$ such that $A \wedge A \in O_C \subseteq \texttt{Form} \setminus \{A\}$, then $O_{A \wedge A} \subseteq O_C \subseteq \texttt{Form} \setminus \{A\}$, but $A \in O_{A \wedge A}$ and $A \notin \texttt{Form} \setminus \{A\}$.

Proposition 2.13.4. *The Gödel-Gentzen translation* $^g : \texttt{Form} \to \texttt{Form}$ *and the Kolmogorov translation* $^k : \texttt{Form} \to \texttt{Form}$ *are continuous functions from* $\mathcal{F}$ *to* $\mathcal{F}$.

Proof. We prove the continuity of the Gödel-Gentzen translation and, because of Corollary 2.12.12(ii), the proof of the continuity of the Kolmogorov translation is similar.

By definition, a function $f : X \to Y$ between two topological spaces X, Y is continuous, if the inverse image $f^{-1}(O)$ of every open set O in Y is open in X. If $\mathcal{B}$ is a basis for Y, it is easy to see that f is continuous if and only if the inverse image $f^{-1}(B)$ of every basic open set B in $\mathcal{B}$ is open in X. Clearly, ${}^{g^{-1}}(\texttt{Form}) = \texttt{Form} \in \mathcal{T}(\mathcal{B})$ and ${}^{g^{-1}}(\emptyset) = \emptyset \in \mathcal{T}(\mathcal{B})$. If $A \in \texttt{Form}$,

$${}^{g^{-1}}(O_A) = \{B \in \texttt{Form} \mid B^g \in O_A\} = \{B \in \texttt{Form} \mid \ A \vdash B^g\}.$$

Let $B \in {}^{g^{-1}}(O_A)$ i.e., $A \vdash B^g$. We show that

$$B \in O_B \subseteq {}^{g^{-1}}(O_A),$$

hence the set ${}^{g^{-1}}(O_A)$ is open, as the union of the open sets O_B, for every $B \in {}^{g^{-1}}(O_A)$. The membership $B \in O_B$ follows from Lemma 2.13.2(i). Next we fix some $C \in O_B$ i.e., $\vdash B \to C$, and we show that $C \in {}^{g^{-1}}(O_A)$ i.e., $A \vdash C^g$. By Corollary 2.12.1(ii) we get

$$\vdash B \to C \Rightarrow \vdash B^g \to C^g,$$

hence the following derivation tree

$$\dfrac{\begin{matrix} \\ \mid M^g \\ B^g \to C^g \end{matrix} \qquad \begin{matrix} A \\ \mid N \\ B^g \end{matrix}}{C^g} \to^-$$

is a derivation $A \vdash C^g$. □

[3] Here we use the fact that if a collection $\mathcal{B}$ of subsets of some set X satisfies the property: "for every $x \in X$ and $B_i, B_j \in \mathcal{B}$ with $x \in B_i \cap B_j$, there is some $B_k \in \mathcal{B}$ such that $x \in B_k \subseteq B_i \cap B_j$", then $\mathcal{B}$ is a basis for some topology $\mathcal{T}(\mathcal{B})$ on X. This topology $\mathcal{T}(\mathcal{B})$ is unique and the smallest topology on X that includes $\mathcal{B}$ (see [7], Theorem 3.2).

Chapter 3

Models

It is an obvious question to ask whether the logical rules we have been considering suffice i.e., whether we have forgotten some necessary rules. To answer this question we first have to fix the *meaning* of a formula i.e., provide a semantics for the syntax developed in the previous chapters. This will be done here by means of fan models. Using this concept of a model we will prove soundness and completeness.

3.1 Trees, fans, and spreads

Definition 3.1.1. *Let X be an inhabited set i.e., a set with a given element (such a set is non-empty set in a strong, positive sense). We define*

$$X^n = \begin{cases} \{\emptyset\} & , n = 0 \\ \mathbb{F}(\{0, \dots, n-1\}, X) & , n > 0, \end{cases}$$

where $\mathbb{F}(\{0, \dots, n-1\}, X)$ denotes the set of functions u from $\{0, \dots, n-1\}$ to X. Such a function is also understood as an n-tuple of elements of X i.e.,

$$u = \big(u(0), u(1), \dots, u(n-1)\big) = \big(u_0, u_1, \dots, u_{n-1}\big),$$

and we call u a node of elements of X, or a node of $X^{<\mathbb{N}}$, where

$$X^{<\mathbb{N}} = \bigcup_{n \in \mathbb{N}} X^n.$$

We also use the symbol $\langle\rangle$ for the empty node. The length $|u|$ of a node of $X^{<\mathbb{N}}$ is defined by

$$|u| = \begin{cases} 0 & , u = \emptyset \\ n & , u \in X^n \ \&\ n > 0. \end{cases}$$

If $u, w \in X^{<\mathbb{N}}$, the relation "u is a (strict) initial segment of w" is defined by

$$u \prec w \Leftrightarrow |u| < |w| \ \&\ \forall_{i \in \{0, \dots, |u|-1\}}(u_i = w_i).$$

If $u \prec w$, then w is a (proper, or strict) successor of u. The relation $u \preceq w \Leftrightarrow u \prec w$ or $u = w$ is a partial order. If $u, w \in X^{<\mathbb{N}} \setminus \{\emptyset\}$, their concatenation $u * w$ is the node

$$u * w = (u_0, \dots u_{|u|-1}, w_0, \dots, w_{|w|-1}).$$

If one of them is the empty node, then their concatenation is the other node. A sequence of elements of X is an element $\alpha \in X^{\mathbb{N}} = \mathbb{F}(\mathbb{N}, X)$, and if $n \in \mathbb{N}$, the n-th initial part $\bar{\alpha}(n)$ of α is defined by

$$\bar{\alpha}(n) = \begin{cases} \emptyset & , n = 0 \\ (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) & , n > 0. \end{cases}$$

A tree T on X is a subset of $X^{<\mathbb{N}}$, which is closed under initial segments i.e.,

$$\forall_{u,w \in X^{<\mathbb{N}}}(w \in T \ \& \ u \prec w \Rightarrow u \in T).$$

An element of T is called a node of T. An infinite path of T is a sequence α of elements of X i.e., $\alpha \in X^{\mathbb{N}}$ such that

$$\forall_{n \in \mathbb{N}}(\bar{\alpha}(n) \in T).$$

The body $[T]$ of T is the set of its infinite paths. If $u \in T$, the set $\mathrm{Succ}(u)$ *of immediate successor nodes of u is defined by*

$$\mathrm{Succ}(u) = \{w \in T \mid u \prec w \ \& \ |w| = |u| + 1\}.$$

A tree T is (in)finite, if T is an (in)finite set. A tree Tit is called well-founded, if it has no infinite path. A tree T is called finitely branching, or a fan, if $\mathrm{Succ}(u)$ *is a finite set, for every $u \in T$. Otherwise, T is called infinitely branching.*

Example 3.1.2. The set $X^{<\mathbb{N}}$ is a tree on X. Its body $\left[X^{<\mathbb{N}}\right]$ is the set $X^{\mathbb{N}}$.

Example 3.1.3. If $X = \mathbb{N}$, the tree $\mathbb{N}^{<\mathbb{N}}$ on $\mathbb{N}$ is called the *Baire tree*. Its body $\left[N^{<\mathbb{N}}\right]$ is the set $\mathbb{N}^{\mathbb{N}}$, which is called the *Baire space*. Clearly, the Baire tree is infinitely branching.

Example 3.1.4. If $X = 2 = \{0, 1\}$, the tree $2^{<\mathbb{N}}$ on $\mathbb{N}$ is called the *Cantor tree*. Its body $\left[2^{<\mathbb{N}}\right]$ is the set $2^{\mathbb{N}}$, which is called the *Cantor space*. Clearly, the Cantor tree is a fan.

Trivially, $\emptyset \prec u$, for every $u \in X^{<\mathbb{N}} \setminus \{\emptyset\}$, while a tree T on X is inhabited if and only if $\emptyset \in T$. Notice that a node of a tree may have more than one immediate successors, but it has always a unique immediate predecessor (defined in the obvious way). If T is a finite tree on X, then, trivially, T is well-founded, but the converse is not true.

Proposition 3.1.5. (i) *There is a well-founded, infinite tree.*

(ii) *An infinite fan F has an infinite path.*

Proof. The proof of (i) is an exercise. The proof of (ii) uses classical logic. If $u \in F$, let u be "good", if there are infinitely many nodes $w \in F$ with $u \prec w$. Let u be "bad", if u is not good. What we want follows from the observation that if all immediate successor nodes of u are bad, then u is also bad. The completion of the proof is an exercise. □

Definition 3.1.6. *A binary relation $R \subseteq X \times X$ on X has an infinite descending chain, if there is $\alpha \in X^{\mathbb{N}}$ such that $\forall_{n \in \mathbb{N}}(\alpha_{n+1} \ R \ \alpha_n)$ i.e.,*

$$\ldots \alpha_3 \ R \ \alpha_2 \ R \ \alpha_1 \ R \ \alpha_0,$$

and $\prec$ is called well-founded, if it has no infinite descending chain.

Clearly, $<_{\mathbb{N}}$ is a well-founded relation on $\mathbb{N}$, while $<_{\mathbb{Z}}$ is not a well-founded relation on $\mathbb{Z}$. If T is a well-founded tree on X, the relation $w \ R \ u \Leftrightarrow u \prec w$ is well-founded relation on T.

Proposition 3.1.7 (Well-founded induction). *If $\prec$ is a well-founded relation on X, then*

$$\Big(\forall_{x\in X}\big(\forall_{y\in X}(y \prec x \Rightarrow P(y)) \Rightarrow P(x)\big)\Big) \Rightarrow \forall_{x\in X}(P(x)).$$

Proof. Suppose that there is some $x \in X$ such that $\neg P(x)$. This implies the (classical) existence of some $x_1 \prec x$ such that $\neg P(x_1)$. By repeating this step (and using some form of the axiom of choice), we get that $\prec$ has an infinitely descending chain, which contradicts our hypothesis. □

Definition 3.1.8. *Let T be a tree on some inhabited set X. A leaf of T is a node of T without proper successors (equivalently, without immediate successors). We denote by* $\mathrm{Leaf}(T)$ *the set of leaves of T. We call T a spread, if* $\mathrm{Leaf}(T) = \emptyset$*, or equivalently, if every node of T has an immediate successor*[1]*. A subtree T' of T, in symbols $T' \leq T$, is a subset T' of T which is also a tree on X. A branch A of T is a linearly ordered subtree of T i.e.,*

$$\forall_{u,w\in A}(u \preceq w \text{ or } w \preceq u).$$

A finite path of T is a finite branch of T. A bar B of a spread S on X is some $B \subseteq S$, such that every infinite path of S "hits" the bar B i.e.,

$$\forall_{\alpha\in[S]}\exists_{n\in\mathbb{N}}\big(\bar{\alpha}(n) \in B\big).$$

If $\bar{\alpha}(n) \in B$, we say that the infinite path α hits the bar B at the node $\bar{\alpha}(n)$. A bar B of S is called uniform, if there is a uniform bound on the length of the initial part of an infinite path that hits B i.e.,

$$\exists_{n\in\mathbb{N}}\forall_{\alpha\in[S]}\exists_{m\leq n}\big(\bar{\alpha}(m) \in B\big).$$

Clearly, a(n infinite) path is an infinite branch.

Example 3.1.9. *The Baire and the Cantor tree are spreads, and for every $n \in N$ the sets*

$$B_n = \{u \in 2^{<\mathbb{N}} \mid |u| = n\}$$

are uniform bars of $2^{<\mathbb{N}}$. Note that $B_0 = \{\emptyset\}$ is a uniform bar of every spread.

Proposition 3.1.10. *A tree T on X is a subtree of a spread S on X.*

Proof. Since X is inhabited by some x_0, we define

$$S = T \cup \bigcup_{u\in\mathrm{Leaf}(T)} u(x_0),$$

$$u(x_0) = \{u * (\underbrace{x_0, x_0, \ldots, x_0}_{n}) \mid n \in \mathbb{N}^+\}$$

where $u * (\underbrace{x_0, x_0, \ldots, x_0})$ is the concatenation of u and the node $(x_0, x_0, \ldots, x_0)$. It is immediate to see that S is a spread having T as a subtree. □

[1] Clearly, the body of a spread is always non-empty.

Proposition 3.1.11. *Let F be a fan on an inhabited set X, and G a fan and a spread on X.*
(i) *If all branches of F are finite, then F has a branch of maximal length.*
(ii) *If B is a bar of G, then B is uniform.*

Proof. The proof of (i) rests on Proposition 3.1.5(ii). For the proof of both (i) and (ii) we use classical reasoning. □

Proposition 3.1.12. *Let X, Y be inhabited sets, and let F be a a fan on X and G a fan on Y such that F, G are spreads.*
(i) *If $u \in F$, and if $B(u) = \{\alpha \in [F] \mid u \prec \alpha\}$, where $u \prec \alpha \Leftrightarrow \exists_{n\in\mathbb{N}}(\bar{\alpha}(n) = u)$, then the family $\{B(u) \mid u \in F\} \cup \{\emptyset\}$ is a basis for a topology T_F on $[F]$.*
(ii) *Let $\phi : F \to G$ satisfying the following properties:*

$$\forall_{u,w\in F}(u \preceq w \Rightarrow \phi(u) \preceq \phi(w)),$$

$$\forall_{\alpha\in[F]}\big(\lim_{n\to+\infty} |\phi(\bar{\alpha}(n))| = +\infty\big).$$

Then, the function $[\phi] : [F] \to [G]$, defined by

$$[\phi](\alpha) = \bigvee_{n\in\mathbb{N}} \phi(\bar{\alpha}(n)),$$

where $u \vee w = \sup_{\preceq}\{u, w\}$, is continuous with respect to the topologies T_F and T_G.

Proof. Exercise. □

3.2 Fan models

For the rest $\mathcal{L}$ is a countable formal language i.e., the sets $\texttt{Rel}, \texttt{Fun}$ are countable.

Definition 3.2.1. *Let the following sets*

$$\boldsymbol{n} = \begin{cases} \emptyset & , n = 0 \\ \{\mathbf{0}, \dots, \boldsymbol{n-1}\} & , n > 0. \end{cases}$$

If D is an inhabited set, the set $D^{\mathbf{n}} = \mathbb{F}(\boldsymbol{n}, D)$ of all functions $f : \mathbf{n} \to D$ can be identified with the product set D^n. Moreover, we define

$$\texttt{Rel}^{(n)}(D) = \mathcal{P}(D^{\mathbf{n}}),$$

$$\texttt{Rel}(D) = \bigcup_{n\in\mathbb{N}} \texttt{Rel}^{(n)}(D),$$

$$\texttt{Fun}^{(n)}(D) = \mathbb{F}(D^{\mathbf{n}}, D),$$

$$\texttt{Fun}(D) = \bigcup_{n\in\mathbb{N}} \texttt{Fun}^{(n)}(D).$$

If $n > 0$, an element of $\texttt{Rel}^{(n)}(D)$ is a relation on D of arity n, and an element of $\texttt{Fun}^{(n)}(D)$ is a function $f : D^n \to D$. Since $D^{\mathbf{0}} = \{\emptyset\}$, we get $\texttt{Rel}^{(0)}(D) = \mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\} = \mathbf{2}$. The value $\mathbf{0} = \emptyset$ represents falsity, and the value $\mathbf{1} = \{\emptyset\}$ represents truth. Moreover, the set $\texttt{Fun}^{(0)}(D) = \mathbb{F}(\{\emptyset\}, D)$ can be identified with D.

Definition 3.2.2. *A* fan model *of $\mathcal{L}$ is a structure $\mathcal{M} = (D, F, X, \mathbf{i}, \mathbf{j})$ satisfying the following clauses:*

(i) D, X are inhabited sets. We may also use the notation $|\mathcal{M}|$ for D.

(ii) *F is a fan on X.*

(iii) $\mathbf{i} : \mathtt{Fun} \to \mathtt{Fun}(D)$ *such that for every $n \in \mathbb{N}$*

$$\mathbf{i}_n = \mathbf{i}_{|\mathtt{Fun}^{(n)}} : \mathtt{Fun}^{(n)} \to \mathtt{Fun}^{(n)}(D).$$

(iv) $\mathbf{j} : \mathtt{Rel} \times F \to \mathtt{Rel}(D)$ *such that for every $n \in \mathbb{N}$*

$$\mathbf{j}_n = \mathbf{j}_{|\mathtt{Rel}^{(n)} \times F} : \mathtt{Rel}^{(n)} \times F \to \mathtt{Rel}^{(n)}(D),$$

and for every $R \in \mathtt{Rel}$ the following monotonicity condition is satisfied:

$$\forall_{u,w \in F} \big(u \preceq w \Rightarrow \mathbf{j}(R, u) \subseteq \mathbf{j}(R, w)\big).$$

We also write

$$R^{\mathcal{M}}(\vec{d}, u) \Leftrightarrow \vec{d} \in \mathbf{j}(R, u),$$

where $\vec{d} = (d_1, \ldots, d_n)$ and $\mathbf{j}(R, u) \in \mathtt{Rel}^{(n)}(D)$.

From the above definition of a fan model we notice the following:

- If $n = 0$ and $f \in \mathtt{Fun}^{(0)} = \text{Const}$, we have that $\mathbf{i}(f) \in \mathtt{Fun}^{(0)}(D)$ i.e.,

$$\mathbf{i}(f) \in D\ .$$

- If $n > 0$ and $f \in \mathtt{Fun}^{(n)}$, we have that $\mathbf{i}(f) \in \mathtt{Fun}^{(n)}(D)$ i.e.,

$$\mathbf{i}(f) : D^n \to D\ .$$

- If $n = 0$, $u \in F$ and $R \in \mathtt{Rel}^{(0)}$, we have that $\mathbf{j}(R, u) \in \mathtt{Rel}^{(0)}(D)$ i.e.,

$$\mathbf{j}(R, u) \in \mathbf{2},$$

  hence $\mathbf{j}(R, u)$ is either true or false.
- We set no special requirement on the value $\mathbf{j}(\bot, u) \in \mathbf{2}$, as minimal logic places no particular constraints on falsum $\bot$.
- If $n > 0$, $u \in F$ and $R \in \mathtt{Rel}^{(n)}$, we have that $\mathbf{j}(R, u) \in \mathtt{Rel}^{(n)}(D)$ i.e.,

$$\mathbf{j}(R, u) \text{ is an } n\text{-ary relation on } D.$$

If $\mathcal{M} = (D, F, X, \mathbf{i}, \mathbf{j})$ is a fan model of $\mathcal{L}$, we can give the following interpretations:

* A node $u \in F$ is interpreted as a "possible world", and its length $|u|$ is its "level".
* The relation $u \prec w$ is interpreted as: "the possible world w is a possible future of the possible world u".

* If $R \in \mathtt{Rel}^{(0)}$, the monotonicity condition of $\mathbf{j}_R \colon F \to \mathbf{2}$, defined by the rule

$$u \mapsto \mathbf{j}_R(u) = \mathbf{j}(R, u),$$

for every $u \in F$, is interpreted as: "if R is true at u, it is true at w", since, if $\mathbf{j}(R, u) = \emptyset$, then we always have that $\mathbf{j}(R, u) \subseteq \mathbf{j}(R, w)$, while if $\mathbf{j}(R, u) = \{\emptyset\}$, the monotonicity $\mathbf{j}(R, u) \subseteq \mathbf{j}(R, w)$, implies that $\mathbf{j}(R, w) = \{\emptyset\}$ too.

The next fact explains why no generality is lost if the fan in a fan model is a spread.

Proposition 3.2.3. *If $\mathcal{M} = (D, F, X, \mathbf{i}, \mathbf{j})$ is a fan model of $\mathcal{L}$, there is a fan model $\mathcal{M}^* = (D, S, X, \mathbf{i}, \mathbf{j}^*)$ of $\mathcal{L}$ such that S is a spread on X.*

Proof. If $x_0 \in X$, we consider S to be the spread of Proposition 3.1.10 on X. We then define

$$\mathbf{j}^*\big(R, u * \underbrace{(x_0, x_0, \ldots, x_0}_{n}\big) = \mathbf{j}(R, u); \qquad n \in \mathbb{N},\ u \in \mathrm{Leaf}(F),$$

$$\mathbf{j}^*(R, u) = \mathbf{j}(R, u); \qquad u \notin \mathrm{Leaf}(F).$$

By case distinction on the nodes of S it is straightforward to show that $\mathbf{j}^*$ satisfies the monotonicity condition, and hence $\mathcal{M}^* = (D, S, X, \mathbf{i}, \mathbf{j}^*)$ is a fan model of $\mathcal{L}$. □

3.3 The Tarski-Beth definition of truth in a fan model

The first step in the assignment of a mathematical meaning to a formula of $\mathcal{L}$, in the sense of Tarski and Beth, is to associate an element of an inhabited set to every variable of $\mathcal{L}$.

Definition 3.3.1. *If D is a set inhabited by d_0, a variable assignment in D is a map*

$$\eta : \mathtt{Var} \to D.$$

We denote by $[x_1 \mapsto d_1, \ldots, x_n \mapsto d_n]$ the variable assignment defined by

$$[x_1 \mapsto d_1, \ldots, x_n \mapsto d_n](x) = \begin{cases} d_i & , x = x_i \in \{x_1, \ldots, x_n\} \\ d_0 & , x \notin \{x_1, \ldots, x_n\}. \end{cases}$$

It might be that $d_i = d_j$, for some $i, j \in \{1, \ldots, n\}$. If $\eta \in \mathbb{F}(\mathtt{Var}, D)$ and $d \in D$, let η_x^d be the variable assignment in D defined by η and d as follows[2]*:*

$$\eta_x^d(y) = \begin{cases} \eta(y) & , \text{ if } y \neq x, \\ d & , \text{ if } y = x. \end{cases}$$

The next step is to associate an element of D to every term of $\mathcal{L}$. This we can do with the use of an assignment routine and a fixed fan model of $\mathcal{L}$.

[2] If we use classical logic in our metatheory, then the use of this instance of the principle of the excluded middle, $x = y$ or $\neg(x = y)$, is legitimate. If we use constructive logic though, we need to equip the set of variables $\mathtt{Var}$ of $\mathcal{L}$ with a decidable equality i.e., an equality satisfying such a disjunction.

Definition 3.3.2. *Let $\mathcal{M} = (D, F, X, \mathbf{i}, \mathbf{j})$ be a fan model of $\mathcal{L}$, and let η be a variable assignment in D. The term assignment in D generated by $\mathcal{M}$ and η is the function*

$$\eta_{\mathcal{M}} : \texttt{Term} \to D,$$

defined by recursion on Term *through the following clauses:*

$$\begin{aligned}
\eta_{\mathcal{M}}(x) &= \eta(x), \\
\eta_{\mathcal{M}}(c) &= \mathbf{i}(c), \\
\eta_{\mathcal{M}}(f(t_1, \ldots, t_n)) &= \mathbf{i}(f)(\eta_{\mathcal{M}}(t_1), \ldots, \eta_{\mathcal{M}}(t_n)),
\end{aligned}$$

for every $x \in \texttt{Var}$, $c \in \text{Const}$, $f \in \texttt{Fun}^{(n)}$, $t_1, \ldots, t_n \in \texttt{Term}$ and $n \in \mathbb{N}^+$. We often write

$$t^{\mathcal{M}}[\eta] = \eta_{\mathcal{M}}(t),$$

and when $\mathcal{M}$ is fixed, we may even use the same symbol $\eta(t)$ for $\eta_{\mathcal{M}}(t)$. If $\vec{t} \in \texttt{Term}^{<\mathbb{N}}$, let

$$\eta_{\mathcal{M}}(\vec{t}) = \begin{cases} \emptyset & , \text{ if } \vec{t} = \emptyset, \\ (\eta_{\mathcal{M}}(t_0), \ldots, \eta_{\mathcal{M}}(t_{|\vec{t}|-1})) & , \text{ if } \vec{t} = (t_0, \ldots, t_{|\vec{t}|-1}). \end{cases}$$

Now we are ready to formulate the Tarski-Beth definition of truth of a formula of $\mathcal{L}$ in a fan model of $\mathcal{L}$. In the rest of this chapter we use the following notation for some formula ϕ of our metalanguage (the language of our metatheory):

$$\forall_{u' \succeq_n u}(\phi) \Leftrightarrow \forall_{u' \succeq u}(|u'| = |u| + n \Rightarrow \phi).$$

Definition 3.3.3 (Tarski, Beth)**.** *Let $\mathcal{M} = (D, F, X, \mathbf{i}, \mathbf{j})$ be a fan model of $\mathcal{L}$, such that F is a spread. We define inductively the relation*

"the formula A is true in $\mathcal{M}$ at the node u under the variable assignment η", or "u forces A under η in $\mathcal{M}$",

in symbols

$$\mathcal{M}, u \Vdash A[\eta], \quad \big(\text{or simpler } u \Vdash A[\eta]\big),$$

by the following rules[3]*:*

$$\begin{aligned}
u \Vdash (R(\vec{t}))[\eta] &\Leftrightarrow \exists_{n \in \mathbb{N}} \forall_{u' \succeq_n u}\Big(R^{\mathcal{M}}(\vec{t}^{\mathcal{M}}[\eta], u')\Big), \\
u \Vdash (A \vee B)[\eta] &\Leftrightarrow \exists_{n \in \mathbb{N}} \forall_{u' \succeq_n u}(u' \Vdash A[\eta] \text{ or } u' \Vdash B[\eta]), \\
u \Vdash (\exists_x A)[\eta] &\Leftrightarrow \exists_{n \in \mathbb{N}} \forall_{u' \succeq_n u} \exists_{d \in D}\big(u' \Vdash A[\eta_x^d]\big), \\
u \Vdash (A \to B)[\eta] &\Leftrightarrow \forall_{u' \succeq u}(u' \Vdash A[\eta] \Rightarrow u' \Vdash B[\eta]), \\
u \Vdash (A \wedge B)[\eta] &\Leftrightarrow u \Vdash A[\eta] \;\&\; u \Vdash B[\eta], \\
u \Vdash (\forall_x A)[\eta] &\Leftrightarrow \forall_{d \in D}\big(u \Vdash A[\eta_x^d]\big).
\end{aligned}$$

If $A_1, \ldots, A_n \in \texttt{Form}$, we also use the notation

$$u \Vdash \{A_1, \ldots, A_n\}[\eta] :\Leftrightarrow u \Vdash A_1[\eta] \;\&\; \ldots \;\&\; u \Vdash A_n[\eta].$$

[3]These rules, which are written as equivalences, are pairs of inductive rules, in the usual sense i.e., in the first rule of the pair the formula on the left is the nominator and the formula on the right is the denominator, while in the second rule of the pair it is the other way around.

In this definition, the logical connectives $\to, \wedge, \vee, \forall, \exists$ on the left hand side are part of the object language $\mathcal{L}$, whereas the same connectives on the right hand side are to be understood in the usual sense: they belong to the metalanguage. It should always be clear from the context whether a formula is part of the object or the metalanguage. Regarding the Beth-Tarski definition of truth, we make the following remarks.

- If $R \in \mathtt{Rel}^{(0)}$ and $\vec{t} = \emptyset$, then by the first clause of the definition

$$\begin{aligned} u \Vdash R[\eta] &\Leftrightarrow \exists_{n\in\mathbb{N}}\forall_{u'\succeq_n u}\Big(R^{\mathcal{M}}(\emptyset^{\mathcal{M}}[\eta], u')\Big) \\ &\Leftrightarrow \exists_{n\in\mathbb{N}}\forall_{u'\succeq_n u}\Big(R^{\mathcal{M}}(\emptyset, u')\Big) \\ &\Leftrightarrow \exists_{n\in\mathbb{N}}\forall_{u'\succeq_n u}(\emptyset \in \mathbf{j}(R, u') \\ &\Leftrightarrow \exists_{n\in\mathbb{N}}\forall_{u'\succeq_n u}(\mathbf{j}(R, u') = \mathbf{1}). \end{aligned}$$

  If $R \in \mathtt{Rel}^{(n)}$, for some $n > 0$, then

$$\begin{aligned} u \Vdash (R(\vec{t}))[\eta] &\Leftrightarrow \exists_{n\in\mathbb{N}}\forall_{u'\succeq_n u}\Big(R^{\mathcal{M}}(\vec{t}^{\mathcal{M}}[\eta], u')\Big) \\ &\Leftrightarrow \exists_{n\in\mathbb{N}}\forall_{u'\succeq_n u}(\vec{t}^{\mathcal{M}}[\eta] \in \mathbf{j}(R, u') \\ &\Leftrightarrow \exists_{n\in\mathbb{N}}\forall_{u'\succeq_n u}((\eta_{\mathcal{M}}(t_0), \ldots, \eta_{\mathcal{M}}(t_{|\vec{t}|-1})) \in \mathbf{j}(R, u')). \end{aligned}$$

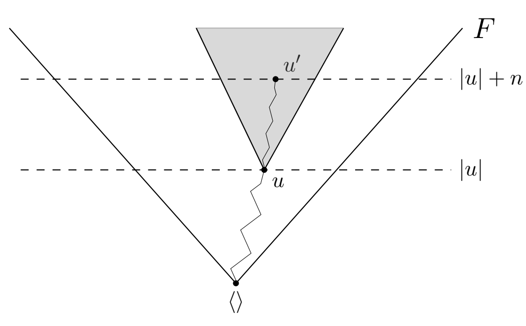
  Hence, R (or $R(\vec{t})$) is true in $\mathcal{M}$ at u under η if and only if there is a level of possible worlds in F such that R is true in $\mathcal{M}$ at all possible future worlds of u of that level under η. If $u \Vdash (R(\vec{t}))[\eta]$, and if

$$S_F(u) = \{w \in F \mid u \preceq w\} \cup \{w \in F \mid w \preceq u\},$$

  then

$$B_F(u) = \big\{w \in S_F(u) \mid R^{\mathcal{M}}(\vec{t}^{\mathcal{M}}[\eta], w)\big\}$$

  is a uniform bar of the spread subfan $S_F(u)$ of F, with $|u| + n$ as a uniform bound.



- The formula $A \vee B$ is true in $\mathcal{M}$ at u under η if and only if for every possible future u' of u of level $|u| + n$ either A is true at u' or B is true at u', for some $n \in \mathbb{N}$.
- The formula $A \to B$ is true in $\mathcal{M}$ at u under η if and only if for every possible future u' of u if A is true in $\mathcal{M}$ at u' under η, then B is true in $\mathcal{M}$ at u' under η.

- The formula $\forall_x A$ is true in $\mathcal{M}$ at u under η if and only if the formula A is true in $\mathcal{M}$ at u under η_x^d, for every $d \in D$. E.g., if $A = R(x)$, where $R \in \mathtt{Rel}^{(1)}$. If $d \in D$, then

$$u \Vdash (R(x))[\eta_x^d] \Leftrightarrow \exists_{n\in\mathbb{N}}\forall_{u' \succeq_n u}\big(\eta_x^d(x) \in \mathbf{j}(R, u')\big)$$
$$\Leftrightarrow \exists_{n\in\mathbb{N}}\forall_{u' \succeq_n u}(d \in \mathbf{j}(R, u'))$$

i.e., there is a level of future words of u such that $d \in \mathbf{j}(R, u')$, and this is the case for every $d \in D$. As any possible interpretation d of x is in all $\mathbf{j}(R, u')$, for some level of possible future worlds of u, it is natural to define then that $\forall_x R(x)$ is true in $\mathcal{M}$ under η. The use of η_x^a in the definition of $u \Vdash (\exists_x A)[\eta]$ and $u \Vdash (\forall_x A)[\eta]$ reflects that no capture occurs when we infer $u \Vdash (\exists_x A)[\eta]$ and $u \Vdash (\forall_x A)[\eta]$ from $\exists_{n\in\mathbb{N}}\forall_{u'\succeq_n u}\exists_{d\in D}(u' \Vdash A[\eta_x^d])$ and $\forall_{d\in D}(u \Vdash A[\eta_x^d])$, respectively.

Proposition 3.3.4 (Extension). *Let $A \in \mathtt{Form}$, $\mathcal{M} = (D, F, X, \mathbf{i}, \mathbf{j})$ be a fan model of $\mathcal{L}$, F is a spread, η a variable assignment in D, and $u, w \in F$. Then*

$$u \preceq w \ \& \ u \Vdash A[\eta] \Rightarrow w \Vdash A[\eta].$$

Proof. Exercise. □

Proposition 3.3.5. *Let $\mathcal{M} \equiv (D, F, X, \mathbf{i}, \mathbf{j})$ be a fan model of $\mathcal{L}$, F is a spread, η a variable assignment in D, and $A, B \in \mathtt{Form}$.*

(i) *The set*

$$[\![A]\!]_{\mathcal{M},\eta} = \big\{\alpha \in [F] \mid \exists_{n\in\mathbb{N}}\big(\bar{\alpha}(n) \Vdash A[\eta]\big)\big\}$$

is open in T_F, where the topology T_F on $[F]$ is defined in Proposition 3.1.12.

(ii) *The following hold:*

$$[\![A \wedge B]\!]_{\mathcal{M},\eta} = [\![A]\!]_{\mathcal{M},\eta} \cap [\![B]\!]_{\mathcal{M},\eta},$$
$$[\![A \vee B]\!]_{\mathcal{M},\eta} = [\![A]\!]_{\mathcal{M},\eta} \cup [\![B]\!]_{\mathcal{M},\eta}.$$

Proof. Exercise. □

Next proposition is a kind of converse to Proposition 3.3.4. According to it, in order to infer the truth of A at some node u from the truth of A in the possible future u' of u, we need to know that A is true at all future-nodes of u' of some level above (or equal to) the level of u'.

Proposition 3.3.6 (Covering). *If $A \in \mathtt{Form}$, $\mathcal{M} = (D, F, X, \mathbf{i}, \mathbf{j})$ is a fan model of $\mathcal{L}$, F is a spread, and η is a variable assignment in D, then*

$$\big[\exists_{n\in\mathbb{N}}\forall_{u'\succeq_n u}\big(u' \Vdash A[\eta]\big)\big] \Rightarrow u \Vdash A[\eta].$$

Proof. By induction on $\mathtt{Form}$. *Case* $R(\vec{s})$. Assume $\forall_{u'\succeq_n u}\big(u' \Vdash (R(\vec{s}))[\eta]\big)$. Since F is a fan, there are finitely many nodes u' such that $u' \succeq_n u$. Let their set be $N = \{u_1, \ldots, u_l\}$. By definition we have that for each $u_k \in N$

$$\exists_{n_k\in\mathbb{N}}\forall_{w_k\succeq_{n_k} u_k}\big(R^{\mathcal{M}}(\vec{s}^{\mathcal{M}}[\eta], w_k)\big).$$

Let $m = \max\{n_1, \ldots, n_l\}$. Then we have that

$$\forall_{w\succeq_{n+m} u}\bigg(R^{\mathcal{T}}(\vec{s}^{\mathcal{T}}[\eta], w)\bigg),$$

hence by the corresponding clause of the Tarski-Beth definition we get $u \Vdash \big(R(\vec{s})[\eta]\big)$. For this we argue as follows. If $w \succeq_{n+m} u$, then $w \succeq w_k \succeq_{n_k} u_k$, for some $k \in \{1, \ldots, l\}$. Since by hypothesis, $\eta_{\mathcal{M}}(\vec{s}) \in \mathbf{j}(R, w_k)$, by the monotonicity of $\mathbf{j}_R$ we get $\eta_{\mathcal{M}}(\vec{s}) \in \mathbf{j}(R, w) \Leftrightarrow R^{\mathcal{M}}(\vec{s}^{\mathcal{M}}[\eta], w)$. The cases $A \vee B$ and $\exists_x A$ are handled similarly.

Case $A \to B$. Let $N = \{u_1, \ldots, u_l\}$ be the set of all $u' \succeq u$ with $|u'| = |u| + n$ such that $u' \Vdash (A \to B)[\eta]$. We show that

$$\forall_{w \succeq u}(w \Vdash A[\eta] \Rightarrow w \Vdash B[\eta]).$$

Let $w \succeq u$ and $w \Vdash A[\eta]$. We must show $w \Vdash B[\eta]$. If $|w| \geq |u| + n$, then $w \succeq u_k$, for some $k \in \{1, \ldots, l\}$. Hence, by the hypothesis on u_k and the definition of $u_k \Vdash (A \to B)[\eta]$, we get $w \Vdash B[\eta]$. If $|u| \leq |w| < |u| + n$, then by Proposition 3.3.4 for the set N' of all elements u_j of N that extend w we have that each $u_j \Vdash A[\eta]$. Hence, we also have that $u_j \Vdash B[\eta]$. But N' is the set of all successors of w with length $|w| + m$, where $m = |u| + n - |w|$. By the induction hypothesis on the formula B, we get the required $w \Vdash B[\eta]$. The cases $A \wedge B$ and $\forall_x A$ are straightforward to show. □

3.4 Soundness of minimal logic

Lemma 3.4.1 (Coincidence). *Let* $\mathcal{M} = (D, F, X, \mathbf{i}, \mathbf{j})$ *be a fan model of* $\mathcal{L}$*,* $t \in \texttt{Term}$*,* $A \in \texttt{Form}$*, and* η*,* ξ *variable assignments in* D*.*

(i) *If* $\eta(x) = \xi(x)$ *for all* $x \in \mathrm{FV}(t)$*, then* $\eta_{\mathcal{M}}(t) = \xi_{\mathcal{M}}(t)$*.*

(ii) *If* $\eta(x) = \xi(x)$ *for all* $x \in \mathrm{FV}(A)$*, then* $\mathcal{M}, u \Vdash A[\eta]$ *if and only if* $\mathcal{M}, u \Vdash A[\xi]$*.*

Proof. By Induction on `Term` and `Form`, respectively. The details are left to the reader. □

Lemma 3.4.2 (Substitution). *Let* $\mathcal{M} = (D, F, X, \mathbf{i}, \mathbf{j})$ *be a fan model of* $\mathcal{L}$*,* $t, r(x) \in \texttt{Term}$*,* $A(x) \in \texttt{Form}$ *with* $\mathrm{Free}_{t,x}(A) = 1$*, and* η *a variable assignment in* D*.*

(i) $\eta_{\mathcal{M}}(r(t)) = \eta_x^{\eta_{\mathcal{M}}(t)}(r(x))$*.*

(ii) $\mathcal{M}, u \Vdash A(t)[\eta]$ *if and only if* $\mathcal{M}, u \Vdash A(x)[\eta_x^{\eta_{\mathcal{M}}(t)}]$*.*

Proof. By Induction on `Term` and `Form`, respectively. The details are left to the reader. □

Next theorem expresses that minimal derivations are sound with respect the Beth-Tarski notion of truth of a formula in a fan model i.e., they respect truth in a fan model.

Theorem 3.4.3 (Soundness of minimal logic). *Let* $\mathcal{M} = (D, F, X, \mathbf{i}, \mathbf{j})$ *be a fan model of* $\mathcal{L}$*,* $u \in F$ *and* η *a variable assignment in* D*. If* $M \in \mathfrak{D}_V(A)$ *such that* $u \Vdash \{C_1, \ldots, C_n\}[\eta]$*, where* $\{C_1, \ldots, C_n\} = \texttt{Form}(V)$*, then* $u \Vdash A[\eta]$*.*

Proof. We fix $\mathcal{M}$ and we prove by induction on derivations the formula

$$\forall_{M \in \mathfrak{D}_V(A)}\Big(\forall_{\eta \in \mathbb{F}(\texttt{Var}, D)} \forall_{u \in F}\big(u \Vdash \texttt{Form}(V)[\eta] \Rightarrow u \Vdash A[\eta]\big)\Big).$$

Case 1_A. The validity of $u \Vdash A[\eta] \Rightarrow u \Vdash A[\eta]$ is immediate.

Case $\to^+$. Let the derivation

$$\dfrac{\begin{array}{c}[A],\ C_1,\ldots,C_n\\ \mid N\\ B\end{array}}{A \to B}\to^+$$

and suppose $u \Vdash \{C,\ldots,C_n\}[\eta]$. We show $u \Vdash (A \to B)[\eta] \Leftrightarrow \forall_{u' \succeq u}(u' \Vdash A[\eta] \Rightarrow u' \Vdash B[\eta])$ under the inductive hypothesis on N:

$$\mathrm{IH}(N): \quad \forall_\eta \forall_w (w \Vdash \{A, C_1,\ldots,C_n\}[\eta] \Rightarrow w \Vdash B[\eta]).$$

We fix u' such that $u' \succeq u$ and we suppose $u' \Vdash A[\eta]$. By Extension (Proposition 3.3.4) we get $u' \Vdash \{C_1,\ldots,C_n\}[\eta]$, hence $u' \Vdash \{A, C_1,\ldots,C_n\}[\eta]$. Hence, by IH($N$) we get $u' \Vdash B[\eta]$.

Case ($\to^-$). Let the derivation

$$\dfrac{\begin{array}{c}C_1,\ldots,C_n\\ \mid N\\ A \to B\end{array} \qquad \begin{array}{c}D_1,\ldots,D_m\\ \mid K\\ A\end{array}}{B}\to^-$$

and suppose $u \Vdash \{C_1,\ldots,C_n, D_1,\ldots,D_m\}[\eta]$. We show $u \Vdash B[\eta]$ under the inductive hypotheses on N and K:

$$\mathrm{IH}(N): \quad \forall_\eta \forall_w (w \Vdash \{C_1,\ldots,C_n\}[\eta] \Rightarrow w \Vdash (A \to B)[\eta]),$$

$$\mathrm{IH}(K): \quad \forall_\eta \forall_w (w \Vdash \{D_1,\ldots,D_m\}[\eta] \Rightarrow w \Vdash A[\eta]).$$

By IH(N) we have that $u \Vdash (A \to B)[\eta]$, and by IH(K) we get $u \Vdash A[\eta]$, hence $u \Vdash B[\eta]$.

Case ($\forall^+$) Let the derivation

$$\dfrac{\begin{array}{c}C_1,\ldots,C_n\\ \mid N\\ A\end{array}}{\forall_x A}\forall^+ x$$

with the variable condition $x \notin \mathrm{FV}(C_1)$ & … & $x \notin \mathrm{FV}(C_n)$, and suppose $u \Vdash \{C_1,\ldots,C_n\}[\eta]$. We show $u \Vdash (\forall_x A)[\eta] \Leftrightarrow \forall_{d \in D}(u \Vdash A[\eta_x^d])$ under the inductive hypothesis on N:

$$\mathrm{IH}(N): \quad \forall_\eta \forall_w (w \Vdash \{C_1,\ldots,C_n\}[\eta] \Rightarrow w \Vdash A[\eta]).$$

Let $d \in D$. By the variable condition we get $\eta_{|\mathrm{FV}(C_i)} = (\eta_x^d)_{|\mathrm{FV}(C_i)}$, for every $i \in \{1,\ldots,n\}$, hence by Coincidence (Lemma 3.4.1) we conclude that $u \Vdash \{C_1,\ldots,C_n\}[\eta_x^d]$. By IH($N$) on η_x^d and u we get $u \Vdash A[\eta_x^d]$.

Case ($\forall^-$). Let the derivation

$$\dfrac{\begin{array}{c}C_1,\ldots,C_n\\ \mid N\\ \forall_x A\end{array} \qquad r \in \texttt{Term}}{A(r)}\forall^-$$

and suppose $u \Vdash \{C_1,\ldots,C_n\}[\eta]$. We show $u \Vdash A(r)[\eta]$ under the inductive hypotheses on N:

$$\mathrm{IH}(N): \quad \forall_\eta \forall_w (w \Vdash \{C_1,\ldots,C_n\}[\eta] \Rightarrow w \Vdash (\forall_x A)[\eta]).$$

Applying IH(N) on u we get $\forall_{d\in D}(u \Vdash A[\eta_x^d])$. If we consider $d = \eta_{\mathcal{M}}(r)$, we get $u \Vdash A\big[\eta_x^{\eta_{\mathcal{M}}(r)}\big]$, and by Substitution (Lemma 3.4.2) we conclude that $u \Vdash A(r)[\eta]$.

Case $(\wedge^+)$ and *Case* $(\wedge^-)$ are straightforward.

Case $(\vee_0^+)$ and *Case* $(\vee_1^+)$ are straightforward.

Case $(\vee^-)$. Let the derivation

$$\dfrac{\begin{array}{c} C_1,\dots,C_n \\ \mid N \\ A\vee B \end{array} \qquad \begin{array}{c} [A],D_1,\dots,D_m \\ \mid K \\ C \end{array} \qquad \begin{array}{c} [B],E_1,\dots,E_l \\ \mid L \\ C \end{array}}{C}\ \vee^-$$

and suppose $u \Vdash \{C_1,\dots,C_n,D_1,\dots,D_m,E_1,\dots,E_l\}[\eta]$. We show $u \Vdash C[\eta]$ under the inductive hypotheses on N, K and L:

$$\text{IH}(N):\quad \forall_\eta\forall_w(w \Vdash \{C_1,\dots,C_n\}[\eta] \Rightarrow w \Vdash (A\vee B)[\eta]),$$

$$\text{IH}(K):\quad \forall_\eta\forall_w(w \Vdash \{A,D_1,\dots,D_m\}[\eta] \Rightarrow w \Vdash C[\eta]),$$

$$\text{IH}(L):\quad \forall_\eta\forall_w(w \Vdash \{B,E_1,\dots,E_l\}[\eta] \Rightarrow w \Vdash C[\eta]).$$

By IH(N) we get $u \Vdash (A\vee B)[\eta] \Leftrightarrow \exists_n\forall_{u'\succeq_n u}(u' \Vdash A[\eta]$ or $u' \Vdash B[\eta])$. By Covering (Proposition 3.3.6) it suffices to show for this $n\in\mathbb{N}$:

$$\forall_{u'\succeq_n u}(u' \Vdash C[\eta]).$$

We fix u' such that $u' \succeq_n u$. If $u' \Vdash A[\eta]$, then by Extension and IH(K) we get $u' \Vdash C[\eta]$. If $u' \Vdash B[\eta]$, then by Extension and IH(L) we get $u' \Vdash C[\eta]$.

Case $(\exists^+)$ is straightforward.

Case $(\exists^-)$. Let the derivation

$$\dfrac{\begin{array}{c} C_1,\dots,C_n \\ \mid N \\ \exists_x A \end{array} \qquad \begin{array}{c} [A],D_1,\dots,D_m \\ \mid K \\ B \end{array}}{B}\ \exists^- x$$

with the variable condition $x \notin \text{FV}(D_1)$ & … & $x \notin \text{FV}(D_m)$, and $x \notin \text{FV}(B)$, and suppose $u \Vdash \{C,\dots,C_n,D_1,\dots,D_m\}$. We show $u \Vdash B[\eta]$ under the inductive hypotheses on N, K:

$$\text{IH}(N):\quad \forall_\eta\forall_w(w \Vdash \{C_1,\dots,C_n\}[\eta] \Rightarrow w \Vdash (\exists_x A)[\eta]),$$

$$\text{IH}(K):\quad \forall_\eta\forall_w(w \Vdash \{A,D_1,\dots,D_m\}[\eta] \Rightarrow w \Vdash B[\eta]).$$

By IH(N) we get that $u \Vdash (\exists_x A)[\eta] \Leftrightarrow \exists_n\forall_{u'\succeq_n u}\exists_{d\in D}(u' \Vdash A[\eta_x^d])$. By Covering it suffices to show for this $n\in\mathbb{N}$:

$$\forall_{u'\succeq_n u}(u' \Vdash B[\eta]).$$

We fix u' such that $u' \succeq_n u$, and let $d \in D$ such that $u' \Vdash A[\eta_x^d]$. Since by the variable condition we get $\eta_{|FV(D_i)} = (\eta_x^d)_{|FV(D_i)}$, and since by Extension $u' \Vdash \{D_1,\dots,D_m\}[\eta]$, by Coincidence we get $u' \Vdash \{A,D_1,\dots,D_m\}[\eta_x^d]$. By IH($K$) on η_x^d and u' we get $u' \Vdash B[\eta_x^d]$. Since by the variable condition we get $\eta_{|FV(B)} = (\eta_x^d)_{|FV(B)}$, we conclude that $u' \Vdash B[\eta]$. $\square$

Corollary 3.4.4. *Let $\Gamma \cup \{A\} \subseteq \texttt{Form}$ such that $\Gamma \vdash A$. If $\mathcal{M} = (D, F, X, \mathbf{i}, \mathbf{j})$ is a fan model of $\mathcal{L}$, $u \in F$ and η is a variable assignment in D, the following hold:*
(i) $\mathcal{M}, u \Vdash \Gamma[\eta] \Rightarrow \mathcal{M}, u \Vdash A[\eta]$.
(ii) *If $\Gamma = \emptyset$, then $u \Vdash A[\eta]$, and $[\![A]\!]_{\mathcal{M},\eta} = [F]$.*
(iii) *The set*

$$[\![u]\!]_{\mathcal{M},\eta} = \{A \in \texttt{Form} \mid u \Vdash A[\eta]\}$$

is open in the topology $\mathcal{T}(\mathcal{B})$ on $\texttt{Form}$, defined in Proposition 2.13.3.

Proof. Exercise. □

3.5 Countermodels and intuitionistic fan models

The main application of the soundness theorem is its use in the proof of underivability results.

Definition 3.5.1. *A countermodel to some derivation $\Gamma \vdash A$ is a triple $(\mathcal{M}, \eta, u)$, where $\mathcal{M} = (D, F, X, \mathbf{i}, \mathbf{j})$ is a fan model of $\mathcal{L}$, η is a variable assignment in D, and $u \in F$ such that*

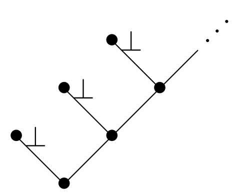
$$\mathcal{M}, u \Vdash \Gamma[\eta] \text{ and } \mathcal{M}, u \nVdash A[\eta].$$

By Corollary 3.4.4 of the soundness theorem, if $(\mathcal{M}, \eta, u)$ is a countermodel to the derivation $\Gamma \vdash A$, we can conclude that $\Gamma \nvdash A$, since if there was such a derivation we should have $\mathcal{M}, u \Vdash \Gamma[\eta] \Rightarrow \mathcal{M}, u \Vdash A[\eta]$, which contradicts the existence of a countermodel.

Example 3.5.2 (Consistency of minimal logic, or minimal underivability of falsum)**.** Suppose that there is a derivation $\vdash \bot$. By Corollary 3.4.4(ii), if $u = \langle\rangle = \emptyset$, we have that

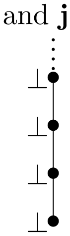
$$\langle\rangle \Vdash \bot[\eta] \Leftrightarrow \exists_n \forall_{u \in F}\big(|u| = n \Rightarrow \mathbf{j}(\bot, u) = \mathbf{1}\big).$$

To every node of the following fan we write all propositions forced at that node (the nodes where falsum is forced are considered to be extended, and at every extension-node falsum is also forced).



This is a fan model because monotonicity holds trivially. Clearly, the above condition is not satisfied, hence $\vdash$ is consistent. As minimal, intuitionistic, and classical logic are equiconsistent (Corollary 2.12.4), we conclude that intuitionistic and classical logic are also consistent.

Example 3.5.3 (Minimal underivability of Ex falsum)**.** A countermodel to the derivation $\vdash \bot \to R$, where $R \in \texttt{Rel}^{(0)} \setminus \{\bot\}$, is constructed as follows: take $F = \{x_0\}^{<\mathbb{N}}$, D any inhabited set, and define $\mathbf{j}(\bot, \emptyset) = \mathbf{1}$, and $\mathbf{j}(R, \emptyset) = \mathbf{0}$.



By extension we get $\mathbf{j}(\bot, u) = \mathbf{1}$, for every $u \in F$. Moreover, we get $\mathbf{j}(R, u) = \mathbf{0}$, for every $u \in F$; if there was some $u \in F \setminus \{\emptyset\}$ such that $\mathbf{j}(R, u) = \mathbf{1}$, then, since this is the only node $u' \in F$ such that $u' \succeq_{|u'|} \emptyset$, by Covering we would get $\mathbf{j}(R, \emptyset) = \mathbf{1}$ too. We show that $\emptyset \nVdash (\bot \to R)[\eta]$, where η is arbitrary. Suppose that $\emptyset \Vdash (\bot \to R)[\eta] \Leftrightarrow \forall_u (u \Vdash \bot[\eta] \Rightarrow u \Vdash R[\eta])$. For every $u \in F$ though, we have that $u \Vdash \bot[\eta]$ and $u \nVdash R[\eta]$.

Definition 3.5.4. *An intuitionistic fan model of a countable first-order language $\mathcal{L}$ is a fan model $\mathcal{M}_i = (D, F, X, \mathbf{i}, \mathbf{j})$ of $\mathcal{L}$ such that*

$$\forall_{u \in F} \big(\mathbf{j}(\bot, u) = \mathbf{0}\big).$$

It is easy to see that if $\mathcal{M}_i$ is an intuitionistic fan model, then

$$\mathcal{M}_i, u \Vdash (\bot \to A)[\eta],$$

for every $A, \in \texttt{Form}, u \in F$ and assignment η in D. Notice that an intuitionistic fan model provides an immediate proof that $\vdash_i$ is consistent, hence by Corolary 2.12.4 we get the consistency of $\vdash, \vdash_c$ once more. Notice that the fan model used in Example 3.5.2 is not intuitionistic.

Lemma 3.5.5. *A fan model $\mathcal{M} = (D, F, X, \mathbf{i}, \mathbf{j})$ of $\mathcal{L}$, where $F$is a spread, is intuitionistic if and only if*

$$\forall_\eta \forall_{u \in F} \big(u \nVdash \bot[\eta]\big).$$

Proof. Exercise. □

Proposition 3.5.6. *Let $\mathcal{M}_i = (D, F, X, \mathbf{i}, \mathbf{j})$ be an intuitionistic fan model of $\mathcal{L}$, η a variable assignment, $u \in F$ and $A \in \texttt{Form}$.*

(i) $u \Vdash (\neg A)[\eta] \Leftrightarrow \forall_{u' \succeq u} (u' \nVdash A[\eta])$.

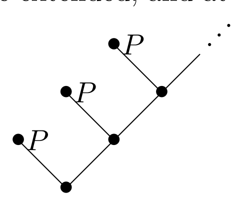
(ii) $u \Vdash (\neg\neg A)[\eta] \Leftrightarrow \forall_{u' \succeq u} \neg \forall_{u'' \succeq u'} (u'' \nVdash A[\eta])$.

Proof. Exercise. □

Definition 3.5.7. *An intuitionistic countermodel to some derivation $\Gamma \vdash_i A$ is a triple $(\mathcal{M}_i, \eta, u)$, where $\mathcal{M}_i = (D, F, X, \mathbf{i}, \mathbf{j})$ is an intuitionistic fan model, η is a variable assignment in D, and $u \in F$ such that $\mathcal{M}_i u \Vdash \Gamma[\eta]$ and $\mathcal{M}_i u \nVdash A[\eta]$.*

Since the soundness theorem of intuitionistic logic follows immediately from the soundness theorem of minimal logic, we can use it to conclude an intuitionistic underivability $\Gamma \nvdash_i A$ from an intuitionistic countermodel to $\Gamma \vdash_i A$.

Example 3.5.8 (Intuitionistic underivability of DNE)**.** We give an intuitionistic countermodel to the derivation $\vdash_i \neg\neg P \to P$. We describe the desired fan model by means of a diagram below. Next to every node we write all propositions forced at that node (again the nodes where P is forced are considered to be extended, and at every extension-node P is also forced).



This is a fan model because monotonicity clearly holds. Observe also that $\mathbf{j}(\bot, u) = \mathbf{0}$, for every node u i.e., it is an intuitionistic fan model, and moreover $\emptyset \nVdash P[\eta]$. Using Proposition 3.5.6(ii), it is easily seen that $\emptyset \Vdash (\neg\neg P)[\eta]$. Thus $\emptyset \nVdash (\neg\neg P \to P)[\eta]$, and hence $\nvdash_i (\neg\neg P \to P)$.

3.6 Completeness of minimal logic

Theorem 3.6.1 (Completeness of minimal logic). *Let $\Gamma \cup \{A\} \subseteq$ `Form`. The following are equivalent.*

(i) $\Gamma \vdash A$.

(ii) $\Gamma \Vdash A$, *i.e., for all fan models $\mathcal{M}$, assignments η in $|\mathcal{M}|$ and nodes u in the fan of $\mathcal{M}$*

$$\mathcal{M}, u \Vdash \Gamma[\eta] \Rightarrow \mathcal{M}, u \Vdash A[\eta].$$

Proof. (Harvey Friedman) Soundness of minimal logic already gives "(i) implies (ii)". The main idea in the proof of the other direction is the construction of a fan model $\mathcal{M}$ over the Cantor tree $2^{<\mathbb{N}}$ with domain D the set `Term` of all terms of the underlying language such that the following property holds:

$$\Gamma \vdash B \Leftrightarrow \mathcal{M}, \emptyset \Vdash B[\mathrm{id}_{\mathtt{Var}}].$$

We assume here that $\Gamma \cup \{A\}$ is a set of closed formulas. In order to define $\mathcal{M}$, we will need an enumeration $A_0, A_1, A_2, \ldots$ of the underlying language $\mathcal{L}$ (assumed countable), in which every formula occurs infinitely often. We also fix an enumeration $x_0, x_1, \ldots$ of distinct variables. Since Γ is countable it can we written $\Gamma = \bigcup_n \Gamma_n$ with finite sets Γ_n such that $\Gamma_n \subseteq \Gamma_{n+1}$. With every node $u \in 2^{<\mathbb{N}}$, we associate a finite set Δ_u of formulas and a set V_u of variables, by induction on the length of u. We write $\Delta \vdash_n B$ to mean that there is a derivation of height $\leq n$ of B from Δ.

Let $\Delta_\emptyset = \emptyset$ and $V_\emptyset = \emptyset$. Take a node u such that $|u| = n$ and suppose that Δ_u, V_u are already defined. We define Δ_{u*0}, V_{u*0} and Δ_{u*1}, V_{u*1} as follows:

Case 0. $\mathrm{FV}(A_n) \not\subseteq V_u$. Then let

$$\Delta_{u*0} = \Delta_{u*1} = \Delta_u \quad \text{and} \quad V_{u*0} = V_{u*1} = V_u.$$

Case 1. $\mathrm{FV}(A_n) \subseteq V_u$ and $\Gamma_n, \Delta_u \nvdash_n A_n$. Let

$$\Delta_{u*0} = \Delta_u \quad \text{and} \quad \Delta_{u*1} = \Delta_u \cup \{A_n\},$$
$$V_{u*0} = V_{u*1} = V_u.$$

Case 2. $\mathrm{FV}(A_n) \subseteq V_u$ and $\Gamma_n, \Delta_u \vdash_n A_n = A'_n \vee A''_n$. Let

$$\Delta_{u*0} = \Delta_u \cup \{A_n, A'_n\} \quad \text{and} \quad \Delta_{u*1} = \Delta_u \cup \{A_n, A''_n\},$$
$$V_{u*0} = V_{u*1} = V_u.$$

Case 3. $\mathrm{FV}(A_n) \subseteq V_u$ and $\Gamma_n, \Delta_u \vdash_n A_n = \exists_x A'_n(x)$. Let

$$\Delta_{u*0} = \Delta_{u*1} = \Delta_u \cup \{A_n, A'_n(x_i)\} \quad \text{and} \quad V_{u*0} = V_{u*1} = V_u \cup \{x_i\},$$

where x_i is the first variable $\notin V_u$.

Case 4. $\mathrm{FV}(A_n) \subseteq V_u$ and $\Gamma_n, \Delta_u \vdash_n A_n$, with A_n neither a disjunction nor an existentially quantified formula. Let

$$\Delta_{u*0} = \Delta_{u*1} = \Delta_u \cup \{A_n\} \quad \text{and} \quad V_{u*0} = V_{u*1} = V_u.$$

The following remarks (R1)-(R3) are clear.

(R1) Δ_u, V_u are finite sets.

(R2) $\mathrm{FV}(\Delta_u) \subseteq V_u$.

(R3) $u \preceq w \Rightarrow \Delta_u \subseteq \Delta_w$ and $V_u \subseteq V_w$.

(R4) $\forall_{x_i \in \mathtt{Var}} \exists_m \forall_{u \in 2^{<\mathbb{N}}} (|u| = m \Rightarrow x_i \in V_u)$.

Remark (R4) is shown as follows: Let the derivation $\vdash \exists_x(\bot \to \bot)$ with height m_0. Suppose that for every x_j with $j < i$, there is some m_j such that $\forall_{u \in 2^{<\mathbb{N}}}(|u| = m_j \Rightarrow x_j \in V_u)$. Let $n \geq \max\{m_0, m_1, \dots, m_{i-1}\}$ such that $A_n \Leftrightarrow \exists_x(\bot \to \bot)$ (this n can be found, as the formula $\exists_x(\bot \to \bot)$ occurs infinitely often in the fixed enumeration of formulas). Sinve $n \geq m_0$, if $|u| = n$, then $\Gamma_n, \Delta_u \vdash_n \exists_x(\bot \to \bot)$. By definition of n and (R3) we get that $x_1, \dots, x_{i-1} \in V_u$. If $x_i \in V_u$, then $x_i \in V_{u*j}$, with $j \in \mathbf{2}$. If $x_i \notin V_u$, and since $\mathrm{FV}(\exists_x(\bot \to \bot)) = \emptyset \subseteq V_u$, by Case 3 we have that $x_i \in V_{u*j}$, since x_i is the first variable in the fixed enumeration of $\mathtt{Var}$ that does not occur in V_u. Hence $m_i = n + 1$ satisfies the required property.

We also have the following:

$$\forall_{u' \succeq_n u} (\Gamma, \Delta_{u'} \vdash B) \Rightarrow \Gamma, \Delta_u \vdash B, \quad \text{provided } \mathrm{FV}(B) \subseteq V_u. \tag{3.1}$$

It is sufficient to show that, for $\mathrm{FV}(B) \subseteq V_u$,

$$(\Gamma, \Delta_{u*0} \vdash B) \wedge (\Gamma, \Delta_{u*1} \vdash B) \Rightarrow (\Gamma, \Delta_u \vdash B).$$

In cases 0, 1 and 4, this is obvious. For case 2, the claim follows immediately from the axiom schema $\vee^-$. In case 3, we have $\mathrm{FV}(A_n) \subseteq V_u$ and $\Gamma_n, \Delta_u \vdash_n A_n \Leftrightarrow \exists_x A'_n(x)$. Assume $\Gamma, \Delta_u \cup \{A_n, A'_n(x_i)\} \vdash B$ with $x_i \notin V_u$, and $\mathrm{FV}(B) \subseteq V_u$. Then $x_i \notin \mathrm{FV}(\Delta_u \cup \{A_n, B\})$, hence $\Gamma, \Delta_u \cup \{A_n\} \vdash B$ by $\exists^-$ and therefore $\Gamma, \Delta_u \vdash B$.

Next, we show

$$\Gamma, \Delta_u \vdash B \Rightarrow \exists_n \forall_{u' \succeq_n u} (B \in \Delta_{u'}), \quad \text{provided } \mathrm{FV}(B) \subseteq V_u. \tag{3.2}$$

Choose $n \geq |u|$ such that $B = A_n$ and $\Gamma_n, \Delta_u \vdash_n A_n$. For all $u' \succeq u$, if $|u'| = n + 1$ then $A_n \in \Delta_{u'}$ (we work as above for Cases 2-4).

Using the sets Δ_u we define the fan model $\mathcal{M} = (\mathtt{Term}, 2^{<\mathbb{N}}, 2, \mathbf{i}, \mathbf{j})$ as follows. If $f \in \mathtt{Fun}^{(n)}$, then $\mathbf{i}(f) : \mathtt{Term}^n \to \mathtt{Term}$ is defined by

$$\mathbf{i}(f)(t_1, \dots, t_n) = f(t_1, \dots, t_n).$$

Obviously, $t^{\mathcal{M}}[\mathrm{id}_{\mathtt{Var}}] = t$ for all $t \in \mathtt{Term}$. If $R \in \mathtt{Rel}^{(n)}$, then $\mathbf{j}(R, u) \subseteq \mathtt{Term}^n$ is defined by

$$\mathbf{j}(R, u) = \{(t_1, \dots, t_n) \in \mathtt{Term}^n \mid R(t_1, \dots, t_n) \in \Delta_u\}.$$

Hence, if $R \in \mathtt{Rel}^{(0)}$, $\mathbf{j}(R, u) = \mathbf{0}$, for every $u \in 2^{<\mathbb{N}}$. We write $u \Vdash B$ for $\mathcal{M}, u \Vdash B[\mathrm{id}_{\mathtt{Var}}]$, and we show:

CLAIM. $\Gamma, \Delta_u \vdash B \Leftrightarrow u \Vdash B$, provided $\mathrm{FV}(B) \subseteq V_u$.

The proof is by induction on the well-founded relation $C \lhd_* B$, "C is a proper Gentzen subformula[4] of B" (see Proposition 3.1.7). I.e., if

$$P(B) \Leftrightarrow \forall_u \big(\mathrm{FV}(B) \subseteq V_u \Rightarrow (\Gamma, \Delta_u \vdash B \Leftrightarrow u \Vdash B)\big),$$

[4]The relation "a formula B is a Gentzen subformula of the formula A" is defined inductively by the rules:

$$\frac{}{A \lhd A}\,(R) \qquad \frac{B \Box C \lhd A}{B \lhd A, \quad C \lhd A}\,(\Box \in \{\to, \wedge, \vee\}),$$

$$\frac{\triangle_x B \lhd A, \quad \mathrm{Free}_{s,x}(B) = 1}{B(s) \lhd A}\,(\triangle \in \{\exists, \forall\}),$$

we show by induction on Form that

$$\forall_{B\in\texttt{Form}}\big(\forall_{C\triangleleft_* B}(P(C)) \Rightarrow P(B)\big),$$

and we conclude that $\forall_{B\in\texttt{Form}}(P(B))$.

Case $R\vec{s}$. Assume $\mathrm{FV}(R\vec{s}) \subseteq V_u$. The following are equivalent:

$$\begin{array}{ll} \Gamma, \Delta_u \vdash R\vec{s}, & \\ \exists_n \forall_{u'\succeq_n u}\,(R\vec{s}\in\Delta_{u'}) & \text{by (3.2) and (3.1)}, \\ \exists_n \forall_{u'\succeq_n u}\,R^{\mathcal{M}}(\vec{s},u') & \text{by definition of } \mathcal{M}, \\ k \Vdash R\vec{s} & \text{by definition of } \Vdash, \text{ since } t^{\mathcal{M}}[\mathrm{id}_{\texttt{Var}}] = t. \end{array}$$

Case $B\vee C$. Assume $\mathrm{FV}(B\vee C)\subseteq V_u$. For the implication ($\Rightarrow$) let $\Gamma,\Delta_u \vdash B\vee C$. Choose an $n \geq |u|$ such that $\Gamma_n, \Delta_u \vdash_n A_n = B\vee C$. Then, for all $u'\succeq u$ such that $|u'| = n$,

$$\Delta_{u*0} = \Delta_{u'} \cup \{B\vee C, B\} \quad \text{and} \quad \Delta_{u'*1} = \Delta_{u'} \cup \{B \vee C, C\},$$

and therefore by hypothesis on B and C

$$u' * 0 \Vdash B \quad \text{and} \quad u' * 1 \Vdash C.$$

Then by definition we have $u \Vdash B\vee C$. For the reverse implication ($\Leftarrow$) we argue as follows:

$$\begin{array}{ll} u \Vdash B\vee C, & \\ \exists_n \forall_{u'\succeq_n u}(u' \Vdash B \vee u' \Vdash C), & \\ \exists_n \forall_{u'\succeq_n u}((\Gamma,\Delta_{u'} \vdash B)\vee(\Gamma,\Delta_{u'}\vdash C)) & \text{by hypothesis on } B, C, \\ \exists_n \forall_{u'\succeq_n u}\,(\Gamma,\Delta_{u'}\vdash B\vee C), & \\ \Gamma,\Delta_u \vdash B\vee C & \text{by (3.1)}. \end{array}$$

Case $B\wedge C$. This is easy.

Case $B\to C$. Assume $\mathrm{FV}(B\to C)\subseteq V_k$. For ($\Rightarrow$) let $\Gamma,\Delta_u\vdash B\to C$. We must show $u \Vdash B\to C$, i.e.,

$$\forall_{u'\succeq u}(u'\Vdash B \to u' \Vdash C).$$

Let $u'\succeq u$ be such that $u'\Vdash B$. By hypothesis on B, it follows that $\Gamma,\Delta_{u'}\vdash B$. Hence $\Gamma,\Delta_{u'}\vdash C$ follows by assumption. Then again by hypothesis on C we get $u'\Vdash C$.

For ($\Leftarrow$) let $u\Vdash B\to C$, i.e., $\forall_{u'\succeq u}(u'\Vdash B\to u'\Vdash C)$. We show that $\Gamma,\Delta_u\vdash B\to C$, using Choose $n\geq \mathrm{lh}k$ such that $B = A_n$. For all $u'\succeq_m u$ with $m = n-|u|$ we show that $\Gamma,\Delta_{u'}\vdash B\to C$.

If $\Gamma_n,\Delta_{u'}\vdash_n A_n$, then $u'\Vdash B$ by induction hypothesis, and $u'\Vdash C$ by assumption. Hence $\Gamma,\Delta_{u'}\vdash C$ again by hypothesis on C and thus $\Gamma,\Delta_{u'}\vdash B\to C$.

If $\Gamma_n,\Delta_{u'}\not\vdash_n A_n$, then by definition $\Delta_{u'*1} = \Delta_{u'}\cup\{B\}$. Hence $\Gamma,\Delta_{u'*1}\vdash B$, and thus $u'*1\Vdash B$ by hypothesis on B. Now $u'*1\Vdash C$ by assumption, and finally $\Gamma,\Delta_{u'*1}\vdash C$ by hypothesis on C. From $\Delta_{u'*1}=\Delta_{u'}\cup\{B\}$ it follows that $\Gamma,\Delta_{u'}\vdash B\to C$.

Case $\forall_x B(x)$. Assume $\mathrm{FV}(\forall_x B(x))\subseteq V_u$. For ($\Rightarrow$) let $\Gamma,\Delta_u\vdash\forall_x B(x)$. Fix a term t. Then $\Gamma,\Delta_u\vdash B(t)$. Choose $n\geq|k|$ such that $\mathrm{FV}(B(t))\subseteq V_{u'}$ for all u' with $|u'| = n$. Then $\forall_{u'\succeq_m u}\,(\Gamma,\Delta_{u'}\vdash B(t))$ with $m = n-|k|$, hence $\forall_{u'\succeq_m u}\,(u'\Vdash B(t))$ by hypothesis on $B(t)$, hence $u\Vdash B(t)$ by the covering lemma. This holds for every term t, hence $k\Vdash\forall_x B(x)$.

For ($\Leftarrow$) assume $u \Vdash \forall_x B(x)$. Pick $u' \succeq_n u$ such that $A_m \Leftrightarrow \exists_x(\bot \to \bot)$, for $m = |u| + n$. Then at height m we put some x_i into the variable sets: for $u' \succeq_n u$ we have $x_i \notin V_{u'}$ but $x_i \in V_{u'*j}$. Clearly $u' * j \Vdash B(x_i)$, hence $\Gamma, \Delta_{u'*j} \vdash B(x_i)$ by hypothesis on $B(x_i)$), hence (since at this height we consider the trivial formula $\exists_x(\bot \to \bot)$) also $\Gamma, \Delta_{u'} \vdash B(x_i)$. Since $x_i \notin V_{u'}$ we obtain $\Gamma, \Delta_{u'} \vdash \forall_x B(x)$. This holds for all $u' \succeq_n u$, hence $\Gamma, \Delta_u \vdash \forall_x B(x)$ by (3.1).

Case $\exists_x B(x)$. Assume $\mathrm{FV}(\exists_x B(x)) \subseteq V_u$. For ($\Rightarrow$) let $\Gamma, \Delta_u \vdash \exists_x B(x)$. Choose an $n \geq |u|$ such that $\Gamma_n, \Delta_u \vdash_n A_n = \exists_x B(x)$. Then, for all $u' \succeq u$ with $|u'| = n$

$$\Delta_{u'*0} = \Delta_{u'*1} = \Delta_{u'} \cup \{\exists_x B(x), B(x_i)\}$$

where $x_i \notin V_{u'}$. Hence by hypothesis on $B(x_i)$ (applicable since $\mathrm{FV}(B(x_i)) \subseteq V_{u'*j}$)

$$u' * 0 \Vdash B(x_i) \quad \text{and} \quad u' * 1 \Vdash B(x_i).$$

It follows by definition that $u \Vdash \exists_x B(x)$.

For ($\Leftarrow$) assume $u \Vdash \exists_x B(x)$. Then $\forall_{u' \succeq_n u} \exists_{t \in \mathtt{Term}} (u' \Vdash B(x)[(\mathrm{id}_{\mathtt{Var}})^t_x])$ for some n, hence $\forall_{u' \succeq_n u} \exists_{t \in \mathtt{Term}} (u' \Vdash B(t))$. For each of the finitely many $u' \succeq_n u$ pick an m such that $\forall_{u'' \succeq_m u'} (\mathrm{FV}(B(t)) \subseteq V_{u''})$. Let m_0 be the maximum of all these m. Then

$$\forall_{u'' \succeq_{m_0+n} u} \exists_{t \in \mathtt{Term}} ((u'' \Vdash B(t)) \wedge \mathrm{FV}(B(t)) \subseteq V_{u''}).$$

The hypothesis on $B(t)$ yields

$$\begin{aligned}
&\forall_{u'' \succeq_{m_0+n} k} \exists_{t \in \mathtt{Term}} (\Gamma, \Delta_{u''} \vdash B(t)), \\
&\forall_{u'' \succeq_{m_0+n} k} (\Gamma, \Delta_{u''} \vdash \exists_x B(x)), \\
&\Gamma, \Delta_u \vdash \exists_x B(x) \qquad\qquad \text{by (3.1)},
\end{aligned}$$

and this completes the proof of the claim.

Now we finish the proof of the completeness theorem by showing that (b) implies (a). We apply (b) to the tree model $\mathcal{M}$ constructed above from Γ, the empty node $\emptyset$ and the assignment $\eta = \mathrm{id}_{\mathtt{Var}}$. Then $\mathcal{M}, \emptyset \Vdash \Gamma[\mathrm{id}_{\mathtt{Var}}]$ by the claim (since each formula in Γ is derivable from Γ). Hence $\mathcal{M}, \emptyset \Vdash A[\mathrm{id}_{\mathtt{Var}}]$ by (b) and therefore $\Gamma \vdash A$ by the claim again. □

Completeness of intuitionistic logic follows as a corollary.

Corollary 3.6.2 (Completeness of intuitionistic logic)**.** *Let* $\Gamma \cup \{A\} \subseteq \mathtt{Form}$. *The following are equivalent:*

(i) $\Gamma \vdash_i A$.

(ii) $\Gamma, \mathrm{Efq} \Vdash A$, *i.e., for all intuitionistic fan models* $\mathcal{M}_i$, *assignments* η *in* $|\mathcal{M}_i|$ *and nodes* u *in the fan of* $\mathcal{M}_i$

$$\mathcal{M}_i, u \Vdash \Gamma[\eta] \Rightarrow \mathcal{M}_i, u \Vdash A[\eta].$$

Proof. It follows immediately from Theorem 3.6.1. □

3.7 $\mathcal{L}$-models and classical models

For the rest of this section, fix a countable formal language $\mathcal{L}$; we do not mention the dependence on $\mathcal{L}$ in the notation. Since we deal with classical logic, we only consider formulas built without $\vee, \exists$ i.e. formulas in $\texttt{Form}^*$ (see Definition 2.2.9). We define the notion of an $\mathcal{L}$-model, and what the value of a term and the meaning of a formula in an $\mathcal{L}$-model should be.

Definition 3.7.1. *An $\mathcal{L}$-model is a structure $\mathcal{M} = (D, \mathbf{i}, \mathbf{j})$, where*

(i) *D is an inhabited set.*

(ii) *For every n-ary function symbol f, $\mathbf{i}$ assigns to f a map $\mathbf{i}(f)\colon D^n \to D$.*

(iii) *For every n-ary relation symbol R, $\mathbf{j}$ assigns to R an n-ary relation on D^n. In case $n = 0$, $\mathbf{j}(R)$ is either true or false. We require that $\mathbf{j}(\bot)$ is false i.e., $\mathbf{j}(\bot) = \mathbf{0}$.*

We may write $|\mathcal{M}|$ for the carrier set D of $\mathcal{M}$ and $f^{\mathcal{M}}$, $R^{\mathcal{M}}$ for the interpretations $\mathbf{i}(f)$, $\mathbf{j}(R)$ of the function and relation symbols. Assignments η and their extensions on $\texttt{Term}$ are defined as in Section 3.2. We also write $t^{\mathcal{M}}[\eta]$ for $\eta_{\mathcal{M}}(t)$.

Definition 3.7.2 (Validity)**.** *For every $\mathcal{L}$-model $\mathcal{M} = (D, \mathbf{i}, \mathbf{j})$, assignment η in D and formula $A \in \texttt{Form}^*$ we define the relation "A is valid in $\mathcal{M}$ under the assignment η", in symbols $\mathcal{M} \models A[\eta]$ inductively, with respect only formulas without $\vee$ and $\exists$ as follows:*

$$\begin{aligned}
\mathcal{M} \models R[\eta] \quad & \Leftrightarrow \mathbf{j}(R) = \mathbf{1}; \quad R \in \texttt{Rel}^{(0)}, \\
\mathcal{M} \models (R\vec{s})[\eta] \quad & \Leftrightarrow R^{\mathcal{M}}(\vec{s}^{\mathcal{M}}[\eta]);; \quad R \in \texttt{Rel}^{(n)}, n > 0 \\
\mathcal{M} \models (A \to B)[\eta] & \Leftrightarrow ((\mathcal{M} \models A[\eta]) \Rightarrow (\mathcal{M} \models B[\eta])), \\
\mathcal{M} \models (A \wedge B)[\eta] & \Leftrightarrow ((\mathcal{M} \models A[\eta]) \;\&\; (\mathcal{M} \models B[\eta])), \\
\mathcal{M} \models (\forall_x A)[\eta] \quad & \Leftrightarrow \forall_{d \in D}(\mathcal{M} \models A[\eta_x^d]).
\end{aligned}$$

Since $\mathbf{j}(\bot)$ is false, we have $\mathcal{M} \not\models \bot[\eta]$.

Lemma 3.7.3 (Coincidence)**.** *Let $\mathcal{M} = (D, \mathbf{i}, \mathbf{j})$ be an $\mathcal{L}$-model, t a term, $A \in \texttt{Form}^*$, and η, ξ assignments in D.*

(i) *If $\eta(x) = \xi(x)$ for all $x \in \mathrm{FV}(t)$, then $\eta(t) = \xi(t)$.*

(ii) *If $\eta(x) = \xi(x)$ for all $x \in \mathrm{FV}(A)$, then $\mathcal{M} \models A[\eta]$ if and only if $\mathcal{M} \models A[\xi]$.*

Proof. By induction on $\texttt{Term}$ and on $\texttt{Form}^*$. □

Lemma 3.7.4 (Substitution)**.** *Let $\mathcal{M} = (D, \mathbf{i}, \mathbf{j})$ be an $\mathcal{L}$-model, $t, r(x)$ terms, $A(x) \in \texttt{Form}^*$, and η an assignment in D.*

(i) *$\eta(r(t)) = \eta_x^{\eta(t)}(r(x))$.*

(ii) *$\mathcal{M} \models A(t)[\eta]$ if and only if $\mathcal{M} \models A(x)[\eta_x^{\eta(t)}]$.*

Proof. By induction on $\texttt{Term}$ and on $\texttt{Form}^*$. □

Definition 3.7.5. *An $\mathcal{L}$-model $\mathcal{M}_c = (D, \mathbf{i}, \mathbf{j})$ is called classical, if for every $A \in \texttt{Form}^*$, and every assignment η in D we have that*

$$\mathcal{M}_c \models (\neg\neg A)[\eta] \Rightarrow \mathcal{M}_c \models A[\eta].$$

If the weaker classical derivation $\vdash_c^*$ is only considered, then for the *constructive* proof of completeness theorem of classical logic it suffices to assume for $\mathcal{M}_c$ that

$$\neg\neg R^{\mathcal{M}_c}(\vec{d}) \Rightarrow R^{\mathcal{M}_c}(\vec{d})$$

for all relation symbols R and all $\vec{d} \in D^{|\vec{d}|}$. If classical logic is used in our metatheory, then every $\mathcal{L}$-model is classical. To show this, we suppose that $\mathcal{M}_c \models (\neg\neg A)[\eta]$, and we show that $\mathcal{M}_c \models A[\eta]$ by showing $\neg\neg(\mathcal{M}_c \models A[\eta])$. For that, suppose $\neg(\mathcal{M}_c \models A[\eta])$. Then we get

$$\mathcal{M}_c \models (\neg A)[\eta] \Leftrightarrow \mathcal{M}_c \models (A \to \bot)[\eta] \Leftrightarrow \big(\mathcal{M}_c \models A[\eta] \Rightarrow \mathcal{M}_c \models \bot[\eta]\big),$$

as the premiss in the last implication is false by our second hypothesis. By our first hypothesis

$$\mathcal{M}_c \models (\neg\neg A)[\eta] \Leftrightarrow \mathcal{M}_c \models (\neg A \to \bot)[\eta] \Leftrightarrow \big(\mathcal{M}_c \models (\neg A)[\eta] \Rightarrow \mathcal{M}_c \models \bot[\eta]\big)$$

and since the premiss in the last implication holds, we get $\mathcal{M}_c \models \bot[\eta]$, which contradicts $\neg(\mathcal{M}_c \models \bot[\eta])$, hence we showed that $\neg\neg(\mathcal{M}_c \models A[\eta])$. With DNE we get $\mathcal{M}_c \models A[\eta]$. Moreover, one can show constructively (exercise) that

$$\mathcal{M}_c \models (\neg A)[\eta]) \Leftrightarrow \neg(\mathcal{M}_c \models A[\eta]).$$

3.8 Soundness theorem of classical logic

Theorem 3.8.1 (Soundness of classical logic)**.** *Let $\mathcal{M}_c = (D, \mathbf{i}, \mathbf{j})$ be a classical model of $\mathcal{L}$ and η a variable assignment in D. Let $\mathfrak{D}_V^{c,-}(A)$ be the set of classical derivations without the rules for $\vee$ and $\exists$. If $M_c \in \mathfrak{D}_V^{c,-}(A)$ such that $\mathcal{M}_c \models \{C_1, \ldots, C_n\}[\eta]$, where $\{C_1, \ldots, C_n\} =$ $\texttt{Form}(V)$, then $\mathcal{M}_c \models A[\eta]$.*

Proof. We fix $\mathcal{M}_c$ and we prove by induction the following formula

$$\forall_{M \in \mathfrak{D}_V^{c,-}(A)}\Big(\forall_{\eta \in \mathbb{F}(\texttt{Var},D)}\big(\mathcal{M}_c \models \texttt{Form}(V)[\eta] \Rightarrow \mathcal{M}_c \models A[\eta]\big)\Big).$$

Case DNE_A. It follows immediately from the classicality of $\mathcal{M}_c$.
Case 1_A. The validity of $\mathcal{M}_c \models A[\eta] \Rightarrow \mathcal{M}_c \models A[\eta]$ is immediate.
Case $\to^+$. Let the derivation

$$\dfrac{\begin{array}{c}[A],\ C_1, \ldots, C_n \\ \mid N \\ B\end{array}}{A \to B} \to^+$$

and suppose $\mathcal{M}_c \models \{C, \ldots, C_n\}[\eta]$. We show $\mathcal{M}_c \models (A \to B)[\eta] \Leftrightarrow \mathcal{M}_c \models A[\eta] \Rightarrow \mathcal{M}_c \models B[\eta]$ under the inductive hypothesis on N:

$$\text{IH}(N): \quad \forall_\eta(\mathcal{M}_c \models \{A, C_1, \ldots, C_n\}[\eta] \Rightarrow \mathcal{M}_c \models B[\eta]).$$

If $\mathcal{M}_c \models A[\eta]$, then $\mathcal{M}_c \models \{A, C_1, \ldots, C_n\}[\eta]$, hence by IH$(N)$ we get $\mathcal{M}_c \models B[\eta]$.
Case $(\to^-)$. Let the derivation

$$\dfrac{\begin{array}{c}C_1, \ldots, C_n \\ \mid N \\ A \to B\end{array} \qquad \begin{array}{c}D_1, \ldots, D_m \\ \mid K \\ A\end{array}}{B} \to^-$$

and suppose $\mathcal{M}_c \models \{C_1, \ldots, C_n, D_1, \ldots, D_m\}[\eta]$. We show $\mathcal{M}_c \models B[\eta]$ under the inductive hypotheses on N and K:

$$\text{IH}(N): \quad \forall_\eta\big(\mathcal{M}_c \models \{C_1, \ldots, C_n\}[\eta] \Rightarrow w\mathcal{M}_c \models (A \to B)[\eta]\big),$$

$$\text{IH}(K): \quad \forall_\eta\big(\mathcal{M}_c \models \{D_1, \ldots, D_m\}[\eta] \Rightarrow \mathcal{M}_c \models A[\eta]\big).$$

By IH(N) $\mathcal{M}_c \models (A \to B)[\eta]$, and by IH($K$) we get $\mathcal{M}_c \models A[\eta]$, hence $\mathcal{M}_c \models B[\eta]$.
Case ($\forall^+$) Let the derivation

$$\begin{array}{c} C_1, \ldots, C_n \\ \mid N \\ \dfrac{A}{\forall_x A}\ \forall^+ x \end{array}$$

with the variable condition $x \notin \text{FV}(C_1)$ & ... & $x \notin \text{FV}(C_n)$, and suppose $\mathcal{M}_c \models \{C_1, \ldots, C_n\}[\eta]$. We show $\mathcal{M}_c \models (\forall_x A)[\eta] \Leftrightarrow \forall_{d \in D}(\mathcal{M}_c \models A[\eta_x^d])$ under the inductive hypothesis on N:

$$\text{IH}(N): \quad \forall_\eta(\mathcal{M}_c \models \{C_1, \ldots, C_n\}[\eta] \Rightarrow \mathcal{M}_c \models A[\eta]).$$

Let $d \in D$. By the variable condition $\eta_{|\text{FV}(C_i)} = (\eta_x^d)_{|\text{FV}(C_i)}$, for every $i \in \{1, \ldots, n\}$, hence by Coincidence we conclude that $\mathcal{M}_c \models \{C_1, \ldots, C_n\}[\eta_x^d]$. By IH($N$) on η_x^d we get $\mathcal{M}_c \models A[\eta_x^d]$.
Case ($\forall^-$). Let the derivation

$$\begin{array}{c} C_1, \ldots, C_n \\ \mid N \\ \dfrac{\forall_x A \qquad\qquad r \in \texttt{Term}}{A(r)}\ \forall^- \end{array}$$

and let $\mathcal{M}_c \models \{C_1, \ldots, C_n\}[\eta]$. We show $\mathcal{M}_c \models A(r)[\eta]$ under the inductive hypotheses on N:

$$\text{IH}(N): \quad \forall_\eta(\mathcal{M}_c \models \{C_1, \ldots, C_n\}[\eta] \Rightarrow \mathcal{M}_c \models (\forall_x A)[\eta]).$$

By IH(N) we have that $\forall_{d \in D}(\mathcal{M}_c \models A[\eta_x^d])$. If we consider $d = \eta_{\mathcal{M}}(r)$, we get $\mathcal{M}_c \models A\big[\eta_x^{\eta_{\mathcal{M}}(r)}\big]$, and by Substitution we conclude that $\mathcal{M}_c \models A(r)[\eta]$.
Case ($\wedge^+$) and *Case* ($\wedge^-$) are straightforward. □

Corollary 3.8.2. *Let $\Gamma \cup \{A\} \subseteq \texttt{Form}^*$ such that $\Gamma \vdash_c A$. If $\mathcal{M}_c = (D, \mathbf{i}, \mathbf{j})$ is a classical model of $\mathcal{L}$, and η is a variable assignment in D, the following hold:*
(i) $\mathcal{M}_c \models \Gamma[\eta] \Rightarrow \mathcal{M}_c \models A[\eta]$.
(ii) *If $\Gamma = \emptyset$, then $\mathcal{M}_c \models A[\eta]$.*

Proof. Exercise. □

3.9 Completeness of classical logic

Theorem 3.9.1 (Completeness of classical logic)**.** *Let $\Gamma \cup \{A\} \subseteq \texttt{Form}^*$. Assume that*

$$\Gamma \models A$$

i.e., for all classical models $\mathcal{M}_c$ and assignments η in $|\mathcal{M}_c|$ we have that

$$\mathcal{M}_c \models \Gamma[\eta] \Rightarrow \mathcal{M}_c \models A[\eta].$$

Then "there must exist" a derivation of A from $\Gamma \cup \mathrm{Dne}$, in other words,

$$\neg\neg\big(\Gamma \cup \mathrm{Dne} \vdash A\big) \Leftrightarrow \neg\neg(\Gamma \vdash_c^* A).$$

Proof. (Ulrich Berger, with constructive logic) The proof is based on the proof of completeness of minimal logic. According to it, a contradiction is derived from the assumption $\Gamma \cup \mathrm{Dne} \nvdash A$. By the completeness theorem for minimal logic, there must be a fan model $\mathcal{M} = (\mathtt{Term}, 2^{<\mathbb{N}}, 2, \mathbf{i}, \mathbf{j})$ and a node u_0 such that $u_0 \Vdash \Gamma, \mathrm{Dne}$ and $u_0 \nVdash A$. The details of the proof are found in [19]. □

Since in the above proof the carrier set of the classical model in question is the countable set `Term`, the following holds immediately.

Remark 3.9.2. *The hypothesis $\Gamma \models A$ of completeness theorem can be replaced by*

$$\Gamma \models^{\aleph_0} A$$

i.e., "for all classical models $\mathcal{M}_c$ with a countable carrier set $|\mathcal{M}_c|$, for all assignments η, $\mathcal{M}_c \models \Gamma[\eta] \Rightarrow \mathcal{M}_c \models A[\eta]$".

Definition 3.9.3. *We call a classical models $\mathcal{M}_c$ with a countable carrier set $|\mathcal{M}_c|$ a countable (classical) model. Similarly, a finite model $\mathcal{M}_c$ is a model with a finite carrier set $|\mathcal{M}_c|$. In general, the cardinality of a classical model $\mathcal{M}_c$ is the cardinality of its carrier set $|\mathcal{M}_c|$.*

Corollary 3.8.2(i) of the soundness theorem for classical logic can take the form

$$\Gamma \vdash_c A \Rightarrow \Gamma \models A,$$

while the completeness theorem can be written as the implication

$$\Gamma \models A \Rightarrow \neg\neg(\Gamma \vdash_c^* A).$$

As the implication

$$\Gamma \vdash_c^* A \Rightarrow \Gamma \vdash_c A$$

implies constructively the implication

$$\neg\neg(\Gamma \vdash_c^* A) \Rightarrow \neg\neg(\Gamma \vdash_c A),$$

we get with constructive logic the implication

$$\Gamma \models A \Rightarrow \neg\neg(\Gamma \vdash_c A),$$

hence with classical logic we get the implication

$$\Gamma \models A \Rightarrow \Gamma \vdash_c A$$

i.e., the converse implication that expresses the soundness theorem for classical logic.

3.10 The compactness theorem

Definition 3.10.1. *A set of formulas* Γ *(included in* $\texttt{Form}^*$*) is consistent, if* $\Gamma \nvdash_c \bot$*, and it is satisfiable, if there is (in the weak sense) a classical model* $\mathcal{M}_c$ *and an assignment* η *in* $|\mathcal{M}_c|$ *such that* $\mathcal{M}_c \models \Gamma[\eta]$*. I.e.,*

$$\Gamma \textit{ is consistent } \Leftrightarrow \neg(\Gamma \vdash_c \bot),$$

$$\Gamma \textit{ is satisfiable } \Leftrightarrow \neg\neg\big(\exists_{\mathcal{M}_c}\exists_{\eta\in\mathbb{F}(\texttt{Var},|\mathcal{M}_c|)}(\mathcal{M}_c \models \Gamma[\eta])\big).$$

Notice that we use the equivalence between $\tilde{\exists}_x A$ and $\neg\neg\exists_x A$ in the above formulation of satisfiability (Proposition 2.6.2(vi). The consistency of Γ is a so-called *syntactical* notion, while satisfiability of Γ is a so-called *semantical* one. As classical logic is consistent, the empty set $\emptyset$ is consistent.

Corollary 3.10.2. *If* $\Gamma \subseteq \texttt{Form}^*$*, then* Γ *is consistent if and only if* Γ *is satisfiable.*

Proof. (with constructive logic) We show only that if Γ is consistent, then Γ is satisfiable, and the converse implication is an exercise. Assume $\Gamma \nvdash_c \bot$, and also assume that Γ is not satisfiable i.e.,

$$\neg\neg\neg\big(\exists_{\mathcal{M}_c}\exists_{\eta\in\mathbb{F}(\texttt{Var},|\mathcal{M}_c|)}(\mathcal{M}_c \models \Gamma[\eta])\big).$$

By Brouwer's theorem we get

$$\neg\big(\exists_{\mathcal{M}_c}\exists_{\eta\in\mathbb{F}(\texttt{Var},|\mathcal{M}_c|)}(\mathcal{M}_c \models \Gamma[\eta])\big),$$

which implies constructively

$$\forall_{\mathcal{M}_c}\forall_{\eta\in\mathbb{F}(\texttt{Var},|\mathcal{M}_c|)}(\mathcal{M}_c \not\models \Gamma[\eta])\big).$$

Hence, for every every classical model $\mathcal{M}_c$ and every assignment $\eta\colon \texttt{Var} \to |\mathcal{M}_c|$ we have that

$$\mathcal{M}_c \models \Gamma[\eta] \Rightarrow \mathcal{M}_c \models \bot[\eta].$$

By the completeness theorem for classical logic there must be a derivation $\Gamma \vdash_c \bot$ i.e.,

$$\neg\neg(\Gamma \vdash_c \bot).$$

This, together with the assumption $\neg(\Gamma \vdash_c \bot)$, lead to a contradiction. hence, we showed

$$\neg\neg\neg\neg\big(\exists_{\mathcal{M}_c}\exists_{\eta\in\mathbb{F}(\texttt{Var},|\mathcal{M}_c|)}(\mathcal{M}_c \models \Gamma[\eta])\big).$$

By Brouwer's theorem again we get

$$\neg\neg\big(\exists_{\mathcal{M}_c}\exists_{\eta\in\mathbb{F}(\texttt{Var},|\mathcal{M}_c|)}(\mathcal{M}_c \models \Gamma[\eta])\big)$$

i.e., Γ is satisfiable. □

Of course, the above proof is considerably simplified, if classical logic is used. Among the many important corollaries of the completeness theorem the compactness and Löwenheim-Skolem theorems stand out as particularly important. Although their classical proofs are much simpler, we also show these theorems constructively.

Corollary 3.10.3 (Compactness theorem)**.** *Let* $\Gamma \subseteq \texttt{Form}^*$*. If every finite subset of* Γ *is satisfiable, then* Γ *is satisfiable.*

Proof. (with constructive logic) Assume that Γ is not satisfiable i.e.,

$$\neg\neg\neg\big(\exists_{\mathcal{M}_c}\exists_{\eta\in\mathbb{F}(\texttt{Var},|\mathcal{M}_c|)}(\mathcal{M}_c \models \Gamma[\eta])\big).$$

Working as in the proof of Corollary 3.10.2, by the completeness theorem for classical logic there must be a derivation $\Gamma \vdash_c \bot$ i.e., $\neg\neg(\Gamma \vdash_c \bot)$. As

$$\Gamma \vdash_c \bot \Rightarrow \exists_{\Gamma_0 \subseteq^{\text{fin}} \Gamma}\big(\Gamma_0 \vdash_c \bot\big),$$

we get

$$\neg\neg(\Gamma \vdash_c \bot) \Rightarrow \neg\neg\exists_{\Gamma_0 \subseteq^{\text{fin}} \Gamma}\big(\Gamma_0 \vdash_c \bot\big),$$

By definition

$$\Gamma_0 \text{ is satisfiable } \Leftrightarrow \neg\neg\big(\exists_{\mathcal{M}_c}\exists_{\eta\in\mathbb{F}(\texttt{Var},|\mathcal{M}_c|)}(\mathcal{M}_c \models \Gamma_0[\eta])\big).$$

The following implication holds:

$$\neg\neg\exists_{\Gamma_0 \subseteq^{\text{fin}} \Gamma}\big(\Gamma_0 \vdash_c \bot\big) \Rightarrow \neg\big(\exists_{\mathcal{M}_c}\exists_{\eta\in\mathbb{F}(\texttt{Var},|\mathcal{M}_c|)}(\mathcal{M}_c \models \Gamma_0[\eta])\big),$$

which contradicts the satisfiability of Γ_0. To show that implication, suppose

$$Q = \exists_{\mathcal{M}_c}\exists_{\eta\in\mathbb{F}(\texttt{Var},|\mathcal{M}_c|)}\big(\mathcal{M}_c \models \Gamma_0[\eta]\big).$$

For that classical model $\mathcal{M}_c$ and assignment η the following implication holds:

$$\exists_{\Gamma_0 \subseteq^{\text{fin}} \Gamma}\big(\Gamma_0 \vdash_c \bot\big) \Rightarrow \mathcal{M}_c \models \bot[\eta],$$

as by the soundness theorem for classical logic

$$\mathcal{M}_c \models \Gamma_0[\eta] \Rightarrow \mathcal{M}_c \models \bot[\eta],$$

and by our hypothesis Q we have that $\mathcal{M}_c \models \Gamma_0[\eta]$. Hence we get

$$\neg\neg\exists_{\Gamma_0 \subseteq^{\text{fin}} \Gamma}\big(\Gamma_0 \vdash_c \bot\big) \Rightarrow \neg\neg(\mathcal{M}_c \models \bot[\eta]),$$

which contradicts $\neg(\mathcal{M}_c \models \bot[\eta])$. Hence we showed

$$\neg\neg\neg\neg\big(\exists_{\mathcal{M}_c}\exists_{\eta\in\mathbb{F}(\texttt{Var},|\mathcal{M}_c|)}(\mathcal{M}_c \models \Gamma[\eta])\big).$$

By Brouwer's theorem again we get

$$\neg\neg\big(\exists_{\mathcal{M}_c}\exists_{\eta\in\mathbb{F}(\texttt{Var},|\mathcal{M}_c|)}(\mathcal{M}_c \models \Gamma[\eta])\big)$$

i.e., Γ is satisfiable. □

Corollary 3.10.4 (Löwenheim, Skolem)**.** *Let* $\Gamma \subseteq \texttt{Form}^*$ *in a countable language* $\mathcal{L}$*. If* Γ *is satisfiable, then* Γ *is satisfiable in a countable classical model.*

Proof. The proof with classical logic is straightforward. The constructive proof is an exercise. □

Hence, however large a model of a satisfiable Γ can be, we can always find a small model i.e., a countable one. In the spirit of the converse direction, one can show with compactness that if there are arbitrarily large finite models of Γ, then there is also an infinite model of Γ. Before showing this result we interpolate some related notions and facts on equality in $\mathcal{L}$.

Definition 3.10.5. *Let the underlying language $\mathcal{L}$ contain a binary relation symbol $\approx$ i.e., $\approx \in \mathtt{Rel}^{(2)}$. The set $\mathrm{Eq}_{\mathcal{L}}$ of $\mathcal{L}$-equality axioms consists of (the universal closures of)*

(Eq_1) $x \approx x$,

(Eq_2) $x \approx y \to y \approx x$,

(Eq_3) $x \approx y \;\&\; y \approx z \to x \approx z$,

(Eq_4) $x_1 \approx y_1 \;\wedge\; \ldots \;\wedge\; x_n \approx y_n \to f(x_1, \ldots, x_n) \approx f(y_1, \ldots, y_n)$,

(Eq_5) $x_1 \approx y_1 \;\wedge\; \ldots \;\wedge\; x_n \approx y_n \;\wedge\; R(x_1, \ldots, x_n) \to R(y_1, \ldots, y_n)$,

for all n-ary function symbols f, for all relation symbols R of $\mathcal{L}$, and $n \in \mathbb{N}$.

Note that the equality axioms are formulas of $\mathcal{L}$. If f is a 0-ary function symbol, then Eq_4 has as special case the axiom $c \approx c$. Notice that this equality is the given "internal" equality of $\mathcal{L}$ and must not be confused with the "external" equality $x = y$, which is the metatheoretical equality of the set `Var`. Consequently, if $t, s \in \mathtt{Term}$, we get the following formula of $\mathcal{L}$:

$$t \approx s.$$

Lemma 3.10.6 (Equality)**.** *Let $r, s, t \in \mathtt{Term}$ and $A \in \mathtt{Form}^*$.*

(i) $\mathrm{Eq}_{\mathcal{L}} \vdash t \approx s \to r(t) \approx r(s)$.

(ii) *If* $\mathrm{Free}_{t,x}(A) = \mathrm{Free}_{s,x}(A)$, *then* $\mathrm{Eq}_{\mathcal{L}} \vdash t \approx s \to (A(t) \leftrightarrow A(s))$.

Proof. (i) By induction on `Term` we prove the following formula

$$\forall_{r \in \mathtt{Term}} \Big(\mathrm{Eq}_{\mathcal{L}} \vdash t \approx s \to r(t) \approx r(s) \Big).$$

(ii) By Induction on $\mathtt{Form}^*$ we prove the following formula

$$\forall_{A \in \mathtt{Form}^*} \Big(\mathrm{Eq}_{\mathcal{L}} \vdash t \approx s \to (A(t) \leftrightarrow A(s)) \Big). \qquad \square$$

Note that the expressions

$$t \approx s \to r(t) \approx r(s),$$
$$t \approx s \to (A(t) \leftrightarrow A(s))$$

are formulas of $\mathcal{L}$. An $\mathcal{L}$-model $\mathcal{M}$ satisfies the equality axioms if and only if $\approx^{\mathcal{M}}$ is a *congruence relation* (i.e., an equivalence relation compatible with the functions and relations of $\mathcal{M}$).

Proposition 3.10.7. *Let $\mathcal{L}$ be a countable language with equality $\approx$, and let $\Gamma \subseteq \mathtt{Form}^*$.*

(i) *If for every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ with $m > n$ and there are a classical model M_c with cardinality m and an assignment in $|\mathcal{M}_c|$ such that $\mathcal{M}_c \models \Gamma[\eta]$, then there is an infinite classical model $\mathcal{N}_c$ and an assignment θ in $|\mathcal{N}_c|$ such that $\mathcal{N}_c \models \Gamma[\theta]$.*

(ii) *If for every classical model $\mathcal{M}_c$ and every assignment η in $|\mathcal{M}_c|$ such that $\mathcal{M}_c \models \Gamma[\eta]$ we have that the cardinality of $\mathcal{M}_c$ is finite, then there is $m \in \mathbb{N}$ that bounds their cardinality.*

(iii) *There is no set Γ that is modeled exactly by all finite finite classical models.*

Proof. (with classical logic) (i) Let $C = \{c_n \mid n \in \mathbb{N}\}$ be a set of constants such that $c_n \neq c_m$, for every $n \neq m$. Let also the new countable language

$$\mathcal{L}' = \mathcal{L} \cup C.$$

We extend the equality of $\mathcal{L}$ to $\mathcal{L}'$ by keeping for simplicity the same symbol $\approx$. Let the following set Γ' of formulas in $\mathcal{L}'$:

$$\Gamma' = \Gamma \cup \{\neg(c_n \approx c_m) \mid n, m \in \mathbb{N} \ \&\ n \neq m.\}$$

If ${\Gamma_0}'$ is a finite subset of Γ', it is of the form

$${\Gamma_0}' = \Gamma_0 \cup \Sigma_0,$$

where Γ_0 is a finite subset of Γ, and

$$\Sigma_0 = \{\neg(c_{n_1} \approx c_{m_1}), \ldots, \neg(c_{n_k} \approx c_{m_k})\},$$

for some $k \in \mathbb{N}$. Clearly, we can find a finite model $\mathcal{M}_c$ with cardinality m and an assignment in $|\mathcal{M}_c|$, such that $\mathcal{M}_c \models \Gamma[\eta]$, hence $\mathcal{M}_c \models \Gamma_0[\eta]$, and $m > 2k$. We can extend η to some η' such that all constant occurring in Σ_0 are assigned to pairwise distinct element of $|\mathcal{M}_c|$ under η' (clearly, the equality on the carrier set is a congruence). Hence $\mathcal{M}_c \models {\Gamma_0}'[\eta']$. By the compactness theorem for the countable language $\mathcal{L}'$ there is a model classical model $\mathcal{N}_c$ and an assignment θ in $|\mathcal{N}_c|$ such that $\mathcal{N}_c \models \Gamma'[\theta]$. Consequently, $\mathcal{N}_c$ is infinite, and clearly $\mathcal{N}_c \models \Gamma[\theta]$. (ii) and (iii) follow immediately from (i). $\square$

With classical logic one can also show that the compactness theorem implies the completeness theorem for classical logic (exercise).

Chapter 4

Gödel's incompleteness theorems

4.1 Elementary functions

The elementary functions are those number-theoretic functions that can be defined explicitly by compositional terms built up from variables and the constants $0, 1$ by repeated applications of addition $+$, modified subtraction $\dot{-}$, bounded sums and bounded products.

Definition 4.1.1. *The set of elementary functions of type $\mathbb{N}^k \to \mathbb{N}$, where $k > 1$, is defined inductively by the following rules:*

(Elem$_1$)

$$\frac{}{\overline{0}^1 \in \mathrm{Elem}^{(1)}}, \qquad \frac{}{\overline{1}^1 \in \mathrm{Elem}^{(1)}},$$

where $\overline{0}^1$ is the constant function 0 on $\mathbb{N}$ and $\overline{1}^1$ is the constant function 1 on $\mathbb{N}$.

(Elem$_2$)

$$\frac{k \in \mathbb{N}^+, i \in \{1, \ldots, k\}}{\mathtt{pr}_i^k \in \mathrm{Elem}^{(k)}},$$

where the projection function $\mathtt{pr}_i^k$ is defined by $\mathtt{pr}_i^k(x_1, \ldots, x_k) = x_i$.

(Elem$_3$)

$$\frac{}{+ \in \mathrm{Elem}^{(2)}},$$

where $+(x, y) = x + y$ is the addition of natural numbers.

(Elem$_4$)

$$\frac{}{\dot{-} \in \mathrm{Elem}^{(2)}},$$

where the modified subtraction $\dot{-}(x, y) = x \dot{-} y$ is defined by

$$x \dot{-} y = \begin{cases} x - y & , x \geq y \\ 0 & , \textit{otherwise}. \end{cases}$$

(Elem$_5$)

$$\frac{n, k \in \mathbb{N}^+, f \in \mathrm{Elem}^{(n)}, f_1 \ldots, f_n \in \mathrm{Elem}^{(k)}}{f \circ (f_1, \ldots, f_n) \in \mathrm{Elem}^{(k)}},$$

where the composite function $f \circ (f_1, \ldots, f_n)$ is defined by

$$\big[f \circ (f_1, \ldots, f_n)\big](x_1, \ldots, x_k) = f\big(f_1(x_1, \ldots, x_k), \ldots, f_n(x_1, \ldots, x_k)\big).$$

(Elem$_6$)
$$\frac{r \in \mathbb{N}, f \in \mathrm{Elem}^{(r+1)}}{\Sigma f \in \mathrm{Elem}^{(r+1)}},$$

where
$$\big(\Sigma f\big)(x_1, \ldots, x_r, y) = \sum_{z<y} f(x_1, \ldots, x_r, z),$$

and
$$\big(\Sigma f\big)(x_1, \ldots, x_r, 0) = 0.$$

(Elem$_7$)
$$\frac{r \in \mathbb{N}, f \in \mathrm{Elem}^{(r+1)}}{\Pi f \in \mathrm{Elem}^{(r+1)}},$$

where
$$\big(\Pi f\big)(x_1, \ldots, x_r, y) = \prod_{z<y} f(x_1, \ldots, x_r, z),$$

and
$$\big(\Pi f\big)(x_1, \ldots, x_r, 0) = 1.$$

We also define
$$\mathrm{Elem} = \bigcup_{k=1}^{\infty} \mathrm{Elem}^{(k)}.$$

The function Σf is the bounded sum of f, and the function Πf is the bounded product of f. By omitting bounded products, one obtains the so-called subelementary functions.

Proposition 4.1.2. *The following functions are elementary:*

(i) $\overline{0}^k \colon \mathbb{N}^k \to \mathbb{N}$, *defined by* $\overline{0}^k(x_1, \ldots, x_k) = 0$.

(ii) $\overline{1}^k \colon \mathbb{N}^k \to \mathbb{N}$, *defined by* $\overline{1}^k(x_1, \ldots, x_k) = 1$.

(iii) *The identity function* $\mathrm{id}_{\mathbb{N}} \colon \mathbb{N} \to \mathbb{N}$.

(iv) *The maximum function* $\max_2 \colon \mathbb{N}^2 \to \mathbb{N}$, *where* $\max_2(x, y) = \max\{x, y\}$.

(v) *The successor function* $\mathrm{Succ} \colon \mathbb{N} \to \mathbb{N}$, *where* $\mathrm{Succ}(x) = x + 1$.

(vi) *The predecessor function* $\mathrm{Pred} \colon \mathbb{N} \to \mathbb{N}$, *where*
$$\mathrm{Pred}(x) = \begin{cases} x - 1 & , x \geq 1 \\ 0 & , x = 0. \end{cases}$$

(vii) *The product function* $\cdot \colon \mathbb{N}^2 \to \mathbb{N}$, *where* $\cdot(x, y) = x \cdot y$.

(viii) *The factorial function* $! \colon \mathbb{N} \to \mathbb{N}$, *where* $!(x) = x!$.

(ix) *The exponential function* $\exp_2 \colon \mathbb{N}^2 \to \mathbb{N}$, *where* $\exp_2(x, y) = x^y$.

Proof. We show only (vii) and (viii), and the rest is an exercise. We have that
$$\cdot(x, y) = x \cdot y = \sum_{z<y} \mathtt{pr}_1^2(x, z) = \sum_{z<y} x,$$
$$!(x) = x! = \prod_{y<x} \mathrm{Succ}(y) = \prod_{y<x} (y + 1).$$
□

Proposition 4.1.3. *Let $k \in \mathbb{N}^+$ and $n, r \in \mathbb{N}$.*
(i) *If $f, g \in \mathrm{Elem}^{(k)}$, then $f + g \in \mathrm{Elem}^{(k)}$, $f \dot{-} g \in \mathrm{Elem}^{(k)}$, and $f \cdot g \in \mathrm{Elem}^{(k)}$.*
(ii) *$\overline{n}^k \colon \mathbb{N}^k \to \mathbb{N}$, defined by $\overline{n}^k(x_1, \ldots, x_k) = n$.*
(iii) *The function $x \mapsto x^m$, where $m \in \mathbb{N}$, is in $\mathrm{Elem}^{(1)}$.*
(iv) *A polynomial on $\mathbb{N}$ is in $\mathrm{Elem}^{(1)}$.*
(v) *The elementary functions are closed under "definition by cases" i.e., if $h, g_0, g_1 \in \mathrm{Elem}^{(k)}$, "the case-distinction function $\mathrm{Case}(g_0, g_1; h)$ of g_0 and g_1 with respect to h" is in $\mathrm{Elem}^{(k)}$, where*

$$\mathrm{Case}(g_0, g_1; h)(x_1, \ldots, x_k)\,) = \begin{cases} g_0(x_1, \ldots, x_k)\,) & , \textit{if } h(x_1, \ldots, x_k)\,) = 0 \\ g_1(x_1, \ldots, x_k)\,) & , \textit{otherwise} \end{cases}$$

(vi) *The elementary functions are closed under "bounded minimisation" i.e., if $f \in \mathrm{Elem}^{(r+1)}$, then $\mu f \in \mathrm{Elem}^{(r+1)}$, where*

$$(\mu f)(x_1, \ldots, x_r, y) = \mu_{z<y}(f(x_1, \ldots, x_r, z) = 0)$$

where $\mu_{z<y}(f(x_1, \ldots, x_r, z) = 0)$ denotes the least $z < y$ such that $f(x_1, \ldots, x_r, z) = 0$. If there is no $z < y$ such that $f(x_1, \ldots, x_r, z) = 0$, then $\mu f(x_1, \ldots, x_r, y) = y$.

Proof. Case (iv) can be shown with the use of modified subtraction, and case (v) with the use of modified subtraction and the bounded sum. Hence, not only the elementary, but in fact the subelementary functions are closed under bounded minimization. The rest is an exercise. □

Furthermore, we define $\mu_{z \leq y}\big(f(x_1, \ldots, x_r, z) = 0\big)$ as $\mu_{z<y+1}\big(f(x_1, \ldots, x_r, z) = 0\big)$.

4.2 A non-elementary function

The existence of non-elementary functions is easily justified on cardinality grounds; the set Elem is countable, while the set

$$\mathbb{F}^{\infty}(\mathbb{N}) = \bigcup_{k=1}^{\infty} \mathbb{F}(\mathbb{N}^k, \mathbb{N})$$

has the cardinality of the set of real numbers. Next we show how to find a non-elementary function, which is defined explicitly by some rule.

Definition 4.2.1. *If $k \in \mathbb{N}$, the function $2_k \colon \mathbb{N} \to \mathbb{N}$ is defined by*

$$2_0(m) = m; \qquad m \in \mathbb{N},$$
$$2_{k+1}(m) = 2^{2_k(m)}; \qquad m \in \mathbb{N}.$$

If $m \in \mathbb{N}$, then

$$2_1(m) = 2^{2_0(m)} = 2^m,$$
$$2_2(m) = 2^{2_1(m)} = 2^{2^m},$$
$$2_3(m) = 2^{2_2(m)} = 2^{2^{2^m}},$$
$$2_{k+1}(m) = 2^{2_k(m)} = 2^{\cdot^{\cdot^{2^m}}},$$

where there are $k + 1$-many 2's in the above tower of powers.

Lemma 4.2.2. *For every elementary function $f \colon \mathbb{N}^s \to \mathbb{N}$ there is $k \in \mathbb{N}$ such that for all $(x_1, \ldots, x_s) \in \mathbb{N}^s$ we have that*

$$f(x_1, \ldots, x_s)\,) < 2_k\big(\max\{x_1, \ldots, x_s\}\,\big)$$

Proof. By the induction principle that corresponds to the definition of elementary functions of arity k. If $f = \overline{0}^1$, then $\overline{0}^1(n) = 0 < 2^n = 2_1(n)$. If $f = \overline{1}^1$, then $\overline{1}^1(n) = 1 < 2^{2^n} = 2_2(n)$. If $s \in \mathbb{N}^+$, and $1 \leq i \leq s$, then

$$\begin{aligned}\mathtt{pr}_i^s(x_1, \ldots, x_s) &= x_i \\ &\leq \max\{x_1, \ldots, x_s\} \\ &< 2^{\max\{x_1, \ldots, x_s\}} \\ &= 2_1\big(\max\{x_1, \ldots, x_s\}\big).\end{aligned}$$

For the rest calculations we use the following inequalities:

$$n < 2^n \Rightarrow n^n < (2^n)^n,$$

$$(*) \qquad n^n < (2^n)^n \leq 2^{2^n}, \qquad \text{for every } n > 3.$$

$$(**) \qquad 2n < 2^{2^n}$$

The inequality $(2^n)^n \leq 2^{2^n}$ is shown by induction on $n > 3$, while to show $(**)$, we verify cases $n = 0, \ldots, n = 3$, and for $n > 3$ we have that $2n < n^n$, and we use $(*)$. Hence,

$$\begin{aligned}x + y &\leq 2\max\{x, y\} \\ &< 2 \cdot 2^{\max\{x,y\}} \\ &\overset{(**)}{<} 2^{2^{\max\{x,y\}}} \\ &= 2_2\big(\max\{x, y\}\big),\end{aligned}$$

$$x \dot{-} y \leq \max\{x, y\} < 2^{\max\{x,y\}} = 2_1\big(\max\{x, y\}\big).$$

Let $f_1, \ldots, f_n \in \mathrm{Elem}^{(s)}$ and $f \in \mathrm{Elem}^{(n)}$ such that

$$f_1(x_1, \ldots, x_s) < 2_{k_1}\big(\max\{x_1, \ldots, x_s\}\big),$$

$$\cdots\cdots\cdots\cdots$$

$$f_n(x_1, \ldots, x_s) < 2_{k_n}\big(\max\{x_1, \ldots, x_s\}\big)$$

$$f(y_1, \ldots, y_n) < 2_k\big(\max\{y_1, \ldots, y_n\}\big),$$

for some $k_1, \ldots, k_n, k \in \mathbb{N}$. If

$$l = \max\{k_1, \ldots, k_n, k\},$$

$$\begin{aligned}\big[f \circ (f_1, \ldots, f_n)\big](x_1, \ldots, x_s) &= f\big(f_1(x_1, \ldots, x_s), \ldots, f_n(x_1, \ldots, x_s)\big) \\ &< 2_k\big(\max\{f_1(x_1, \ldots, x_s), \ldots, f_n(x_1, \ldots, x_s)\}\big) \\ &\leq 2_k\Big(\max\Big\{2_{k_1}\big(\max\{x_1, \ldots, x_s\}\big), \ldots, 2_{k_n}\big(\max\{x_1, \ldots, x_s\}\big)\Big\}\Big) \\ &\leq 2_k\big(2_l\big(\max\{x_1, \ldots, x_s\}\big)\big) \\ &\leq 2_l\big(2_l\big(\max\{x_1, \ldots, x_s\}\big)\big) \\ &= 2_{2l}\big(\max\{x_1, \ldots, x_s\}\big)\big).\end{aligned}$$

Next we suppose that

$$f(x_1, \dots, x_r, y)\,) < 2_k\big(\max\{x_1, \dots, x_r, y\}\,\big),$$

for some $k \in \mathbb{N}$. As

$$\max\{x_1, \dots, x_r, 0\}, \dots, \max\{x_1, \dots, x_r, y-1\} \le \max\{x_1, \dots, x_r, y\},$$

and as

$$y \le 2_k(y) \le 2_k\big(\max\{x_1, \dots, x_r, y\}\big),$$

for every $k \in \mathbb{N}$, we have that

$$\begin{aligned}
\big(\Sigma f\big)(x_1, \dots, x_r, y) &= \sum_{z<y} f(x_1, \dots, x_r, z) \\
&< \sum_{z<y} 2_k\big(\max\{x_1, \dots, x_r, z\}\big) \\
&\le \sum_{z<y} 2_k\big(\max\{x_1, \dots, x_r, y\}\big) \\
&= y 2_k\big(\max\{x_1, \dots, x_r, y\}\big) \\
&\le \big[2_k\big(\max\{x_1, \dots, x_r, y\}\big)\big]^2 \\
&< 2_{k+2}\big(\max\{x_1, \dots, x_r, y\}\big),
\end{aligned}$$

as, if $m > 1$, we have that

$$2_n(m)^2 \le 2_n(m)^m < 2_n(m)^{2_n(m)} \overset{(*)}{<} 2^{2^{2_n(m)}} = 2_{n+2}(m).$$

Similarly,

$$\begin{aligned}
\big(\Pi f\big)(x_1, \dots, x_r, y) &= \prod_{z<y} f(x_1, \dots, x_r, z) \\
&< \prod_{z<y} 2_k\big(\max\{x_1, \dots, x_r, z\}\big) \\
&\le \prod_{z<y} 2_k\big(\max\{x_1, \dots, x_r, y\}\big) \\
&= \Big[2_k\big(\max\{x_1, \dots, x_r, y\}\big)\Big]^y \\
&\le \big[2_k\big(\max\{x_1, \dots, x_r, y\}\big)\big]^{2_k\left(\max\{x_1, \dots, x_r, y\}\right)} \\
&< 2_{k+2}\big(\max\{x_1, \dots, x_r, y\}\big),
\end{aligned}$$

as $2_n(m)^{2_n(m)} \overset{(*)}{<} 2^{2^{2_n(m)}} = 2_{n+2}(m)$. □

By Lemma 4.2.2 we can explicitly define a non-elementary function.

Corollary 4.2.3. *The function $f\colon \mathbb{N} \to \mathbb{N}$, defined by $f(n) = 2_n(1)$, for every $n \in \mathbb{N}$, is not elementary.*

Proof. Exercise. □

4.3 Elementary relations

Definition 4.3.1. *A relation $R \subseteq \mathbb{N}^k$ is called elementary if its characteristic function*

$$\chi_R(x_1, \ldots, x_k) = \begin{cases} 1 & , \text{ if } (x_1, \ldots, x_k) \in R \\ 0 & , \text{ otherwise} \end{cases}$$

is elementary.

Example 4.3.2. The equality $=$ on $\mathbb{N}$ and the inequality $<$ on $\mathbb{N}$ are elementary since their characteristic functions can be described as follows:

$$\chi_<(n, m) = 1 \dot{-} (1 \dot{-} (m \dot{-} n)),$$

$$\chi_=(n, m) = 1 \dot{-} (\chi_<(n, m) + \chi_<(m, n)).$$

Notice that the above writing is a simplification of the following formulation

$$\chi_<(n, m) = \overline{1}^2 \dot{-} \big(\overline{1}^2 \dot{-} [\mathtt{pr}_2^2(n, m) \dot{-} \mathtt{pr}_1^2(n, m))]\big).$$

Furthermore if $R \subseteq \mathbb{N}^{s+1}$ is elementary then so is the function

$$\begin{aligned} f(\vec{n}, m) &= \mu_{k<m} R(\vec{n}, k) \\ &= \mu_{k<m}\big(\chi_R(\vec{n}, k) = 1\big) \\ &= \mu_{k<m}\big((\overline{1}^{s+1} \dot{-} \chi_R)(\vec{n}, k) = 0\big) \end{aligned}$$

as

$$(\overline{1}^{s+1} \dot{-} \chi_R)(\vec{n}, k) = 1 - \chi_R(\vec{n}, k) = \begin{cases} 0 & , (\vec{n}, k) \in R \\ 1 & , (\vec{n}, k) \notin R. \end{cases}$$

Next we show that he elementary relations are closed under applications of propositional connectives and bounded quantifiers.

Lemma 4.3.3. *Let $R, S \subseteq \mathbb{N}^k$ and $T \subseteq \mathbb{N}^{k+1}$ be elementary. The following relations*

$$\neg R = \mathbb{N}^k \setminus R, \quad R \;\&\; S = R \cap S, \quad R \vee S = R \cup S, \quad R \Rightarrow S = R \cup (\mathbb{N}^k \setminus S),$$

$$A(\vec{x}, y) = \forall_{z<y}\big(T(\vec{x}, z)\big) = \forall_z\big(z < y \Rightarrow T(\vec{x}, z)\big),$$

$$E(\vec{x}, y) = \exists_{z<y}\big(T(\vec{x}, z)\big) = \exists_z\big(z < y \;\&\; T(\vec{x}, z)\big),$$

are also elementary

Proof. The following equalities hold:

$$\chi_{\neg R} = \overline{1}^k \dot{-} \chi_R, \quad \chi_{R \,\&\, S} = \chi_R \cdot \chi_S,$$

$$\chi_A(\vec{x}, y) = \prod_{z<y} \chi_T(\vec{x}, z),$$

and the result for the rest relations follows from their redundancy to them e.g.,

$$E(\vec{x}, y) \Leftrightarrow \neg\forall_{z<y}\big(\neg T(\vec{x}, z)\big).$$

□

Example 4.3.4. The above closure properties enable us to show that many "natural" functions and relations of number theory are elementary. E.g., the floor of a positive rational, defined as a function on pairs of naturals, and the "remainder function" mod : $\mathbb{N}^2 \to \mathbb{N}$, where mod $(n, m) = n \mod m$ is the remainder of the division of n by m are elementary as

$$\left\lfloor \frac{n}{m} \right\rfloor = \mu_{k<n}(n < (k+1)m),$$
$$n \bmod m = n \dot{-} \left\lfloor \frac{n}{m} \right\rfloor m.$$

The unary relation $\texttt{Prime}$ and the enumeration-function of primes are also elementary, since

$$\texttt{Prime}(n) \Leftrightarrow 1 < n \ \& \ \neg\exists_{m<n}(1 < m \ \& \ n \bmod m = 0),$$
$$p_n = \mu_{m<2^{2^n}} \Big(\texttt{Prime}(m) \ \& \ n = \sum_{i<m} \chi_{\texttt{Prime}}(i)\Big).$$

The values $p_0, p_1, p_2, \ldots$ form the enumeration of primes in increasing order. The inequality

$$p_n \leq 2^{2^n}$$

for the n-th prime p_n can be proved by induction on n: for $n = 0$ this is clear by our convention in Proposition 4.1.3(vi), and for $n \geq 1$ we obtain

$$p_n \leq p_0 p_1 \cdots p_{n-1} + 1 \leq 2^{2^0} 2^{2^1} \cdots 2^{2^{n-1}} + 1 = 2^{2^n - 1} + 1 < 2^{2^n}.$$

4.4 The set of functions $\mathcal{E}$

We define the set of functions $\mathcal{E}$ that is going to be equal to the set of elementary functions Elem. This alternative characterisation of Elem is useful, in order to show that Elem is closed under limited recursion through the closure of $\mathcal{E}$ under limited recursion.

Definition 4.4.1. *The set $\mathcal{E}$ consists of those number theoretic functions that can be defined from the initial functions: constant* 0*, successor* Succ*, projections, addition* $+$*, modified subtraction, multiplication, and exponentiation* 2^x*, by applications of composition and bounded minimisation.*

Corollary 4.4.2. (i) *Every function in $\mathcal{E}$ is elementary.*

(ii) *The characteristic functions of the equality and "less than" relations are in $\mathcal{E}$.*

(iii) *A relation $R \subseteq \mathbb{N}^k$ is an $\mathcal{E}$-relation, if its characteristic function is in $\mathcal{E}$. The $\mathcal{E}$-relations are closed under propositional connectives and bounded quantifiers.*

Proof. (i) By induction on $\mathcal{E}$. All initial functions in $\mathcal{E}$ are elementary. The exponentiation-map $x \mapsto 2^x$ is shown to be in $\text{Elem}^{(1)}$ similarly to the proof for $\exp_2$ (Proposition 4.1.2). Moreover, the elementary functions are closed under composition and bounded minimisation.
(ii) It follows immediately by the writing of their characteristic functions in Example 4.3.2, and by the fact that $\overline{1}^1 = \text{Succ} \circ \overline{0}^1 \in \mathcal{E}$.
(iii) As the closure under bounded products is not mentioned in the definition of $\mathcal{E}$, we write the characteristic function of

$$A(\vec{x}, y) = \forall_{z<y}\big(T(\vec{x}, z)\big) = \forall_z\big(z < y \Rightarrow T(\vec{x}, z)\big),$$

as follows:

$$\chi_A = \chi_= \circ \big(\mathtt{pr}^{k+1}_{k+1}, f\big),$$
$$f(\vec{x}, y) = \mu_{z<y}\big(\chi_T(\vec{x}, z) = 0\big).$$

As

$$\begin{aligned}\chi_A(\vec{x}, y) = 1 &\Leftrightarrow \big[\chi_= \circ \big(\mathtt{pr}^{k+1}_{k+1}, f\big)\big](\vec{x}, y) = 1 \\ &\Leftrightarrow \mathtt{pr}^{k+1}_{k+1}(\vec{x}, y) = \mu_{z<y}\big(\chi_T(\vec{x}, z) = 0\big) \\ &\Leftrightarrow y = \mu_{z<y}\big(\chi_T(\vec{x}, z) = 0\big),\end{aligned}$$

by our convention in Proposition 4.1.3(vi) we have that $\chi_T(\vec{x}, z) = 1$, for every $z < y$. □

Lemma 4.4.3. *There are pairing functions π, π_1, π_2 in $\mathcal{E}$ with the following properties:*
(i) *π maps $\mathbb{N} \times \mathbb{N}$ bijectively onto $\mathbb{N}$.*
(ii) *$\pi(a, b) + b + 2 \leq (a + b + 1)^2$, for $a + b \geq 1$, hence $\pi(a, b) < (a + b + 1)^2$.*
(iii) *$\pi_1(c), \pi_2(c) \leq c$.*
(iv) *$\pi(\pi_1(c), \pi_2(c)) = c$.*
(v) *$\pi_1(\pi(a, b)) = a$.*
(vi) *$\pi_2(\pi(a, b)) = b$.*

Proof. We enumerate the pairs of natural numbers

$$\begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \vdots \\ (0,3) & (1,3) & (2,3) & (3,3) & \dots \\ (0,2) & (1,2) & (2,2) & (3,2) & \dots \\ (0,1) & (1,1) & (2,1) & (3,1) & \dots \\ (0,0) & (1,0) & (2,0) & (3,0) & \dots \end{array}$$

as follows:

$$\begin{array}{lllll} \vdots & & & & \\ 6 & \dots & & & \\ 3 & 7 & \dots & & \\ 1 & 4 & 8 & \dots & \\ 0 & 2 & 5 & 9 & \dots \end{array}$$

I.e., if Δ_n are the diagonals:

$$\Delta_0 = (0, 0),$$
$$\Delta_1 = (0, 1)(1, 0),$$
$$\Delta_2 = (0, 2)(1, 1)(2, 0),$$
$$\Delta_3 = (0, 3)(1, 2)(2, 1)(3, 0),$$

etc., then the above enumeration enumerates the pairs of all diagonals following the route

$$\Delta_0 \to \Delta_1 \to \Delta_2 \to \Delta_3 \to \dots.$$

We remark the following:

- If $(a, b) \in \Delta_n$, then $a + b = n$.
- The number of pairs in Δ_n is $n + 1$.
- The number $\pi(a, b)$ associated to the pair (a, b) counts the number of pairs from $(0, 0)$, the first pair in the enumeration, until reaching (a, b) in the diagonal Δ_{a+b} and having gone through the previous diagonals

$$\Delta_0 \to \Delta_1 \to \Delta_2 \to \Delta_3 \to \ldots \to \Delta_{a+b-1}.$$

As Δ_0 has 1 element, Δ_1 has 2 elements, ..., Δ_{a+b-1} has $a + b$ number of elements we get

$$\begin{aligned} \pi(a, b) &= [1 + 2 + \ldots (a + b)] + a \\ &= \frac{1}{2}(a + b)(a + b + 1) + a \\ &= \Big(\sum_{i \leq a+b} i\Big) + a. \end{aligned}$$

The second equality above shows that π is in $\mathcal{E}$ (the justification of this is an exercise), while the third equality shows that π is subelementary. Clearly $\pi \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is bijective. Moreover, $a, b \leq \pi(a, b)$ and in case $\pi(a, b) \neq 0$ also $a < \pi(a, b)$. Let

$$\begin{aligned} \pi_1(c) &= \mu_{x \leq c} \exists_{y \leq c} (\pi(x, y) = c), \\ \pi_2(c) &= \mu_{y \leq c} \exists_{x \leq c} (\pi(x, y) = c). \end{aligned}$$

As π is in $\mathcal{E}$, we also have that π_1 and π_2 are in $\mathcal{E}$. Moreover, by their definition, and since π is subelementary, we also have that π_1 and π_2 are subelementary. Clearly, $\pi_i(c) \leq c$ for $i \in \{1, 2\}$ and

$$\pi_1(\pi(a, b)) = a, \quad \pi_2(\pi(a, b)) = b, \quad \pi(\pi_1(c), \pi_2(c)) = c.$$

For $\pi(a, b)$ we have the estimate

$$\pi(a, b) + b + 2 \leq (a + b + 1)^2 \quad \text{for } a + b \geq 1.$$

This follows with $n = a + b$ from

$$\frac{1}{2}n(n + 1) + n + 2 \leq (n + 1)^2 \quad \text{for } n \geq 1,$$

which is equivalent to $n(n+1)+2(n+1) \leq 2((n+1)^2-1)$ and hence to $(n+2)(n+1) \leq 2n(n+2)$, which holds for $n \geq 1$. □

Theorem 4.4.4 (Gödel's β-function)**.** *There is in $\mathcal{E}$ a function β with the following property: For every sequence $a_0, \ldots, a_{n-1} < b$ of numbers less than b we can find a number*

$$c \leq 4 \cdot 4^{n(b+n+1)^4},$$

such that $\beta(c, i) = a_i$ for all $i < n$.

Proof. Let

$$a = \pi(b, n) \quad \text{and} \quad d = \prod_{i<n} \big(1 + \pi(a_i, i)a!\big).$$

From $a!$ and d we can, for each given $i < n$, reconstruct the number a_i as the unique $x < b$ such that

$$1 + \pi(x, i)a! \mid d.$$

For clearly a_i is such an x, and if some $x < b$ were to satisfy the same condition, then because $\pi(x, i) < a$ and the numbers $1 + ka!$ are relatively prime for $k \leq a$, we would have $\pi(x, i) = \pi(a_j, j)$ for some $j < n$. Hence $x = a_j$ and $i = j$, thus $x = a_i$. Therefore

$$a_i = \mu_{x<b} \exists_{z<d} ((1 + \pi(x, i)a!)z = d).$$

We can now define Gödel's β-function as

$$\beta(c, i) = \mu_{x<\pi_1(c)} \exists_{z<\pi_2(c)} ((1 + \pi(x, i) \cdot \pi_1(c)) \cdot z = \pi_2(c)).$$

Clearly, β is in $\mathcal{E}$. Furthermore with $c = \pi(a!, d)$ we see that $\beta(c, i) = a_i$. One can then estimate the given bound on c, using $\pi(b, n) < (b + n + 1)^2$ (exercise). □

The above definition of β shows that it is a subelementary function.

Theorem 4.4.5. *The set of functions $\mathcal{E}$ is closed under limited recursion. Thus if g, h, k are given functions in $\mathcal{E}$ and f is defined from them according to the schema*

$$\begin{aligned} f(\vec{m}, 0) &= g(\vec{m}), \\ f(\vec{m}, n + 1) &= h(n, f(\vec{m}, n), \vec{m}), \\ f(\vec{m}, n) &\leq k(\vec{m}, n), \end{aligned}$$

then f is in $\mathcal{E}$.

Proof. Let f be defined from g, h and k in $\mathcal{E}$, by limited recursion as above. Using Gödel's β-function as in the last theorem we can find for any given $\vec{m}, n$ a number c such that $\beta(c, i) = f(\vec{m}, i)$ for all $i \leq n$. Let $R(\vec{m}, n, c)$ be the relation

$$\beta(c, 0) = g(\vec{m}) \ \& \ \forall_{i<n} (\beta(c, i + 1) = h(i, \beta(c, i), \vec{m}))$$

and note that its characteristic function is in $\mathcal{E}$. It is clear, by induction, that if $R(\vec{m}, n, c)$ holds then $\beta(c, i) = f(\vec{m}, i)$, for all $i \leq n$. Therefore we can define f explicitly by the equation

$$f(\vec{m}, n) = \beta(\mu_c R(\vec{m}, n, c), n).$$

The function f is in $\mathcal{E}$, if μ_c can be bounded by some function in $\mathcal{E}$. However, the theorem on Gödel's β-function gives a bound $4 \cdot 4^{(n+1)(b+n+2)^4}$, where in this case b can be taken as the maximum of $k(\vec{m}, i)$ for $i \leq n$. But this can be defined in $\mathcal{E}$ as $k(\vec{m}, i_0)$, where $i_0 = \mu_{i \leq n} \forall_{j \leq n} (k(\vec{m}, j) \leq k(\vec{m}, i))$. Hence μ_c can be bounded by a function in $\mathcal{E}$. □

Note that it is in the previous proof only that the exponential function is required, in providing a bound for μ_c.

Corollary 4.4.6. *The set of functions $\mathcal{E}$ is equal to the set* Elem *of elementary functions.*

Proof. It is sufficient to show that $\mathcal{E}$ is closed under bounded sums and bounded products. Suppose for instance, that f is defined from g in $\mathcal{E}$ by bounded summation: $f(\vec{m}, n) = \sum_{i<n} g(\vec{m}, i)$. Then f can be defined by limited recursion, as follows

$$\begin{aligned} f(\vec{m}, 0) &= 0 \\ f(\vec{m}, n+1) &= f(\vec{m}, n) + g(\vec{m}, n) \\ f(\vec{m}, n) &\leq n \cdot \max_{i<n} g(\vec{m}, i) \end{aligned}$$

and the functions (including the bound) from which it is defined are in $\mathcal{E}$ (why?). Thus f is in $\mathcal{E}$ by the theorem. If f is defined by bounded product, we proceed similarly. □

4.5 Coding finite lists

Computation on lists is a practical necessity, so because we are basing everything here on the single data type $\mathbb{N}$ we must develop some means of *coding* finite lists or sequences of natural numbers into $\mathbb{N}$ itself. There are various ways to do this and we shall adopt one of the most traditional, based on the pairing functions π, π_1, π_2.

Definition 4.5.1. *The empty sequence $\emptyset$ is coded by the number 0 and a sequence n_0, n_1, ..., n_{k-1} is coded by the sequence number*

$$\langle n_0, n_1, \ldots, n_{k-1} \rangle = \pi'(\ldots \pi'(\pi'(0, n_0), n_1), \ldots, n_{k-1})$$

with $\pi'(a, b) = \pi(a, b) + 1$, thus recursively,

$$\langle \emptyset \rangle = 0,$$

$$\langle n_0, n_1, \ldots, n_k \rangle = \pi'(\langle n_0, n_1, \ldots, n_{k-1} \rangle, n_k).$$

Because of the surjectivity of π, every number a can be decoded uniquely as a sequence number $a = \langle n_0, n_1, \ldots, n_{k-1} \rangle$. If a is greater than zero,

$$\mathrm{hd}(a) = \pi_2(a \dot{-} 1)$$

is the head i.e., rightmost element, and

$$\mathrm{tl}(a) = \pi_1(a \dot{-} 1)$$

is the tail of the list. The k-th iterate of tl is denoted $\mathrm{tl}^{(k)}$ and since $\mathrm{tl}(a)$ is less than or equal to a, $\mathrm{tl}^{(k)}(a)$ is elementarily definable by limited recursion. Thus we can define elementarily the length and decoding functions:

$$\begin{aligned} \mathrm{lh}(a) &= \mu_{k \leq a}(\mathrm{tl}^{(k)}(a) = 0), \\ (a)_i &= \mathrm{hd}(\mathrm{tl}^{(\mathrm{lh} a \dot{-} (i+1))}(a)). \end{aligned}$$

We shall write $(a)_{i,j}$ for $((a)_i)_j$ and $(a)_{i,j,k}$ for $(((a)_i)_j)_k$.

If $a = \langle n_0, n_1, \ldots, n_{k-1} \rangle$, it is easy to show that

$$\mathrm{lh}(a) = k \text{ and } (a)_i = n_i, \text{ for each } i < k.$$

Furthermore $(a)_i = 0$ when $i \geq \mathrm{lh}(a)$. This elementary coding machinery will be used at various crucial points in the following. Note that the functions $\mathrm{lh}(\cdot)$ and $(a)_i$ are subelementary, and so is $\langle n_0, n_1, \ldots, n_{k-1} \rangle$ for each fixed k.

Lemma 4.5.2 (Estimate for sequence numbers)**.**

$$(n+1)k \le \langle \underbrace{n,\dots,n}_{k} \rangle < (n+1)^{2^k}.$$

Proof. We prove a slightly strengthened form of the second estimate:

$$\langle \underbrace{n,\dots,n}_{k} \rangle + n + 1 \le (n+1)^{2^k},$$

by induction on k. For $k = 0$ the claim is clear. In the step $k \mapsto k+1$ we have

$$\begin{aligned}
\langle \underbrace{n,\dots,n}_{k+1} \rangle + n + 1 &= \pi(\langle \underbrace{n,\dots,n}_{k} \rangle, n) + n + 2 \\
&\le (\langle \underbrace{n,\dots,n}_{k} \rangle + n + 1)^2 \quad \text{by Lemma 4.4.3} \\
&\le (n+1)^{2^{k+1}} \quad \text{by induction hypothesis.}
\end{aligned}$$

For the first estimate the base case $k = 0$ is clear, and in the step we have

$$\begin{aligned}
\langle \underbrace{n,\dots,n}_{k+1} \rangle &= \pi(\langle \underbrace{n,\dots,n}_{k} \rangle, n) + 1 \\
&\ge \langle \underbrace{n,\dots,n}_{k} \rangle + n + 1 \\
&\ge (n+1)(k+1) \quad \text{by induction hypothesis.}
\end{aligned}$$

□

4.6 Gödel numbers

Definition 4.6.1. *Let $\mathcal{L}$ be a countable first-order language. Assume that we have injectively assigned to every n-ary relation symbol R a symbol number* $\mathrm{sn}(R)$ *of the form $\langle 1, n, i \rangle$ and to every n-ary function symbol f a symbol number* $\mathrm{sn}(f)$ *of the form $\langle 2, n, j \rangle$. Call $\mathcal{L}$ elementarily presented, if the set* $\mathrm{Symb}_{\mathcal{L}}$ *of all these symbol numbers is elementary.*

In what follows we shall always assume that the languages $\mathcal{L}$ considered are elementarily presented. In particular this applies to every language with finitely many relation and function symbols.

Definition 4.6.2 (Gödel numbering)**.** *Let* $\mathrm{sn}(\mathtt{Var}) = \langle 0 \rangle$. *For every $\mathcal{L}$-term r we define recursively its Gödel number $\ulcorner r \urcorner$ by*

$$\begin{aligned}
\ulcorner x_i \urcorner &= \langle \mathrm{sn}(\mathtt{Var}), i \rangle, \\
\ulcorner f(r_1 \dots r_n) \urcorner &= \langle \mathrm{sn}(f), \ulcorner r_1 \urcorner, \dots, \ulcorner r_n \urcorner \rangle.
\end{aligned}$$

Assign numbers to the logical symbols by $\mathrm{sn}(\to) = \langle 3, 0 \rangle$ *and* $\mathrm{sn}(\forall) = \langle 3, 1 \rangle$. *For simplicity we leave out the logical connectives $\wedge$, $\vee$ and $\exists$ here; they could be treated similarly. We define for every $\mathcal{L}$-formula A its Gödel number $\ulcorner A \urcorner$ by*

$$\begin{aligned}
\ulcorner R(r_1 \dots r_n) \urcorner &= \langle \mathrm{sn}(R), \ulcorner r_1 \urcorner, \dots, \ulcorner r_n \urcorner \rangle, \\
\ulcorner A \to B \urcorner &= \langle \mathrm{sn}(\to), \ulcorner A \urcorner, \ulcorner B \urcorner \rangle, \\
\ulcorner \forall_{x_i} A \urcorner &= \langle \mathrm{sn}(\forall), i, \ulcorner A \urcorner \rangle.
\end{aligned}$$

Assume that 0 *is a constant and* Succ *is a unary function symbol in* $\mathcal{L}$*. For every* $a \in \mathbb{N}$ *the numeral* $\underline{a} \in \mathtt{Term}_{\mathcal{L}}$ *is defined by* $\underline{0} = 0$ *and* $\underline{n+1} = \mathrm{Succ}(\underline{n})$*.*

Proposition 4.6.3. *There is an elementary function* s *such that for every formula* $C = C(z)$ *with* $z = x_0$*,*

$$s(\ulcorner C \urcorner, k) = \ulcorner C(\underline{k}) \urcorner;$$

Proof. The proof requires a lot of preparation, and it is omitted. Lemma 4.5.2 is necessary to the proof. □

4.7 Undefinability of the notion of truth

Definition 4.7.1. *Let* $\mathcal{M}$ *be an* $\mathcal{L}$*-structure. A relation* $R \subseteq |\mathcal{M}|^n$ *is called* $\mathcal{L}$*-definable in* $\mathcal{M}$*,or simply definable in* $\mathcal{M}$*, if there is an* $\mathcal{L}$*-formula* $A(x_1, \ldots, x_n)$ *such that*

$$R = \big\{(a_1, \ldots, a_n) \in |\mathcal{M}|^n \mid \mathcal{M} \models A(x_1, \ldots x_n)[x_1 \mapsto a_1, \ldots, x_n \mapsto a_n]\big\}.$$

We assume in this section that $|\mathcal{M}| = \mathbb{N}$*,* 0 *is a constant in* $\mathcal{L}$ *and* Succ *is a unary function symbol in* $\mathcal{L}$ *with* $0^{\mathcal{M}} = 0$ *and* $\mathrm{Succ}^{\mathcal{M}}(a) = a + 1$*.*

Recall that for every $a \in \mathbb{N}$ the numeral $\underline{a} \in \mathtt{Term}_{\mathcal{L}}$ is defined by $\underline{0} = 0$ and $\underline{n+1} = \mathrm{Succ}(\underline{n})$. Observe that in this case the definability of $R \subseteq \mathbb{N}^n$ by $A(x_1, \ldots, x_n)$ is equivalent to

$$R = \big\{(a_1, \ldots, a_n) \in \mathbb{N}^n \mid \mathcal{M} \models A(\underline{a_1}, \ldots, \underline{a_n})\big\}.$$

Definition 4.7.2. *Let* $\mathcal{L}$ *be an elementarily presented language. We assume in this section that every elementary relation is definable in* $\mathcal{M}$*. A set* S *of formulas is called definable in* $\mathcal{M}$*, if*

$$\ulcorner S \urcorner = \{\ulcorner A \urcorner \mid A \in S\}$$

is definable in $\mathcal{M}$*.*

We shall show that already from these assumptions it follows that the notion of truth for $\mathcal{M}$, more precisely the set

$$\mathrm{Th}(\mathcal{M}) = \{A \in \mathtt{Form} \mid \mathrm{FV}(A) = \emptyset \ \&\ M \models A\}$$

of all closed formulas valid in $\mathcal{M}$, is undefinable in $\mathcal{M}$.

Lemma 4.7.3 (Semantical fixed point lemma)**.** *If every elementary relation is definable in* $\mathcal{M}$*, then for every* $\mathcal{L}$*-formula* $B(z)$ *we can find a closed* $\mathcal{L}$*-formula* A *such that*

$$\mathcal{M} \models A \quad \textit{if and only if} \quad \mathcal{M} \models B(\underline{\ulcorner A \urcorner}).$$

Proof. Let s be the elementary function satisfying for every formula $C = C(z)$ with $z = x_0$,

$$s(\ulcorner C \urcorner, k) = \ulcorner C(\underline{k}) \urcorner$$

mentioned above. Then in particular

$$s(\ulcorner C \urcorner, \ulcorner C \urcorner) = \ulcorner C(\underline{\ulcorner C \urcorner}) \urcorner.$$

By assumption the graph G_s of s is definable in $\mathcal{M}$, by $A_s(x_1, x_2, x_3)$ say. Let

$$C(z) = \forall_x (A_s(z, z, x) \to B(x)), \quad A = C(\underline{\ulcorner C \urcorner}),$$

and therefore

$$A = \forall_x (A_s(\underline{\ulcorner C \urcorner}, \underline{\ulcorner C \urcorner}, x) \to B(x)).$$

Hence $\mathcal{M} \models A$ if and only if $\forall_{d \in \mathbb{N}} \big(d = \ulcorner C(\underline{\ulcorner C \urcorner}) \urcorner \Rightarrow \mathcal{M} \models B(\underline{d})\big)$, which is the same as $\mathcal{M} \models B(\underline{\ulcorner A \urcorner})$. □

Theorem 4.7.4 (Tarski's undefinability theorem)**.** *Assume that every elementary relation is definable in $\mathcal{M}$. Then* $\mathrm{Th}(\mathcal{M})$ *is undefinable in $\mathcal{M}$.*

Proof. Assume that $\ulcorner \mathrm{Th}(\mathcal{M}) \urcorner$ is definable by $B_W(z)$. Then for all closed formulas A

$$\mathcal{M} \models A \quad \text{if and only if} \quad \mathcal{M} \models B_W(\underline{\ulcorner A \urcorner}).$$

Now consider the formula $\neg B_W(z)$ and choose by the fixed point lemma a closed $\mathcal{L}$-formula A such that

$$\mathcal{M} \models A \quad \text{if and only if} \quad \mathcal{M} \models \neg B_W(\underline{\ulcorner A \urcorner}).$$

This contradicts the equivalence above. □

4.8 Representable relations and functions

Here we generalize the arguments of the previous section. There we have made essential use of the notion of truth in a structure $\mathcal{M}$, i.e., of the relation $\mathcal{M} \models A$. The set of all closed formulas A such that $\mathcal{M} \models A$ has been called the theory of $\mathcal{M}$, denoted $\mathrm{Th}(\mathcal{M})$. Now, instead of $\mathrm{Th}(\mathcal{M})$, we shall start more generally from an arbitrary theory T.

Definition 4.8.1. *Let $\mathcal{L}$ be a countable first order language with equality, and let $\overline{\mathcal{L}}$ be the set of all closed $\mathcal{L}$-formulas. For every set Γ of formulas let $L(\Gamma)$ be the set of all function and relation symbols occurring in Γ. An axiom system Γ is a set of closed formulas such that* $\mathrm{Eq}_{L(\Gamma)} \subseteq \Gamma$. *A model of an axiom system Γ is an $\mathcal{L}$-model $\mathcal{M}$ such that $L(\Gamma) \subseteq \mathcal{L}$ and $\mathcal{M} \models \Gamma$. For sets Γ of closed formulas we write*

$$\mathrm{Mod}_{\mathcal{L}}(\Gamma) = \{\mathcal{M} \mid \mathcal{M} \textit{ is an } \mathcal{L}\textit{-model} \ \& \ \mathcal{M} \models \Gamma \cup \mathrm{Eq}_{\mathcal{L}}\}.$$

Clearly Γ is satisfiable if and only if Γ has an $\mathcal{L}$-model. A theory T is an axiom system closed under $\vdash_c$, that is, $\mathrm{Eq}_{L(T)} \subseteq T$ *and*

$$T = \{A \in \overline{L(T)} \mid T \vdash_c A\}.$$

A theory T is called complete, if for every formula $A \in \overline{L(T)}$, $T \vdash_c A$ or $T \vdash_c \neg A$.

For every $\mathcal{L}$-model $\mathcal{M}$ satisfying the equality axioms the set $\mathrm{Th}(\mathcal{M})$ of all closed $\mathcal{L}$-formulas A such that $\mathcal{M} \models A$ is a theory. We consider the question as to whether in T there is a *notion of truth* (in the form of a *truth formula* $B(z)$), such that $B(z)$ *means* that z is *true*. A consequence is that we have to explain all notions used without referring to semantical concepts at all.

1. z ranges over closed formulas (or sentences) A, or more precisely over their Gödel numbers $\ulcorner A \urcorner$.
2. A *true* is to be replaced by $T \vdash A$.
3. C *equivalent* to D is to be replaced by $T \vdash C \leftrightarrow D$.

Hence the question now is whether there is a truth formula $B(z)$ such that

$$T \vdash A \leftrightarrow B(\underline{\ulcorner A \urcorner}),$$

for all sentences A. The result will be that this is impossible, under rather weak assumptions on the theory T. Technically, the issue will be to replace the notion of definability by the notion of *representability* within a formal theory. We begin with a discussion of this notion. In this section we assume that $\mathcal{L}$ is an elementarily presented language with 0, Succ and $=$ in $\mathcal{L}$, and T is an $\mathcal{L}$-theory containing the equality axioms $\mathrm{Eq}_{\mathcal{L}}$.

Definition 4.8.2. *A relation $R \subseteq \mathbb{N}^n$ is representable in T if there is a formula $A(x_1, \dots, x_n)$ such that*

$$\begin{aligned} &T \vdash A(\underline{a_1}, \dots, \underline{a_n}) && \text{if } (a_1, \dots, a_n) \in R, \\ &T \vdash \neg A(\underline{a_1}, \dots, \underline{a_n}) && \text{if } (a_1, \dots, a_n) \notin R. \end{aligned}$$

A function $f\colon \mathbb{N}^n \to \mathbb{N}$ is called representable in T if there is a formula $A(x_1, \dots, x_n, y)$ representing the graph $G_f \subseteq \mathbb{N}^{n+1}$ of f, i.e., such that

$$T \vdash A(\underline{a_1}, \dots, \underline{a_n}, \underline{f(a_1, \dots, a_n)}), \tag{4.1}$$

$$T \vdash \neg A(\underline{a_1}, \dots, \underline{a_n}, \underline{c}) \qquad \text{if } c \neq f(a_1, \dots, a_n) \tag{4.2}$$

and such that in addition

$$T \vdash A(\underline{a_1}, \dots, \underline{a_n}, y) \land A(\underline{a_1}, \dots, \underline{a_n}, z) \to y{=}z, \quad \text{for all } a_1, \dots, a_n \in \mathbb{N}. \tag{4.3}$$

If $T \vdash \underline{b} \neq \underline{c}$ for $b < c$ condition (4.2) follows from (4.1) and (4.3).

Lemma 4.8.3. *If the characteristic function χ_R of a relation $R \subseteq \mathbb{N}^n$ is representable in T, then so is the relation R itself.*

Proof. For simplicity assume $n = 1$. Let $A(x, y)$ be a formula representing χ_R. We show that $A(x, \underline{1})$ represents the relation R. Assume $a \in R$. Then $\chi_R(a) = 1$, hence $(a, 1) \in G_{\chi_R}$, hence $T \vdash A(\underline{a}, \underline{1})$. Conversely, assume $a \notin R$. Then $\chi_R(a) = 0$, hence $(a, 1) \notin G_{\chi_R}$, hence $T \vdash \neg A(\underline{a}, \underline{1})$. □

4.9 Undefinability of the notion of truth in formal theories

Lemma 4.9.1 (Fixed point lemma)**.** *Assume that all elementary functions are representable in T. Then for every formula $B(z)$ we can find a closed formula A such that*

$$T \vdash A \leftrightarrow B(\underline{\ulcorner A \urcorner}).$$

Proof. The proof is similar to the proof of the semantical fixed point lemma. Let s be the elementary function introduced there and $A_s(x_1, x_2, x_3)$ a formula representing s in T. Let

$$C(z) = \forall_x (A_s(z, z, x) \to B(x)), \quad A = C(\underline{\ulcorner C \urcorner}),$$

and therefore

$$A = \forall_x (A_s(\underline{\ulcorner C \urcorner}, \underline{\ulcorner C \urcorner}, x) \to B(x)).$$

Because of $s(\ulcorner C \urcorner, \ulcorner C \urcorner) = \ulcorner C(\underline{\ulcorner C \urcorner}) \urcorner = \ulcorner A \urcorner$ we can prove in T

$$A_s(\underline{\ulcorner C \urcorner}, \underline{\ulcorner C \urcorner}, x) \leftrightarrow x = \underline{\ulcorner A \urcorner},$$

hence by definition of A also

$$A \leftrightarrow \forall_x (x = \underline{\ulcorner A \urcorner} \to B(x))$$

and therefore

$$A \leftrightarrow B(\underline{\ulcorner A \urcorner}).$$

□

If $T = \mathrm{Th}(\mathcal{M})$, we obtain the semantical fixed point lemma above as a special case.

Theorem 4.9.2. *Let T be a consistent theory such that all elementary functions are representable in T. Then there cannot exist a formula $B(z)$ defining the notion of truth, i.e., such that for all closed formulas A*

$$T \vdash A \leftrightarrow B(\underline{\ulcorner A \urcorner}).$$

Proof. Assume we would have such a $B(z)$. Consider the formula $\neg B(z)$ and choose by the fixed point lemma a closed formula A such that

$$T \vdash A \leftrightarrow \neg B(\underline{\ulcorner A \urcorner}).$$

For this A we obtain $T \vdash A \leftrightarrow \neg A$, contradicting the consistency of T. □

If $T = \mathrm{Th}(\mathcal{M})$, Tarski's undefinability theorem is a special case of the previous theorem.

4.10 Recursive functions

Definition 4.10.1. *A relation R of arity r is said to be Σ^0_1-definable, if there is an elementary relation E, say of arity $r + l$, such that for all $\vec{n} = n_1, \dots, n_r$,*

$$R(\vec{n}\,) \Leftrightarrow \exists_{k_1} \dots \exists_{k_l} E(\vec{n}, k_1, \dots, k_l).$$

A partial function φ is said to be Σ^0_1-definable, if its graph

$$\{(\vec{n}, m) \mid \varphi(\vec{n}) \textit{ is defined and } \varphi(\vec{n} = m\}$$

is Σ^0_1-definable.

To say that a non-empty relation R is Σ^0_1-definable, or *recursively enumerable*, is equivalent to saying that the set of all sequences $\langle\vec{n}\rangle$ satisfying R can be enumerated (possibly with repetitions) by some elementary function $f\colon \mathbb{N}\to\mathbb{N}$. Such relations are called *elementarily enumerable*. For choose any fixed sequence $\langle a_1,\dots,a_r\rangle$ satisfying R and define

$$f(m) = \begin{cases} \langle (m)_1,\dots,(m)_r\rangle & \text{if } E((m)_1,\dots,(m)_{r+l}) \\ \langle a_1,\dots,a_r\rangle & \text{otherwise.} \end{cases}$$

Conversely, if R is elementarily enumerated by f, then

$$R(\vec{n}\,) \Leftrightarrow \exists_m (f(m) = \langle\vec{n}\rangle)$$

is a Σ^0_1-definition of R.

Definition 4.10.2. *The μ-recursive, or simply recursive functions are those partial functions which can be defined from the initial functions: constant* 0*, successor* S*, projections onto the i-th coordinate, addition* $+$*, modified subtraction* $\dot{-}$ *and multiplication* $\cdot$*, by applications of composition and unbounded minimisation. The latter is the scheme*

$$\frac{f \in \mathrm{Rec}^{(r+1)}}{\mu_y f \in \mathrm{Rec}^{(r)}},$$

where

$$(\mu_y f)(x_1,\dots,x_r) = \mu_y(f(x_1,\dots,x_r,y) = 0)$$

that is, the least number y such that $f(x_1,\dots,x_r,y')$ is defined for every $y' \le y$ and $f(x_1,\dots,x_k,y) = 0$.

Note that it is through unbounded minimisation that partial functions may arise.

Lemma 4.10.3. *Every elementary function is μ-recursive.*

Proof. By removing the bounds on μ one obtains μ-recursive definitions of the pairing functions π, π_1, π_2 and of Gödel's β-function. Then by removing all mention of bounds one sees that the μ-recursive functions are closed under unlimited primitive recursive definitions of the form:

$$f(\vec{m},0) = g(\vec{m}),$$

$$f(\vec{m},n+1) = h(n, f(\vec{m},n)).$$

Thus one can μ-recursively define bounded sums and bounded products, and hence all elementary functions. □

The converse of the previous lemma does not hold (why?). Call a relation R recursive, if its total characteristic function is recursive. One can show that a relation R is recursive if and only if both R and its complement are recursively enumerable.

4.11 Undecidability and incompleteness

Consider a consistent formal theory T with the property that all recursive functions are representable in T. This is a very weak assumption, as it is always satisfied if the theory allows to develop a certain minimum of arithmetic. We shall show that such a theory necessarily is undecidable. Then we prove Gödel's first incompleteness theorem, saying that every axiomatised such theory must be incomplete.

Definition 4.11.1. *In this section let $\mathcal{L}$ be an elementarily presented language with* 0, Succ, $=$ *in $\mathcal{L}$ and T a theory containing the equality axioms $\mathrm{Eq}_{\mathcal{L}}$. A set S of formulas is called recursive (recursively enumerable), if*

$$\ulcorner S \urcorner = \{\ulcorner A \urcorner \mid A \in S\}$$

is recursive (recursively enumerable).

Theorem 4.11.2 (Undecidability)**.** *Assume that T is a consistent theory such that all recursive functions are representable in T. Then T is not recursive.*

Proof. Assume that T is recursive. By assumption there exists a formula $B(z)$ representing $\ulcorner T \urcorner$ in T. Choose by the fixed point lemma a closed formula A such that

$$T \vdash A \leftrightarrow \neg B(\underline{\ulcorner A \urcorner}).$$

We shall prove $(*)$ $T \nvdash A$ and $(**)$ $T \vdash A$; this is the desired contradiction.

Proof of $(*)$. Assume $T \vdash A$. Then $A \in T$, hence $\ulcorner A \urcorner \in \ulcorner T \urcorner$, hence $T \vdash B(\underline{\ulcorner A \urcorner})$ (because $B(z)$ represents in T the set $\ulcorner T \urcorner$). By the choice of A it follows that $T \vdash \neg B(\underline{\ulcorner A \urcorner}$, which contradicts the consistency of T.

Proof of $(**)$. By $(*)$ we know $T \nvdash A$. Therefore $A \notin T$, hence $\ulcorner A \urcorner \notin \ulcorner T \urcorner$ and therefore $T \vdash \neg B(\underline{\ulcorner A \urcorner})$. By the choice of A it follows that $T \vdash A$. □

We now aim at Gödel's first incompleteness theorem.

Definition 4.11.3. *A theory T is consistent, if $\bot \notin T$; otherwise T is inconsistent. Recall that a theory T is complete, if for every closed formula $A \in \mathcal{L}$ we have $A \in T$ or $\neg A \in T$.*

Theorem 4.11.4 (Gödel's First Incompleteness Theorem)**.** *Assume that T is a recursively enumerable consistent theory with the property that all recursive functions are representable in T. Then T is incomplete.*

Proof. Let T be such a theory, which is supposed to be complete. Clearly, the set $F = \{\ulcorner A \urcorner \mid A \in \overline{\mathcal{L}}\}$ is elementary. Since T is complete, we have

$$a \notin \ulcorner T \urcorner \leftrightarrow a \notin F \vee \dot{\neg} a \in \ulcorner T \urcorner$$

with $\dot{\neg} a = \langle \mathrm{sn}(\to), a, \mathrm{sn}(\bot) \rangle$. Hence the complement of $\ulcorner T \urcorner$ is recursively enumerable as well, which means that $\ulcorner T \urcorner$ is recursive. Now the claim follows from the undecidability theorem above. □

There are very simple theories with the property that all recursive functions are representable in them; an example is a finitely axiomatised arithmetical theory Q due to Robinson. One can sharpen the Incompleteness Theorem, as one can produce a formula A such that neither A nor $\neg A$ is provable. The original idea for this sharpening is due to Rosser. Gödel's original first incompleteness theorem provided such an A under the assumption that the theory satisfied a stronger condition than mere consistency, namely ω-consistency. Rosser then improved Gödel's result by showing, with a somewhat more complicated formula, that consistency is all that is required.

A theory T in an elementarily presented language L is *axiomatised*, if it is given by a recursively enumerable axiom system Ax_T. One can show that the set Ax_T is elementary. According to the theorem of Gödel-Rosser, for every axiomatised consistent theory T satisfying certain weak assumptions, there is an undecidable sentence A meaning "for every proof of me there is a shorter proof of my negation". Because A is unprovable, it is clearly true. Gödel's Second Incompleteness Theorem provides a particularly interesting alternative to A, namely a formula Con_T expressing the consistency of T. Again it turns out to be unprovable and therefore true. The proof of this theorem in a sharpened form is due to Löb (see [19], section 3.6.2).

Theorem 4.11.5 (Gödel's Second Incompleteness Theorem)**.** *Let T be an axiomatised consistent extension of Robinson's Q, satisfying certain underivability conditions. Then $T \nvdash \mathrm{Con}_T$.*

Bibliography

[1] P. Aczel, M. Rathjen: *Constructive Set Theory*, book draft, 2010.

[2] S. Awodey: *Category Theory*, Oxford University Press, 2010.

[3] E. Bishop: *Foundations of Constructive Analysis*, McGraw-Hill, 1967.

[4] E. Bishop and D. S. Bridges: *Constructive Analysis*, Grundlehren der Math. Wissenschaften 279, Springer-Verlag, Heidelberg-Berlin-New York, 1985.

[5] D. S. Bridges and F. Richman: *Varieties of Constructive Mathematics*, Cambridge University Press, 1987.

[6] C. C. Chang, A. H. Keisler: *Model Theory*, Dover, 2012.

[7] J. Dugundji: *Topology*, Allyn and Bacon,1966.

[8] G. Gentzen: Über das Verh ältnis zwischen intuitionistischer und klassischer Arithmetik, Galley proof, Mathematische Annalen (received 15th March 1933). First published in English translation in *The collected papers of Gerhard Gentzen*, M.E. Szabo (editor), 53-67, Amsterdam (North-Holland).

[9] G. Gentzen: Untersuchungen über das logische Schließen I, II, Mathematische Zeitschrift, 39, 1935, 176-210, 405-431.

[10] K. Gödel: Zur intuitionistisohen Arithmetik und Zahlentheorie, in : Ergebnisse eines mathematischen Kolloquiums, Heft 4 (for 1931-1932, appeared in 1933), 34-38. Translated into English in *The undecidable*, M. Davis (editor), 75-81, under the title “On intuitionistic arithmetic and number theory”. For corrections of the translation see review in J.S.L. 31 (1966), 484-494.

[11] R. Goldblatt: *Topoi, The Categorial Analysis of Logic*, Dover, 2006.

[12] T. Jech: *Set theory*, Springer, 2002.

[13] A. N. Kolmogorov. On the principle of the excluded middle (in Russian). Mat. Sb., 32, 1925, 646-667.

[14] J. Lambek, P. J. Scott: *Introduction to higher order categorical logic*, Cambridge University Press, 1986.

[15] P. Martin-Löf: An intuitionistic theory of types: predicative part, in H. E. Rose and J. C. Shepherdson (Eds.) *Logic Colloquium’73*, pp.73-118, North-Holland, 1975.

[16] P. Martin-Löf: *Intuitionistic type theory: Notes by Giovanni Sambin on a series of lectures given in Padua, June 1980*, Napoli: Bibliopolis, 1984.

[17] H. Rogers: *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, 1967.

[18] H. Schwichtenberg, A. Troelstra: *Basic Proof Theory*, Cambridge University Press 1996.

[19] H. Schwichtenberg, S. Wainer: *Proofs and Computations*, Cambridge University Press 2012.

[20] T. Streicher: *Realizability*, Lecture Notes, TU Darmstadt, 2018.

[21] The Univalent Foundations Program: *Homotopy Type Theory: Univalent Foundations of Mathematics*, Institute for Advanced Study, Princeton, 2013.