Section 4.3 The fundamental group of the (higher) circle.

The loop-space $\Omega(\mathrm{base}) \equiv (\mathrm{base} =_{S^1} \mathrm{base})$

$\Omega(S^1, \mathrm{base})$

Definition 4.3.1 The integers as an inductive type (in HoTT book as a quotient of $\mathbb{N}$, see also in E. Rijke's Master Thesis p.33) we avoid the theory of quotients here!

$$\begin{array}{c} \vdots \\ \mathrm{neg}(1_{\mathbb{N}}) \\ \mathrm{neg}(0_{\mathbb{N}}) \\ 0_{\mathbb{Z}} \\ \mathrm{pos}(0_{\mathbb{N}}) \\ \mathrm{pos}(1_{\mathbb{N}}) \\ \vdots \end{array}$$

- $\mathrm{Form}_{\mathbb{Z}}$ $\mathbb{Z} : \mathcal{U}$

- $\mathrm{Intro}_{\mathbb{Z}}$ $\dfrac{}{0_{\mathbb{Z}} : \mathbb{Z}}$ $\dfrac{n : \mathbb{N}}{\mathrm{pos}(n) : \mathbb{Z}}$ $\dfrac{n : \mathbb{N}}{\mathrm{neg}(n) : \mathbb{Z}}$

- $\mathrm{Rec}_{\mathbb{Z}}$: $A : \mathcal{U}$ st. $a_0 : A$ $p : \mathbb{N} \to A$ $\nu : \mathbb{N} \to A$

  There is $f : \mathbb{Z} \to A$ st. $f(0_{\mathbb{Z}}) \equiv a_0$
  - $f(\mathrm{pos}(n)) \equiv \cancel{p(f(n))}\ p(n)$ $n : \mathbb{N}$
  - $f(\mathrm{neg}(n)) \equiv \cancel{\nu(f(n))}\ \nu(n)$ $n : \mathbb{N}$

- $\mathrm{Ind}_{\mathbb{Z}}$: $P : \mathbb{Z} \to \mathcal{U}$

  $a_0 : P(0_{\mathbb{Z}})$

  $G : \prod_{n : \mathbb{N}} P(\mathrm{pos}(n))$, $H : \prod_{n : \mathbb{N}} P(\mathrm{neg}(n))$

  There is $F : \prod_{x : \mathbb{Z}} P(x)$ st. $F(0_{\mathbb{Z}}) \equiv a_0$
  - $F(\mathrm{pos}(n)) \equiv G(n)$, $n : \mathbb{N}$
  - $F(\mathrm{neg}(n)) \equiv H(n)$, $n : \mathbb{N}$

- $\mathrm{Ind}_{\mathbb{Z}} \Rightarrow \mathrm{Rec}_{\mathbb{Z}}$: $P : \mathbb{Z} \to A$, $P \equiv A_{\mathbb{Z}}$, then $a_0 : A$, $G : \mathbb{N} \to A$, $H : \mathbb{N} \to A$

  $F(\mathrm{pos}(n)) \equiv G(n)$, $F(\mathrm{neg}(n)) \equiv H(n)$.

Proposition 4.3.2 The function $\mathrm{succ}_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ defined by

$$\boxed{\begin{array}{ll} \cdot\ \mathrm{succ}_{\mathbb{Z}}(0_{\mathbb{Z}}) \equiv \mathrm{pos}(0_{\mathbb{N}}) & \\ \cdot\ \mathrm{succ}_{\mathbb{Z}}(\mathrm{pos}(n)) \equiv \mathrm{pos}(\mathrm{succ}_{\mathbb{N}}(n)) & , n : \mathbb{N} \\ \cdot\ \mathrm{succ}_{\mathbb{Z}}(\mathrm{neg}(n)) \equiv \nu(n) & n : \mathbb{N} \end{array}}$$

where $\nu : \mathbb{N} \to \mathbb{Z}$ is defined by $Rec_{\mathbb{N}}$:

$$\nu(0) \equiv 0_{\mathbb{Z}}$$
$$\nu(succ_{\mathbb{N}}(n)) \equiv neg(n)$$

is well-defined and an equivalence. ($term : \mathbb{Z} \to \mathbb{Z}$)

Proof: We use $Rec_{\mathbb{Z}}$: $0_{\mathbb{Z}} : \mathbb{Z}$, $p : \mathbb{N} \to \mathbb{Z}$, $\nu : \mathbb{N} \to \mathbb{Z}$

where $p \equiv pos \circ succ_{\mathbb{N}} : \mathbb{N} \to \mathbb{Z}$

$$succ_{\mathbb{Z}}(pos(n)) \equiv p(n) \equiv pos(succ_{\mathbb{N}}(n)).$$

Let $pred : \mathbb{Z} \to \mathbb{N}$ defined similarly ($pos \rightleftarrows neg$):

$$\boxed{\begin{aligned} pred_{\mathbb{Z}}(0_{\mathbb{Z}}) &\equiv neg(0_{\mathbb{N}}) \\ pred_{\mathbb{Z}}(neg(n)) &\equiv neg(succ_{\mathbb{N}}(n)) \\ pred_{\mathbb{Z}}(pos(n)) &\equiv \eta(n) \end{aligned}}$$

$$pred_{\mathbb{Z}}(pos(0_{\mathbb{N}})) \equiv 0_{\mathbb{Z}}$$
$$pred_{\mathbb{Z}}(pos(succ_{\mathbb{N}}(n))) \equiv pos(n)$$

and $\eta(0) \equiv 0_{\mathbb{Z}}$

$\eta(succ_{\mathbb{N}}(n)) \equiv pos(n)$

Clearly: $\prod_{x:\mathbb{Z}} succ_{\mathbb{Z}}(pred_{\mathbb{Z}}(x)) = x$ and $\prod_{x:\mathbb{Z}} pred_{\mathbb{Z}}(succ_{\mathbb{Z}}(x)) = x$

By $Ind_{\mathbb{Z}}$: we need $G(n) :\equiv \big(succ_{\mathbb{Z}}(pred_{\mathbb{Z}}(pos(n))) = pos(n)\big) \equiv$

$$succ_{\mathbb{Z}}(\eta(n)) = pos(n)$$

Ind on $\mathbb{N}$: $n = 0$: $succ_{\mathbb{Z}}(0) \equiv pos(0_{\mathbb{N}})$ ✓

If $succ_{\mathbb{Z}}(\eta(n)) = pos(n)$

which is ... $succ_{\mathbb{Z}}(\eta(succ_{\mathbb{N}}(n))) \equiv succ_{\mathbb{Z}}(pos(n)) \equiv pos(succ_{\mathbb{N}}(n))$ □

Proposition 4.3.3 $\mathrm{isSet}(\mathbb{Z}) \equiv \prod_{x,y:\mathbb{Z}} \prod_{p,q:x=_{\mathbb{Z}}y} (p=q)$ (to be done in Munich rebearbeitet)

(like $\mathbb{N}$) (Just mention it.)

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Remark 4.3.4

- $\mathrm{loop}^{-} : \mathbb{Z} \longrightarrow (\mathrm{base} =_{S^1} \mathrm{base})$ is defined by $\mathrm{Rec}_{\mathbb{Z}}$: (in Kijte it is named wind)

$$\mathrm{loop}^{x} : \mathrm{base} =_{S^1} \mathrm{base} \quad , \quad x:\mathbb{Z}$$

$$\left[\ \mathrm{loop}^{0_{\mathbb{Z}}} \equiv \mathrm{refl}_{\mathrm{base}} : \Omega(\mathrm{base})\right.$$

$$\cancel{\mathrm{loop}^{pos(n)} \equiv \mathrm{loop}^{n} * \mathrm{loop}}$$

$$\cancel{\mathrm{loop}^{neg(n)} \equiv}$$

$$\left[\begin{array}{l} \mathrm{loop}^{pos(0)} \equiv \mathrm{loop} \\ \mathrm{loop}^{pos(succ_{\mathbb{N}}(n))} \equiv \mathrm{loop}^{pos(n)} * \mathrm{loop} \equiv \underbrace{\mathrm{loop} * \cdots * \mathrm{loop}}_{succ_{\mathbb{N}}(n)\ \text{times}} \end{array}\right.$$

$$\left[\begin{array}{l} \mathrm{loop}^{neg(0)} \equiv \mathrm{loop}^{-1} \\ \mathrm{loop}^{neg(succ_{\mathbb{N}}(n))} \equiv \underbrace{\mathrm{loop}^{neg(n)}}_{\underbrace{\mathrm{loop}^{-1} * \cdots * \mathrm{loop}^{-1}}_{succ(n)\ \text{times}}} * \mathrm{loop}^{-1} \end{array}\right.$$

- We'd like to know, as it is expected by the intuitive Univalence Principle for paths for the S^1 : that these are all paths-loops at base. From this one can infer (using some defs that the fundamental group of S^1 is $\mathbb{Z}$).

- Remark/Definition 4.3.5 By $\mathrm{Rec}_{(S^1)'}$, there is a function

$$\mathrm{code}' : (S^1)' \to \mathcal{U} \quad \text{st}$$

$$\mathrm{code}'(\mathrm{base}') \equiv \mathbb{Z}$$

$$\mathrm{ap}_{\mathrm{code}'}(\mathrm{loop}') = \mathrm{ua}(\mathrm{succ}_{\mathbb{Z}})$$

$\mathcal{U} : \mathcal{U}'$, $\mathbb{Z} : \mathcal{U}$ $\quad \mathrm{succ}_{\mathbb{Z}} : \mathbb{Z} \simeq_{\mathcal{U}} \mathbb{Z}$, hence

$$\mathrm{ua}(\mathrm{succ}_{\mathbb{Z}}) : \mathbb{Z} =_{\mathcal{U}} \mathbb{Z}$$

Corollary / Def 9.3.6 Through $S^1 \simeq (S^1)'$ there is a function

$$\boxed{\begin{aligned} &\mathrm{code} : S^1 \longrightarrow \mathcal{U} \quad \text{s.t.} \\ &\mathrm{code}(\mathrm{base}) \equiv \mathbb{Z} \quad \text{and} \\ &\mathrm{ap}_{\mathrm{code}}(\mathrm{loop}) = \mathrm{ua}(\mathrm{succ}_{\mathbb{Z}}) \end{aligned}}$$

(code is called the universal cover of S^1.)

Proof: Let $g : S^1 \to (S^1)'$ s.t. $g(\mathrm{base}) \equiv \mathrm{base}'$

$$\begin{array}{l} (S^1)' \\ \big\downarrow \mathrm{code}' \\ \mathcal{U} \end{array} \qquad \mathrm{ap}_g(\mathrm{loop}) = \mathrm{loop}'$$

Define $\mathrm{code} \equiv \mathrm{code}' \circ g$

Then, $\mathrm{code}(\mathrm{base}) \equiv \mathrm{code}'(g(\mathrm{base})) \equiv \mathrm{code}'(\mathrm{base}') \equiv \mathbb{Z}$.

$$\begin{aligned} \mathrm{ap}_{\mathrm{code}' \circ g}(\mathrm{loop}) &= \mathrm{ap}_{\mathrm{code}'}(\mathrm{ap}_g(\mathrm{loop})) \\ &= \mathrm{ap}_{\mathrm{code}'}(\mathrm{loop}') \\ &= \mathrm{ua}_4(\mathrm{succ}_{\mathbb{Z}}) \end{aligned}$$

(use ap_{g} twice)

$\square$

(• Remark: This is sloppy written in the HoTT-book.)

Lemma 9.3.7 $\mathrm{loop}_*^{\mathrm{code}} : \mathrm{code}(\mathrm{base}) \longrightarrow \mathrm{code}(\mathrm{base}) \equiv \mathbb{Z} \to \mathbb{Z}$

$(\mathrm{loop}^{-1})_*^{\mathrm{code}}$

$$\boxed{\begin{aligned} &\text{(i)} \quad \mathrm{loop}_*^{\mathrm{code}}(x) = \mathrm{succ}_{\mathbb{Z}}(x) \\ &\text{(ii)} \quad (\mathrm{loop}^{-1})_*^{\mathrm{code}}(x) = \mathrm{pred}_{\mathbb{Z}}(x) \end{aligned}} \qquad x : \mathbb{Z}$$

Proof: (i) By Corol 3.2.7 ($A : \mathcal{U}$, $P : A \to \mathcal{U}$, $p : x =_A x$, $u : P(x)$, $f : P(x) \to P(x)$ isequiv(f), $\mathrm{ap}_P(p) = \mathrm{ua}(f)$, then $p_*^P(u) = f(u)$)

$\mathrm{loop}_*^{\mathrm{code}}(x) = \mathrm{succ}_{\mathbb{Z}}(x)$ $\qquad \mathrm{ap}_{\mathrm{code}}(\mathrm{loop}) = \mathrm{ua}(\mathrm{succ}_{\mathbb{Z}})$

(ii) By Corol. 2.4.3 $(p^{-1})_*^P : P(y) \to P(x)$ is the inverse of p_*^P.

Hence $(\mathrm{loop}^{-1})_*^{\mathrm{code}}$ is the inverse of $\mathrm{succ}_{\mathbb{Z}}$ i.e. $\mathrm{pred}_{\mathbb{Z}}$. □

Definition 4.3.8 encode : $\prod_{x:S^1} \left((\mathrm{base} =_{S^1} x) \to \mathrm{code}(x) \right)$

$$\boxed{\mathrm{encode}(x, p) \equiv p_*^{\mathrm{code}}(0_{\mathbb{Z}})}$$

where

$p : \mathrm{base} =_{S^1} x$

$p_*^{\mathrm{code}} : \mathrm{code}(\mathrm{base}) \to \mathrm{code}(x) \equiv \mathbb{Z} \to \mathrm{code}(x)$

hence $p_*^{\mathrm{code}}(0_{\mathbb{Z}}) : \mathrm{code}(x)$

~~Definition 4.3.9~~ Proposition There is a dep function $\boxed{\mathrm{decode} : \prod_{x:S^1} \left(\mathrm{code}(x) \to (\mathrm{base} =_{S^1} x) \right)}$

Proof : $P : S^1 \to \mathcal{U}$

$P(x) :\equiv \overset{a_1(x)}{\mathrm{code}(x)} \to (\overset{a_2(x)}{\mathrm{base} =_{S^1} x})$

$P(\mathrm{base}) \equiv \mathbb{Z} \to (\mathrm{base} =_{S^1} \mathrm{base})$

We take $\mathrm{loop}^- : \mathbb{Z} \to (\mathrm{base} =_{S^1} \mathrm{base})$

We also need, in order to apply Ind_{S^1}, some

$$\ell : \left(\mathrm{loop}_*^P(\mathrm{loop}^-) =_{P(\mathrm{base})} \mathrm{loop}^- \right)$$

But, $\mathrm{loop}_*^P(\mathrm{loop}^-) \equiv \mathrm{loop}_*^{a_1 \to a_2}(\mathrm{loop}^-)$

$= \mathrm{loop}_*^{a_2} \circ (\mathrm{loop}^-) \circ (\mathrm{loop}^{-1})_*^{a_1}$ • Prop 2.4.5 $p_*^{P \to Q}(f) = p_*^{Q} \circ f \circ (p^{-1})_*^P$

$= \mathrm{loop}_*^{x \mapsto (\mathrm{base} =_{S^1} x)} \circ \left((\mathrm{loop}^-) \circ \mathrm{pred}_{\mathbb{Z}} \right)$ • Lemma 4.3.7

$= \left(\mathrm{loop}^- \circ \mathrm{pred}_{\mathbb{Z}} \right) \cdot \mathrm{loop}$ • Ex. $p_*^{x \mapsto a = x}(q) = q \cdot p$

$= \left(\lambda(x:\mathbb{Z}).\, \mathrm{loop}^{\mathrm{pred}(x)} \right) \cdot \mathrm{loop}$

$$\overset{(*)}{=} \lambda(x:\mathbb{Z}).\ \mathrm{loop}^{-}(x)$$

$$\equiv \mathrm{loop}^{-}$$

Here we use (a) $\prod_{x:\mathbb{Z}} \left(\mathrm{loop}^{\mathrm{pred}(x)} \cdot \mathrm{loop} = \mathrm{loop}^{x} \right)$

(b) function-extensionality.

For (a) we prove it inductively: $\underline{\underline{(\mathrm{Ex})}}$

$0_{\mathbb{Z}}$: $\quad \mathrm{loop}^{\mathrm{pred}(0_{\mathbb{Z}})} \cdot \mathrm{loop} \equiv \mathrm{loop}^{\mathrm{neg}(0_{\mathbb{N}})} \cdot \mathrm{loop}$

$$\equiv \mathrm{loop}^{-1} \cdot \mathrm{loop}$$

$$= \mathrm{refl}_{\mathrm{base}}$$

$$\equiv \mathrm{loop}^{0_{\mathbb{Z}}}$$

$\mathrm{pos}(n)$-case: $\quad \mathrm{loop}^{\mathrm{pred}(\mathrm{pos}(n))} \cdot \mathrm{loop} = \mathrm{loop}^{\mathrm{pos}(n)}$ is to be shared

and for that we use induction:

$n=0$: $\quad \mathrm{loop}^{\mathrm{pred}(\mathrm{pos}(0_{\mathbb{N}}))} \cdot \mathrm{loop} \equiv \mathrm{loop}^{0_{\mathbb{Z}}} \cdot \mathrm{loop}$

$$\equiv \mathrm{refl}_{\mathrm{base}} \cdot \mathrm{loop}$$

$$\equiv \mathrm{loop}$$

$$\equiv \mathrm{loop}^{\mathrm{pos}(0)}$$

$\mathrm{succ}(n)$-case: $\quad \mathrm{loop}^{\mathrm{pred}(\underbrace{\mathrm{pos}(\mathrm{succ}_{\mathbb{N}}(n))})} \cdot \mathrm{loop} \equiv$

$$\mathrm{loop}^{\mathrm{pos}(n)} \cdot \mathrm{loop} = \mathrm{loop}^{\mathrm{pos}(\underbrace{\mathrm{succ}_{\mathbb{N}}(n)})}$$

Similarly for the $\mathrm{neg}(n)$-case.

By ~~Ind~~ ind_{S^1} $\boxed{\begin{array}{l} \mathrm{decode}(\mathrm{base}) \equiv \mathrm{loop}^{-} \\ \mathrm{apd}_{\mathrm{decode}}(\mathrm{loop}) = \ell \end{array}}$ [illegible] and $\mathrm{apd}_{\mathrm{decode}} : \prod_{p:\mathrm{base}=_{S^1}\mathrm{base}} p_*(\mathrm{decode}(\mathrm{base})) = \mathrm{decode}(b)$

loop^{-} $\ni$

Lemma 4.3.10 Let $x : S^1$, $p : \mathrm{base} =_{S^1} x$. Then

$$\boxed{\mathrm{decode}\left(x, \mathrm{encode}(x,p)\right) = p}$$

Proof: $\mathrm{decode}\left(x, \mathrm{encode}(x,p)\right) \equiv \mathrm{decode}\left(x, p_*^{\mathrm{code}}(0_{\mathbb{Z}})\right) \equiv \mathrm{decode}(x)\left([illegible]\right)$

$$= p_*^{P}(\mathrm{loop}^-)\left(p_*^{\mathrm{code}}(0_{\mathbb{Z}})\right)$$

Since $\mathrm{apd}_{\mathrm{decode}}(p) : \left(p_*^{P}(\mathrm{loop}^-) = \mathrm{decode}(x)\right)$

$$(p \equiv a_1 \to a_2)$$

$$= p_*^{x \mapsto \mathrm{base} \Rightarrow x}\left(\mathrm{loop}^-\left((p^{-1})_*^{\mathrm{code}}\left(p_*^{\mathrm{code}}(0_{\mathbb{Z}})\right)\right)\right)$$

$$= p_*^{x \mapsto \mathrm{base} \Rightarrow x}\left(\mathrm{loop}^-\left(\left(p \cdot p^{-1}\right)_*^{\mathrm{code}}(0_{\mathbb{Z}})\right)\right)$$

$$= p_*^{x \mapsto \mathrm{base} \Rightarrow x}\left(\mathrm{loop}^-\left((\mathrm{refl}_{\mathrm{base}})_*^{\mathrm{code}}(0_{\mathbb{Z}})\right)\right)$$

$$= p_*^{x \mapsto \mathrm{base} \Rightarrow x}\left(\mathrm{loop}^-\left(\mathrm{id}_{\underset{\mathbb{Z}}{\mathrm{code}(\mathrm{base})}}(0_{\mathbb{Z}})\right)\right)$$

$$= p_*^{x \mapsto \mathrm{base} \Rightarrow x}\left(\mathrm{loop}^-\left(0_{\mathbb{Z}}\right)\right)$$

$$\equiv p_*^{x \mapsto \mathrm{base} \Rightarrow x}\left(\mathrm{refl}_{\mathrm{base}}\right)$$

$$= \mathrm{refl}_{\mathrm{base}} \cdot p$$

$$= p.$$

No induction on S^1 is needed here $\square$

Lemma 4.3.11: Let $x : S^1$, $c : \mathrm{code}(x)$. Then

$$\boxed{\mathrm{encode}\left(x, \mathrm{decode}(x,c)\right) = c}$$

$\cancel{\left(\mathrm{encode}(x, \mathrm{decode}(x,c)) = c\right)}$

Proof: We show $\prod_{x:S^1}\left(\underbrace{\prod_{c:\mathrm{code}(x)}\left(\mathrm{encode}(x, \mathrm{decode}(x,c)) = c\right)}_{P(x)}\right)$

We need to find $a_0 : P(\mathrm{base}) \equiv \prod_{c:\mathrm{code}(\mathrm{base})}\left(\mathrm{encode}(\mathrm{base}, \mathrm{decode}(\mathrm{base},c)) = c\right)$

$$\equiv \prod_{x:\mathbb{Z}} \mathrm{encode}(\mathrm{base}, \mathrm{decode}(\mathrm{base}, x)) = x$$

~~Since decode(base) ≡ loop ... by and decode ... (loop)~~

$$\equiv \prod_{x:\mathbb{Z}} \mathrm{encode}(\mathrm{base}, \mathrm{decode}(\mathrm{base})(x)) = x$$

$\mathrm{decode}(\mathrm{base}) \equiv \mathrm{loop}^{-}$

$$\equiv \prod_{x:\mathbb{Z}} \left(\mathrm{encode}(\mathrm{base}, \mathrm{loop}^x) = x\right)$$

$$\equiv \prod_{x:\mathbb{Z}} \left[(\mathrm{encode}(\mathrm{base}))(\mathrm{loop}^x) = x\right]$$

$n \equiv 0_{\mathbb{Z}}$: $\left(\mathrm{encode}(\mathrm{base})\right)\left(\mathrm{loop}^{0_{\mathbb{Z}}}\right) \equiv \mathrm{encode}(\mathrm{base}, \mathrm{refl}_{\mathrm{base}})$

~~transport~~

$$\equiv \left(\mathrm{refl}_{\mathrm{base}}\right)^{\mathrm{code}}_{*}(0_{\mathbb{Z}})$$

$$\equiv \mathrm{id}_{\mathrm{code}(\mathrm{base})}(0_{\mathbb{Z}})$$

$$\equiv \mathrm{id}_{\mathbb{Z}}(0_{\mathbb{Z}})$$

$$\equiv 0_{\mathbb{Z}}.$$

$k = \mathrm{pos}(n)$ Ind on $\mathbb{N}$.

$n = 0_{\mathbb{N}}$: $\mathrm{encode}(\mathrm{base}, \mathrm{loop}^{\mathrm{pos}(0)}) \equiv \mathrm{encode}(\mathrm{base}, \mathrm{loop})$

$$\equiv \mathrm{loop}_*^{\mathrm{code}}(0_{\mathbb{Z}})$$

$$\overset{L.4.37}{=} \mathrm{succ}_{\mathbb{Z}}(0_{\mathbb{Z}})$$

$$\equiv \mathrm{pos}(0_{\mathbb{N}})$$

$\mathrm{succ}_{\mathbb{N}}(n)$: $\mathrm{encode}(\mathrm{base}, \mathrm{loop}^{\mathrm{pos}(\mathrm{succ}_{\mathbb{N}}(n))}) \equiv \mathrm{encode}(\mathrm{base}, \mathrm{loop}^{\mathrm{pos}(n)} \circ \mathrm{loop})$

$$\equiv \left(\mathrm{loop}^{\mathrm{pos}(n)} \cdot \mathrm{loop}\right)_*^{\mathrm{code}}(0_{\mathbb{Z}})$$

$$= \mathrm{loop}_*^{\mathrm{code}}\left(\left(\mathrm{loop}^{\mathrm{pos}(n)}\right)_*^{\mathrm{code}}(0_{\mathbb{Z}})\right)$$

$$\overset{I.H}{=} \mathrm{loop}_*^{\mathrm{code}}(\mathrm{pos}(n))$$

$$\overset{L.4.3.7}{=} \mathrm{succ}_{\mathbb{Z}}(\mathrm{pos}(n))$$

$$\underset{\mathrm{succ}_{\mathbb{Z}}}{\overset{\mathrm{def}}{\equiv}} \underline{\mathrm{pos}(\mathrm{succ}_{\mathbb{N}}(n))}$$

Similarly for

$u = \mathrm{neg}(n)$ $\begin{cases} n = 0 \\ \mathrm{succ}(n) \end{cases}$

So, there is such $a_0 : P(\mathrm{base})$

To apply Ind_{S^1} we need $\mathrm{loop}_*^P(a_0) = a_0$

But a_0 and $\mathrm{loop}_*^P(a_0)$ are term of type $\prod_{x:\mathbb{Z}} (\mathrm{encode}(\mathrm{base}, \mathrm{loop}^x) = x)$

$$a_0(x),\ \mathrm{loop}_*^P(a_0)(x) : \mathrm{encode}(\mathrm{base}, \mathrm{loop}^x) = x$$

Since $\mathbb{Z}$ is a set : $a_0(x) = \mathrm{loop}_*^P(a_0)(x)$.

By Funextensionality $a_0 = \mathrm{loop}_*^P(a_0)$. So, we can use Ind_{S^1} to complete the proof. ∎

Theorem 4.3.12 $\prod_{x:S^1} \left[\left(\mathrm{base} =_{S^1} x \right) \simeq \mathrm{code}(x) \right]$

Proof: Use for $x:S^1$ lemmas 4.3.10, 4.3.11

Corollary 4.3.13: $\boxed{\Omega(S^1, \mathrm{base}) \simeq \mathbb{Z}}$

Proof: From Th. 4.3.12 for $x \equiv \mathrm{base}$.

· With an appropriate def of the fundamental group

$$\pi_1(S^1) = \mathbb{Z}$$

where $\pi_1(S^1) \equiv \| \Omega(S^1, \mathrm{base}) \|_0 = \| \mathbb{Z} \|_0 = \mathbb{Z}$

(and $\|A\|_0$ is the (set) truncation of A) the closest set to A

of HoTT.

· Syntactical things (Canonicity property)

· Semantics of HoTT