Chapter 4: Higher Inductive Types (very recent theory, under development, HITs)

Section 4.1. The higher interval (formal representation of the Interval in Type Theory)

- **Definition 4.1.1**

- $\mathrm{Form}_{\mathbb{I}}$: $\mathbb{I} : \mathcal{U}$

- $\mathrm{Intro}_{\mathbb{I}}$: $\dfrac{}{0_{\mathbb{I}} : \mathbb{I}}$ $\quad \dfrac{}{1_{\mathbb{I}} : \mathbb{I}}$, point-constructors

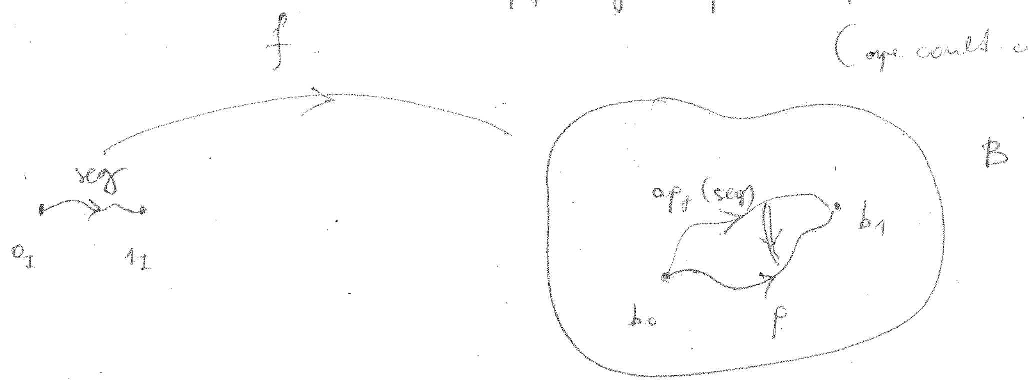
  $\dfrac{}{seg : 0_{\mathbb{I}} = 1_{\mathbb{I}}}$, path-constructor (The new thing wrt. inductive types)

- $\mathrm{Rec}_{\mathbb{I}}$: $B : \mathcal{U}$ (computation type)

  $b_0, b_1 : B$, $p : b_0 =_B b_1$

  There is $f : \mathbb{I} \to B$ s.t.
  - $f(0_{\mathbb{I}}) \equiv b_0$
  - $f(1_{\mathbb{I}}) \equiv b_1$
  - $ap_f(seg) := p$ (propositionally equal by definition)

  (one could consider $\equiv$, cubicalTT)



$$ap_{f\,0,1} : (0_{\mathbb{I}} =_{\mathbb{I}} 1_{\mathbb{I}}) \to (b_0 =_B b_1)$$

- $\mathrm{Ind}_{\mathbb{I}}$: $P : \mathbb{I} \to \mathcal{U}$

  $b_0 : P(0_{\mathbb{I}})$, $b_1 : P(1_{\mathbb{I}})$ $\qquad seg_*^P : P(0_{\mathbb{I}}) \to P(1_{\mathbb{I}})$

  $p : seg_*^P(b_0) =_{P(1_{\mathbb{I}})} b_1$

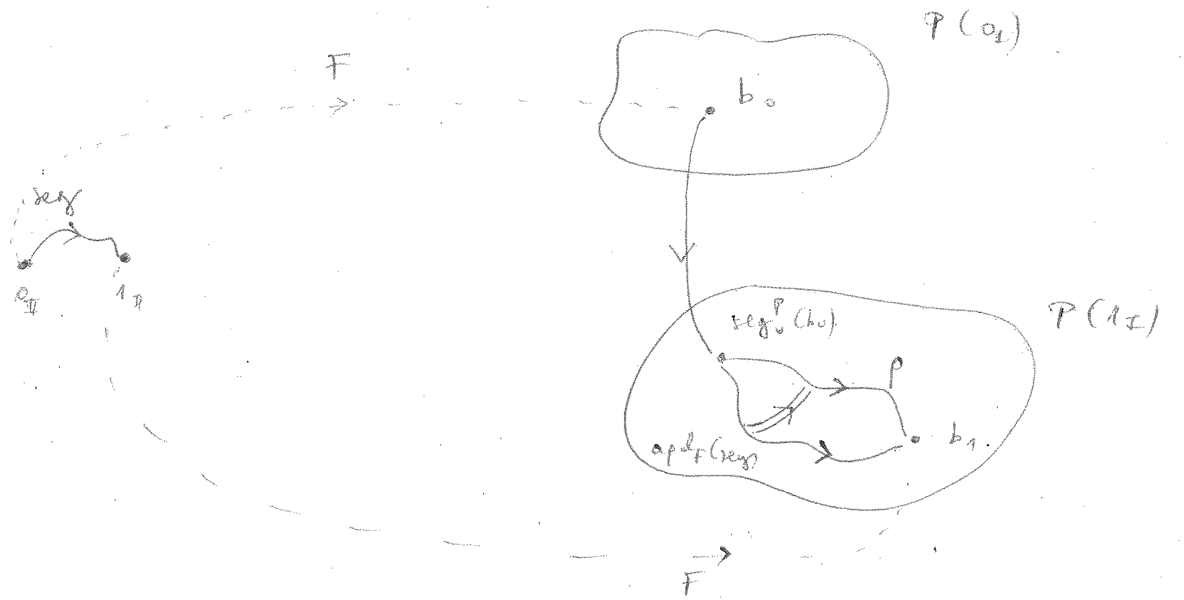
  Then there is $F : \prod_{x:\mathbb{I}} P(x)$ s.t.
  - $F(0_{\mathbb{I}}) \equiv b_0$
  - $F(1_{\mathbb{I}}) \equiv b_1$
  - $apd_F(seg) := p$

• where $\mathrm{apd}_F : \prod_{q : x =_I y} \left(q_*^P(F(x)) =_{P(y)} F(y) \right)$

hence $\mathrm{apd}_F(\mathrm{seg}) : \left(\mathrm{seg}_*^P(F(0_I)) =_{P(1_I)} F(1_I) \right) \equiv \left(\mathrm{seg}_*^P(b_0) =_{P(1_I)} b_1 \right)$

i.e., apd_F is of the same type as p and the equality is meaningful.

Actually $\mathrm{apd}_F(\mathrm{seg})$ is a path from $\mathrm{seg}_*^P(b_0)$ to b_1 in $P(1_I)$ and by Nat-Thesis it has to be equal to p.



• I.e., since you need to have some $F : \prod_{x : I} P(x)$ s.t. $F(0_I) \equiv b_0$, $F(1_I) \equiv b_1$, and since from the data given you can generate only $\mathrm{apd}_F(\mathrm{seg}) : \mathrm{seg}_*^P(b_0) =_{P(1_I)} b_1$ you start from an arbitrary $p : \mathrm{seg}_*^P(b_0) =_{P(1_I)} b_1$ and you require the last clause.

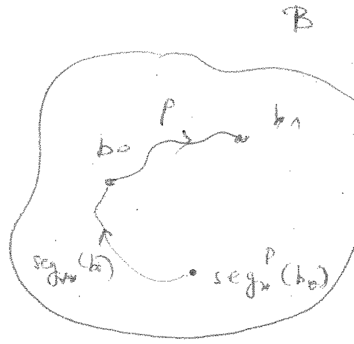
<u>Proposition 4.1.2</u>: $\mathrm{Ind}_I \Rightarrow \mathrm{Rec}_I$

<u>Proof</u>: Let $P \equiv \overline{B}_I$ and $p : b_0 =_B b_1$

$\mathrm{seg}_*^P : P(b_0) \longrightarrow P(b_1) \equiv B \to B$

By Proposition 2.3.2 there is a term:

$\mathrm{seg}_{**}(b_0) : \left(\mathrm{seg}_*^P(b_0) =_B b_0 \right)$



$\boxed{P : A \to U}$ $P(x) \equiv B$, for every $x : A$, $x, y : A$, $p : x =_A y$, $b_0 : B$. Then there is $p_{**}(b_0) : p_*^{\overline{B}_A}(b_0) = b_0$ s.t. $(\mathrm{refl}_x)_{**}(b_0) \equiv \mathrm{refl}_{b_0}$ ┘

Hence $\quad seg_{**}(b_0) * p : (seg_*^{?}(b_0) = b_1)$

By Ind_I there is $F : I \to B$ s.t. $F(0_I) \equiv b_0$, $F(1_I) \equiv b_1$ and

$$apd_F(seg) = seg_{**}(b_0) * p \qquad (1)$$

By Proposition 2.3.4 $\left[f : A \to B,\ p : x =_A y \text{, then } apd_f(p) = p_{**}(f(x)) * ap_f(p) \right]$ which relates the dependent application to the non-dependent one we get

$$apd_F(seg) = seg_{**}(F(0)) * ap_F(seg)$$
$$\equiv seg_{**}(b_0) * ap_F(seg) \qquad (2)$$

$$\xRightarrow{(1),(2)} \quad \cancel{seg_{**}(b_0)} * p = \cancel{seg_{**}(b_0)} * ap_F(seg)$$

$$\xRightarrow[\text{both sides}]{* \, seg_{**}(b_0)^{-1}} \quad p = ap_F(seg) \qquad \square$$

<u>Remark 4.1.3</u> If X is a top. space and $x_0, x_1 \in X$, a path in X from x_0 to x_1 is a continuous map $\gamma : I \to X$ s.t. $\gamma(0) = x_0$ and $\gamma(1) = x_1$. In ITT + HITS a path p from x_0 to x_1 is an object "properly equal" to $ap_f(seg)$. This is the type-theoretic translation of the above topological concept. Actually one shows

$$(x =_A y) \simeq_u \sum_{f : I \to A} \left((f(0_I) =_A x) \times (f(1_I) =_A y) \right)$$

- If $x =_A y$ then by Rec_I there is $f : I \to A$ $f(0_I) \equiv x$ ($\Rightarrow f(0_I) = x$) take refl$_x$ and $f(1_I) \equiv y$.
- If $f : I \to A$ s.t. $f(0_I) =_A x$ and $f(1_I) =_A y$, then

$$ap_f : (0_I =_I 1_I) \to (f(0_I) =_A f(1_I))$$
$$\qquad\qquad\qquad\qquad\qquad \| \qquad\quad \|$$
$$\qquad\qquad\qquad\qquad\qquad x \qquad\quad y$$

$ap_f(seg) : f(0_I) =_A f(1_I)$ and then with the obvious concatenation you take a proof that $x =_A y$.

Corollary 4.1.4 Let $P : \mathbb{I} \to \mathcal{U}$

$b_0 : P(0_{\mathbb{I}})$

$b_1 \equiv seg_*^P(b_0) : P(1_{\mathbb{I}})$ $\qquad seg : 0_{\mathbb{I}} =_{\mathbb{I}} 1_{\mathbb{I}}$

$\qquad seg_*^P : P(0_{\mathbb{I}}) \to P(1_{\mathbb{I}})$

$refl_{b_1} : seg_*^P(b_0) =_{P(1_{\mathbb{I}})} b_1$

By $Ind_{\mathbb{I}}$, there is $F : \prod_{x : \mathbb{I}} P(x)$ s.t.
- $F(0_{\mathbb{I}}) \equiv b_0$
- $F(1_{\mathbb{I}}) \equiv b_1$
- $apd_F(seg) = refl_{b_1}$

Corollary 4.1.5. If $P : \mathbb{I} \to \mathcal{U}$, the inhabitedness of $\prod_{x : \mathbb{I}} P(x)$ rests on the inhabitedness of $P(0_{\mathbb{I}})$. (i.e. if we find $b_0 : P(0_{\mathbb{I}})$, then we get by Cor. 4.1.4 some $F : \prod_{x : \mathbb{I}} P(x)$.)

Proof:

Corollary 4.1.6. TFTH: $\prod_{x : \mathbb{I}} (0_{\mathbb{I}} =_{\mathbb{I}} x)$.

Proof: By Cor. 4.1.5 it suffices to find $b_0 : P(0_{\mathbb{I}}) \equiv 0_{\mathbb{I}} =_{\mathbb{I}} 0_{\mathbb{I}}$.

Take $b_0 \equiv refl_{0_{\mathbb{I}}}$.

Remark 4.1.7: Corol. 4.1.6 expresses that $\mathbb{I}$ is contractible ("shrinkable to a point") where, if $A : \mathcal{U}$

$$isContr(A) \equiv \sum_{x : A} \prod_{y : A} (x =_A y)$$

For $\mathbb{I}$ take e.g. $x \equiv 0_{\mathbb{I}}$.

Corollary 4.1.8 $A : \mathcal{U}$, $f : \mathbb{I} \to A$. Then $f(0_{\mathbb{I}}) =_A f(1_{\mathbb{I}})$.

Proof: $ap_f : 0_{\mathbb{I}} =_{\mathbb{I}} 1_{\mathbb{I}} \to f(0_{\mathbb{I}}) =_A f(1_{\mathbb{I}})$

$ap_f(seg) : f(0_{\mathbb{I}}) =_A f(1_{\mathbb{I}})$

Remark 4.1.9 If we know that $n =_{\mathbb{N}} m \Leftrightarrow n \equiv m$ (Th. 7.2.6), then $f : \mathbb{I} \to \mathbb{N}$ implies that f is constant.
Proof: By Cor 4.1.6 $f(0) =_{\mathbb{N}} f(x)$, for every x (so $f(x) \equiv f(0)$).

Theorem 9.1.10 (function extensionality) Let $f, g : A \to B$ s.t. $\Phi : \prod_{x:A} f(x) =_B g(x)$.

Then $f =_{A \to B} g$

Proof: By Cor. 9.1.8 it suffices to show that there is

$h : \mathbb{I} \to (A \to B)$ s.t. $h(0_{\mathbb{I}}) = f$ and $h(1_{\mathbb{I}}) = g$

- Let $x : A$, $f(x), g(x) : B$ and $\Phi(x) : f(x) =_B g(x)$.

  By $Rec_{\mathbb{I}}$ there is $h_x : \mathbb{I} \to B$ s.t.
  - $h_x(0_{\mathbb{I}}) \equiv f(x)$
  - $h_x(1_{\mathbb{I}}) = g(x)$
  - $ap_{h_x}(seg) = \Phi(x)$

- We define $h(i) \equiv \lambda(x:A).\, h_x(i)$, for every $i : \mathbb{I}$

  Hence $h(0_{\mathbb{I}}) \equiv \lambda(x:A).\, h_x(0_{\mathbb{I}}) \equiv \lambda(x:A).\, f(x) \equiv f$

  $h(1_{\mathbb{I}}) \equiv \lambda(x:A).\, h_x(1_{\mathbb{I}}) \equiv \lambda(x:A).\, g(x) \equiv g$. $\square$

Remark 9.1.11 Similarly one gets that if $P : A \to \mathcal{U}$

$\Phi, \Psi : \prod_{x:A} P(x)$ s.t. $H : \prod_{x:A} (\Phi(x) =_{P(x)} \Psi(x))$,

then $\Phi =_{\prod_{x:A} P(x)} \Psi$.

Ex.

Exercise: $B : \mathcal{U}$, $b_0, b_1 : \mathcal{U}$, $p : b_0 =_B b_1$

$f, g : \mathbb{I} \to B$ s.t.
- $f(0_{\mathbb{I}}) \equiv b_0 \equiv g(0_{\mathbb{I}})$
- $f(1_{\mathbb{I}}) \equiv b_1 \equiv g(1_{\mathbb{I}})$

($\cdot\ ap_f(seg) = p = ap_g(seg)$) not needed.

Then $f =_{\mathbb{I} \to B} g$

Proof: By Th. 9.1.10 it suffices to show $\prod_{x:\mathbb{I}} (f(x) =_B g(x))$

By $f(x) =_B f(0_{\mathbb{I}}) \equiv g(0_{\mathbb{I}}) =_B g(x)$, but use ; Contr($\mathbb{I}$).

Corollary: The function of Rec$_{\mathbb{I}}$ is unique!

• Section 4.2 The higher Circle

• Definition 4.2.1

• Form_{S^1}: $S^1 : \mathcal{U}$

• Intro_{S^1}:

$$\frac{}{base : S^1}$$ point-constructor

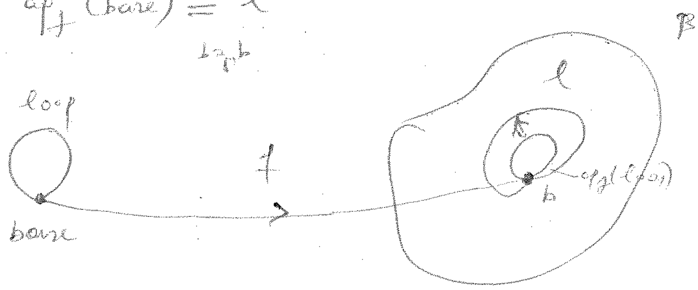
$$\frac{}{loop : base =_{S^1} base}$$ path-constructor

• Rec_{S^1}:

$$\left.\begin{array}{l} B : \mathcal{U} \\ b : B \\ l : b =_B b \end{array}\right\} \text{(computation type)}$$

There is $f : S^1 \to B$ s.t. • $f(base) \equiv b$

• $ap_f(base) =_{b=b} l$



• Ind_{S^1}: $P : S^1 \to \mathcal{U}$

$b : P(base)$

$l : loop_*^P(b) =_{P(base)} b$ $\qquad loop_*^P : P(base) \to P(base)$

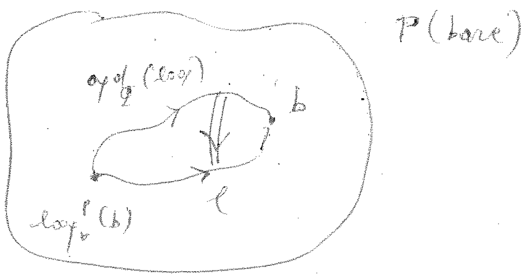
There is $\Phi : \prod_{x:S^1} P(x)$ s.t. • $\Phi(base) \equiv b$

• $apd_\Phi(loop) = l$

$$apd_\Phi : \prod_{p : x =_{S^1} y} p_*^P(\Phi(x)) =_{P(y)} \Phi(y)$$

$$\Downarrow$$

$$apd_\Phi(loop) : loop_*^P(b) = b$$



Remark 4.2.2 Without UA, ITT is compatible with the fact that all loops are equal to refl.
i.e. Without UA we cannot say that $loop \neq refl_{base}$ (There is a model of ITT s.t. all loops = refl)
ITT + K-axiom has a model (Streicher) (K-axiom: every loop = refl)

• Proposition 4.2.3 $Ind_{S^1} \Rightarrow Rec_{S^1}$

Proof: Let $P \equiv \overline{B}_{S^1}$

$\ulcorner B : \mathcal{U}$, $b : B$, $\ell' : b =_B b$ given.
We find $f : S^1 \to B$ s.t. $f(base) \equiv b$ and $ap_f(loop) = \ell'$ $\lrcorner$

$b : P(base) \equiv B$

$loop_*^{\overline{B}_{S^1}} : B \to B$, $loop_*^{\overline{B}_{S^1}}(b) : B$

By Prop. 2.3.2 there is $loop_{**}(b) : loop_*^{\overline{B}_{S^1}}(b) = b$

Take $\ell \equiv loop_{**}(b) * \ell' : loop_*^{\overline{B}_{S^1}}(b) = b$

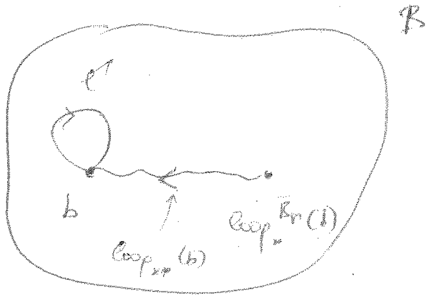
By Ind_{S^1} there is $\Phi : \prod_{x : S^1} B \equiv S^1 \to B$

s.t. $\Phi(base) \equiv b$

$\cdot\ apd_\Phi(loop) = loop_{**}(b) * \ell'$

$\overset{Pr.2.3.4}{=} loop_{**}(b) * ap_\Phi(loop)$

Hence, $ap_\Phi(loop) = \ell'$ □



• Proposition 4.2.4 $A : \mathcal{U}$ $P : A \to \mathcal{U}$ $P(x) \equiv (x =_A x)$

Ex.

$f, g : S^1 \to A$

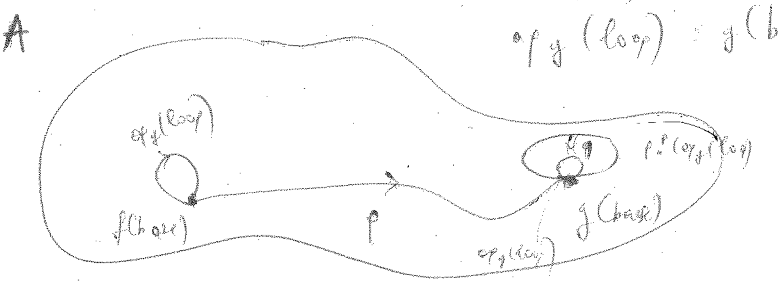
• $p : f(base) =_A g(base)$

• $q : p_*^P(ap_f(loop)) = ap_g(loop)$

$\ulcorner p_*^P : P(f(base)) \to P(g(base)) \equiv (f(base) =_A f(base)) \to (g(base) =_A g(base))$

$loop : base =_{S^1} base \Rightarrow ap_f(loop) : f(base) =_A f(base)$

$ap_g(loop) : g(base) =_A g(base)$ $\lrcorner$



Then $\prod_{x : S^1} (f(x) =_A g(x))$

is inhabited.

Proof: By ind_{S^1}: Let $Q : S^1 \to \mathcal{U}$

$$Q(x) \equiv \left(f(x) =_A g(x)\right)$$

$$p : Q(\mathrm{base}) \equiv \left(f(\mathrm{base}) =_A g(\mathrm{base})\right)$$

We need to find $\boxed{\ell : \mathrm{loop}_*^{Q}(p) = p}$ in order to apply ind_{S^1}.

By Exercise

$$\mathrm{loop}_*^{Q}(p) = [\mathrm{ap}_f(\mathrm{loop})]^{-1} * p * \mathrm{ap}_g(\mathrm{loop}) \implies$$

$$\left(= \mathrm{ap}_f(\mathrm{loop}^{-1}) * p * \mathrm{ap}_g(\mathrm{loop})\right)$$

$$\underbrace{\left(\mathrm{ap}_f(\mathrm{loop}) * \mathrm{loop}_*^{Q}(p) = p * \mathrm{ap}_g(\mathrm{loop})\right)}_{X} \overset{\text{Th. 2.11.5}}{\simeq} \underbrace{p_*^{P}\left(\mathrm{ap}_f(\mathrm{loop})\right) = \mathrm{ap}_g(\mathrm{loop})}_{Y} \quad (*)$$

Since the right type is inhabited by q, the left type is also inhabited

$$\text{But } \left(\mathrm{loop}_*^{Q}(p) = p\right) \simeq \left(\mathrm{ap}_f(\mathrm{loop})^{-1} * p * \mathrm{ap}_g(\mathrm{loop}) = p\right)$$

$$\simeq p * \mathrm{ap}_g(\mathrm{loop}) = \mathrm{ap}_f(\mathrm{loop}) * p$$

$$\simeq X$$

$$\simeq Y$$

(Ex. (*) $p : x =_A y$, $q : x =_A x$, $r : y =_A y$. Then

$$\left(p_*^{z \mapsto z =_A z}(q) = r\right) \simeq \left(q * p = p * r\right)$$

) $\square$

Theorem 4.25 (UA) $\mathrm{loop} \neq \mathrm{refl}_{\mathrm{base}}$

Proof: Let $q : \mathrm{loop} = \mathrm{refl}_{\mathrm{base}}$

and let $A : \mathcal{U}$, $x : A$, $p : x =_A x$

By Rec_{S^1} there is $f : S^1 \to A$ st.
- $f(\mathrm{base}) \equiv x$
- $\mathrm{ap}_f(\mathrm{loop}) = p$

$$\mathrm{ap}_f : (\mathrm{base} =_{S^1} \mathrm{base}) \to (x =_A x)$$

$$\mathrm{ap}_{\mathrm{ap}_f} : (\mathrm{loop} = \mathrm{refl}_{\mathrm{base}}) \to \left(\mathrm{ap}_f(\mathrm{loop}) = \mathrm{ap}_f(\mathrm{refl}_{\mathrm{base}})\right)$$

Hence $ap_{ap_f}(q) = ap_f(loop) = ap_f(refl_{base})$

$$\begin{array}{ccc} ap_{ap_f}(q) = & ap_f(loop) & = ap_f(refl_{base}) \\ & \| & \equiv \\ & p & refl_{f(base)} \equiv refl_x \end{array}$$

i.e. every $p : x =_{\mathcal{U}} x$ is equal to $refl_x$

But in the proof of Proposition 3.2.8 ($\mathcal{U}$ is not a set)

we constructed (with UA) a path $: 2 =_{\mathcal{U}} 2$ which was not equal to $refl_2$.

□

Better writing of the proof $\longrightarrow$ next page

- In analogy to the proof of $0_2 \neq 1_2$ we need the following axiom for the proof of $\mathrm{loop} \neq \mathrm{refl}_{\mathrm{base}}$:

- $S^1 \mapsto (S^1)'$: There is a function $g : S^1 \to S^1$ s.t. $g(\mathrm{base}) \equiv \mathrm{base}'$
  $\cdot\ \mathrm{ap}_g(\mathrm{loop}) = \mathrm{loop}'$

  where $\mathrm{base}', \mathrm{loop}'$ refer to $(S^1)'$ in $\mathcal{U}'$.

- <u>Corollary</u> If $p : \mathrm{loop} = \mathrm{refl}_{\mathrm{base}}$ then there is $p' : \mathrm{loop}' = \mathrm{refl}_{\mathrm{base}'}$

  <u>Proof:</u> $\mathrm{ap}_{\mathrm{ap}_g} : (\mathrm{loop} = \mathrm{refl}_{\mathrm{base}}) \to \mathrm{ap}_g(\mathrm{loop}) = \mathrm{ap}_g(\mathrm{refl}_{\mathrm{base}}) \; (\equiv \mathrm{refl}_{\mathrm{base}'})$

  $$\mathrm{ap}_g(\mathrm{loop}) = \mathrm{loop}'$$

  Hence, $r^{-1} \cdot \mathrm{ap}_{\mathrm{ap}_g}(p) : \mathrm{loop}' = \mathrm{refl}_{\mathrm{base}'}$,

  where $r : \mathrm{ap}_g(\mathrm{loop}) = \mathrm{loop}'$.

- <u>Theorem 4.25</u> $\mathrm{ITT} + \mathrm{UA} + S^1 + (S^1 \mapsto (S^1)') \vdash \mathrm{loop} \neq \mathrm{refl}_{\mathrm{base}}$

  <u>Proof:</u> Let $q : \mathrm{loop} = \mathrm{refl}_{\mathrm{base}}$, and hence $q' : \mathrm{loop}' = \mathrm{refl}_{\mathrm{base}'}$

  Let $\mathcal{M} : \mathcal{U}'$, $2 : \mathcal{M}$, $p : 2 =_{\mathcal{U}} 2$

  By $\mathrm{Rec}_{(S^1)'}$ there is $f : (S^1)' \to \mathcal{U}$ s.t. $f(\mathrm{base}') \equiv 2$
  $\cdot\ \mathrm{ap}_f(\mathrm{loop}') = p$

  Hence, $\mathrm{ap}_{\mathrm{ap}_f} : (\mathrm{loop}' = \mathrm{refl}_{\mathrm{base}'}) \to \mathrm{ap}_f(\mathrm{loop}') = \mathrm{ap}_f(\mathrm{refl}_{\mathrm{base}'})$

  $$\mathrm{ap}_f(\mathrm{loop}') = p, \qquad \mathrm{ap}_f(\mathrm{refl}_{\mathrm{base}'}) \equiv \mathrm{refl}_2$$

  i.e., from q' we get $p = \mathrm{refl}_2$.

  Since in Prop. 3.28 we have $p = \mathrm{refl} \to 0$, for some p (generated by UA from $0_2 \mapsto 1_2,\ 1_2 \mapsto 0_2$) we get a term

  $$q \mapsto t : 0$$ □

Theorem 9.26 S^1 is not contractible (as the topological circle) i.e.

$isContr(S^1) \to 0$ is inhabited.

Proof: Let $u : isContr(S^1) \equiv \sum_{x:S^1} \prod_{y:S^1} (x =_{S^1} y)$. We also define $P : S^1 \to \mathcal{U}$, $P(x) \equiv \prod_{y:S^1} (x =_{S^1} y)$

Hence, $pr_1(u) : S^1$ and $pr_2(u) : \prod_{y:S^1} (pr_1(u) =_{S^1} y)$ then

$$pr_2(u)(y) : pr_1(u) =_{S^1} y \quad \text{and consequently}$$

$$[pr_2(u)(y)]^P_* : P(pr_1(u)) \to P(y) \equiv \left(\prod_{y:S^1} (pr_1(u) =_{S^1} y)\right) \to \left(\prod_{z:S^1} (y =_{S^1} z)\right)$$

hence

$$[pr_2(u)(y)]^P_* (pr_2(u)) : \prod_{z:S^1} (y =_{S^1} z)$$

If $y \equiv base$:

$$\underbrace{[pr_2(u)(base)]^P_* (pr_2(u))}_{W} : \prod_{x:S^1} \overbrace{(base =_{S^1} x)}^{Q(x)}$$

hence $$W(base) : (base =_{S^1} base)$$

and $$apd_W : \prod_{p : base =_{S^1} base} \left(p^Q_* (W(base)) = W(base) \right)$$

hence $$apd_W(loop) : loop_*^{x \mapsto base = x} (W(base)) = W(base)$$

$$\| $$

$$W(base) \cdot loop$$

$\Rightarrow$

$$loop = refl_{base} \quad \text{↯}$$

$\square$

L. 6.4.2. There is $F : \prod_{x:S^1} (x =_{S^1} x)$ s.t.

$$F \neq \mathrm{Refl}_{S^1},$$

where $\mathrm{Refl}_{S^1} : \prod_{x:S^1} (x =_{S^1} x)$

$$\mathrm{Refl}_{S^1}(x) :\equiv \mathrm{refl}_x$$

Proof: We'll use Ind_{S^1}

let $P : S^1 \to \mathcal{U}$ $\quad P(x) \equiv (x =_{S^1} x)$ $\quad x \mapsto (x = x)$

$$P(\mathrm{base}) \equiv (\mathrm{base} =_{S^1} \mathrm{base})$$

let $\mathrm{loop} : P(\mathrm{base})$

We need to find $\ell : \mathrm{loop}_*^P(\mathrm{loop}) = \mathrm{loop}$

$\mathrm{loop}_* : (\mathrm{base} =_{S^1} \mathrm{base}) \to (\mathrm{base} =_{S^1} \mathrm{base})$

$$\mathrm{loop}_*^P(\mathrm{loop}) \equiv \mathrm{loop}_*^{x \mapsto (x =_{S^1} x)}(\mathrm{loop})$$

$$\overset{2.11.2}{=} \mathrm{loop}^{-1} \cdot \mathrm{loop} \cdot \mathrm{loop}$$

$$= \mathrm{refl}_{\mathrm{base}} \cdot \mathrm{loop}$$

$$= \mathrm{loop}$$

i.e. $\ell : \mathrm{loop} = \mathrm{loop}$

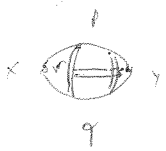
Take $\ell \equiv \mathrm{refl}_{\mathrm{loop}}$

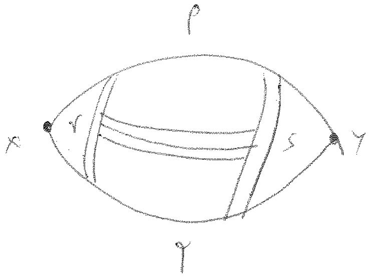
Hence, by Ind_{S^1} there is $F : \prod_{x:S^1} (x =_{S^1} x)$ s.t. $\cdot\ F(\mathrm{base}) \equiv \mathrm{loop}$

$\cdot\ \mathrm{apd}_F(\mathrm{loop}) = \mathrm{refl}_{\mathrm{loop}}$

$\neg (F = \mathrm{Refl}_{S^1})$ since $F(\mathrm{base}) = \mathrm{loop} \neq \mathrm{refl}_{\mathrm{base}} = \mathrm{Refl}_{S^1}(\mathrm{base})$ $\quad \square$

A is 1-type $\stackrel{\text{def}}{=}$ $\prod_{x,y:A} \prod_{p,q:x=_A y} \prod_{r,s:p=q} (r=s)$ is inhabited





$\mathrm{id}_{S^1} =_{S^1 \simeq S^1} \mathrm{id}_{S^1}$ is not a mere proposition (i.e. it is not: $p,q : \mathrm{id}_{S^1} = \mathrm{id}_{S^1}$, then ~~MERE~~ OK (the prop is isequiv plus mereb $p=q$))

Proof: $\left(\mathrm{id}_{S^1} =_{S^1 \simeq S^1} \mathrm{id}_{S^1}\right) \simeq \left(\mathrm{id}_{S^1} =_{(S^1 \to S^1)} \mathrm{id}_{S^1}\right) \stackrel{\text{EXT}}{\simeq} \prod_{x:S^1} (x =_{S^1} x)$

$\mathrm{id}_{S^1} : S^1 \to S^1$ $\qquad$ $\mathrm{id}_{S^1} : S^1 \simeq S^1$

simplified version of isequiv (id_{S^1})

isequiv is a mere prop.

which by 6.4.2 it contains two unequal elements.

- Cor. 6.4.3. $S^1 : \mathcal{U} \Rightarrow \mathcal{U}$ is not a 1-type (we know already that it is not a set i.e. 0-type)

$\prod_{A,B:\mathcal{U}} \prod_{p,q:A=_{\mathcal{U}}B} \prod_{r,s:p=q} (r=s)$ is not inhabited

$A = B = S^1$ $\qquad$ $p,q : S^1 =_{\mathcal{U}} S^1$ $\qquad$ $\left(S^1 =_{\mathcal{U}} S^1\right) \stackrel{\text{UA}}{\simeq} \left(S^1 \simeq S^1\right)$

S^1 $\qquad$ ~~id_{S^1}~~

$\mathrm{id}_{S^1} : S^1 \simeq S^1$

and $\mathrm{id}_{S^1} = \mathrm{id}_{S^1}$ contains ≥ 2 unequal elements

$\square$

Section 4.3 The fundamental group of the (higher) circle.

The loop-space $\Omega(base)$ $\equiv (base =_{S^1} base)$

$\Omega(S^1, base)$

Definition 4.3.1 The integers as a (inductive) type (In HoTT book as a quotient of $\mathbb{N}$, see as in E. Rijke's Master Thesis p.33) we avoid the theory of quotients here!

$neg(1_{\mathbb{N}})$
$neg(0_{\mathbb{N}})$
$0_{\mathbb{Z}}$
$pos(0_{\mathbb{N}})$
$pos(1_{\mathbb{N}})$
$\vdots$

- $\mathrm{Form}_{\mathbb{Z}}$: $\mathbb{Z} : \mathcal{U}$

- $\mathrm{Intro}_{\mathbb{Z}}$: $\dfrac{}{0_{\mathbb{Z}} : \mathbb{Z}}$ $\dfrac{n : \mathbb{N}}{pos(n) : \mathbb{Z}}$ $\dfrac{n : \mathbb{N}}{neg(n) : \mathbb{Z}}$

- $\mathrm{Rec}_{\mathbb{Z}}$: $A : \mathcal{U}$ $a_0 : A$ $p : \mathbb{N} \to A$ $\nu : \mathbb{N} \to A$

  There is $f : \mathbb{Z} \to A$ s.t. $f(0_{\mathbb{Z}}) \equiv a_0$
  - $f(pos(n)) \equiv \cancel{p(f(n))}\, p(n)$ $n : \mathbb{N}$
  - $f(neg(n)) \equiv \cancel{\nu(f(n))}\, \nu(n)$ $n : \mathbb{N}$

- $\mathrm{Ind}_{\mathbb{Z}}$: $P : \mathbb{Z} \to \mathcal{U}$

  $a_0 : P(0_{\mathbb{Z}})$

  $G : \prod_{n : \mathbb{N}} P(pos(n))$, $H : \prod_{n : \mathbb{N}} P(neg(n))$

  There is $F : \prod_{x : \mathbb{Z}} P(x)$ s.t. $F(0_{\mathbb{Z}}) \equiv a_0$
  - $F(pos(n)) \equiv G(n)$, $n : \mathbb{N}$
  - $F(neg(n)) \equiv H(n)$, $n : \mathbb{N}$

- $\mathrm{Ind}_{\mathbb{Z}} \Rightarrow \mathrm{Rec}_{\mathbb{Z}}$: $P : \mathbb{Z} \to A$, $P \equiv \bar{A}_{\mathbb{Z}}$, then $a_0 : A$, $G : \mathbb{N} \to A$, $H : \mathbb{N} \to A$
  $F(pos(n)) \equiv G(n)$, $F(neg(n)) \equiv H(n)$.

Proposition 4.3.2 The function $succ_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ defined by ~~...~~

$$\boxed{\begin{array}{ll} succ_{\mathbb{Z}}(0_{\mathbb{Z}}) \equiv pos(0_{\mathbb{N}}) & \\ succ_{\mathbb{Z}}(pos(n)) \equiv pos(succ_{\mathbb{N}}(n)) & , n : \mathbb{N} \\ succ_{\mathbb{Z}}(neg(n)) \equiv \nu(n) & n : \mathbb{N} \end{array}}$$

where $\nu : \mathbb{N} \to \mathbb{Z}$ is defined by $\mathrm{Rec}_{\mathbb{N}}$:

$$\nu(0) \equiv 0_{\mathbb{Z}}$$

$$\nu(\mathrm{succ}_{\mathbb{N}}(n)) \equiv \mathrm{neg}(n)$$

is well-defined $\overset{\mathrm{term} : \mathbb{Z} \to \mathbb{Z}}{}$ and an equivalence.

<u>Proof:</u> We use $\mathrm{Rec}_{\mathbb{Z}}$: $0_{\mathbb{Z}} : \mathbb{Z}$, $p : \mathbb{N} \to \mathbb{Z}$, $\nu : \mathbb{N} \to \mathbb{Z}$

where $p \equiv \mathrm{pos} \circ \mathrm{succ}_{\mathbb{N}} : \mathbb{N} \to \mathbb{Z}$

$$\mathrm{succ}_{\mathbb{Z}}(\mathrm{pos}(n)) \equiv p(n) \equiv \mathrm{pos}(\mathrm{succ}_{\mathbb{N}}(n)).$$

Let $\mathrm{pred} : \mathbb{Z} \to \mathbb{N}$ defined similarly ($\mathrm{pos} \rightleftarrows \mathrm{neg}$).

$$\boxed{\begin{aligned} &\mathrm{pred}_{\mathbb{Z}}(0_{\mathbb{Z}}) \equiv \mathrm{neg}(0_{\mathbb{N}}) \\ &\mathrm{pred}_{\mathbb{Z}}(\mathrm{neg}(n)) \equiv \mathrm{neg}(\mathrm{succ}_{\mathbb{N}}(n)) \\ &\mathrm{pred}_{\mathbb{Z}}(\mathrm{pos}(n)) \equiv \eta(n) \end{aligned}}$$

$$\mathrm{pred}_{\mathbb{Z}}(\mathrm{pos}(0_{\mathbb{N}})) \equiv 0_{\mathbb{Z}}$$

$$\mathrm{pred}_{\mathbb{Z}}(\mathrm{pos}(\mathrm{succ}_{\mathbb{N}}(n))) \equiv \mathrm{pos}(n)$$

and $\eta(0) \equiv 0_{\mathbb{Z}}$

$$\eta(\mathrm{succ}_{\mathbb{N}}(n)) \equiv \mathrm{pos}(n)$$

<u>Clearly:</u> $\prod_{x : \mathbb{Z}} \mathrm{succ}_{\mathbb{Z}}(\mathrm{pred}_{\mathbb{Z}}(x)) = x$ and $\prod_{x : \mathbb{Z}} \mathrm{pred}_{\mathbb{Z}}(\mathrm{succ}_{\mathbb{Z}}(x)) = x$

By $\mathrm{Ind}_{\mathbb{Z}}$: we need $G(n) : \big(\mathrm{succ}_{\mathbb{Z}}(\mathrm{pred}_{\mathbb{Z}}(\mathrm{pos}(n))) = \mathrm{pos}(n)\big) \equiv$

$$\mathrm{succ}_{\mathbb{Z}}(\eta(n)) = \mathrm{pos}(n)$$

Ind on $\mathbb{N}$ $\quad n = 0$: $\mathrm{succ}_{\mathbb{Z}}(0) \equiv \mathrm{pos}(0_{\mathbb{N}})$ ✓

If $\mathrm{succ}_{\mathbb{Z}}(\eta(n)) = \mathrm{pos}(n)$

which is ... similar... $\mathrm{succ}_{\mathbb{Z}}(\eta(\mathrm{succ}_{\mathbb{N}}(n))) \equiv \mathrm{succ}_{\mathbb{Z}}(\mathrm{pos}(n)) \equiv \mathrm{pos}(\mathrm{succ}_{\mathbb{N}}(n))$ □

Proposition 4.3.3 $\mathrm{isSet}(\mathbb{Z}) \equiv \prod_{x,y:\mathbb{Z}} \prod_{p,q : x =_{\mathbb{Z}} y} (p = q)$ (to be done in Munich verbeuten)

(like IN) (Just mention it.)

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Remark 4.3.4

• $loop^{-} : \mathbb{Z} \longrightarrow (base =_{S^1} base)$ is defined by $Rec_{\mathbb{Z}}$: (in Kijte it is named wind)

$loop^{x} : base =_{S^1} base$, $x : \mathbb{Z}$

$$\left[\ loop^{0_{\mathbb{Z}}} \equiv refl_{base} : \Omega(base)\right.$$

$$\left[\begin{array}{l} loop^{pos(0)} \equiv loop \\ loop^{pos(succ_{N}(n))} \equiv loop^{pos(n)} \cdot loop \end{array}\right.$$

$\equiv \underbrace{loop \cdot \ldots \cdot loop}_{succ_N(n)\text{ times}}$

$$\left[\begin{array}{l} loop^{neg(0)} \equiv loop^{-1} \\ loop^{neg(succ_{N}(n))} \equiv \underbrace{loop^{neg(n)}} \cdot loop^{-1} \end{array}\right.$$

$\underbrace{loop^{-1} \cdot \ldots \cdot loop^{-1}}_{succ(n)\text{-times}}$

• We'd like to know, as it is expected by the intuitive Univalence Principle for paths for the S^1 : that these are all paths-loops at base. From this one can infer (using some defs that the fundamental group of S^1 is $\mathbb{Z}$).

• Remark/Definition 4.3.5 By $Rec_{(S^1)'}$ there is a function

$$code' : (S^1)' \to \mathcal{U} \qquad \text{st.}$$

$$code'(base') \equiv \mathbb{Z}$$

$$ap_{code'}(loop') = ua(succ_{\mathbb{Z}})$$

$\mathcal{U} : \mathcal{U}'$, $\mathbb{Z} : \mathcal{U}$ $succ_{\mathbb{Z}} : \mathbb{Z} \simeq_{\mathcal{U}} \mathbb{Z}$, hence

$$ua(succ_{\mathbb{Z}}) : \mathbb{Z} =_{\mathcal{U}} \mathbb{Z}$$

Corollary / Def 4.3.6 Through $S^1 \simeq (S^1)'$ there is a function

$$\boxed{\begin{aligned} &\mathrm{code} : S^1 \to \mathcal{U} \quad \text{s.t.} \\ &\mathrm{code}(\mathrm{base}) \equiv \mathbb{Z} \quad \text{and} \\ &\mathrm{ap}_{\mathrm{code}}(\mathrm{loop}) = \mathrm{ua}(\mathrm{succ}_{\mathbb{Z}}) \end{aligned}}$$

(code is called the universal cover of S^1)

Proof : Let $g : S^1 \to (S^1)'$ s.t. $g(\mathrm{base}) \equiv \mathrm{base}'$

$$\begin{array}{l} \downarrow \mathrm{code}' \\ \mathcal{U} \end{array} \qquad \mathrm{ap}_g(\mathrm{loop}) = \mathrm{loop}'$$

Define $\mathrm{code} \equiv \mathrm{code}' \circ g$

Then, $\mathrm{code}(\mathrm{base}) \equiv \mathrm{code}'(g(\mathrm{base})) \equiv \mathrm{code}'(\mathrm{base}') \equiv \mathbb{Z}$.

$$\begin{aligned} \mathrm{ap}_{\mathrm{code}' \circ g}(\mathrm{loop}) &= \mathrm{ap}_{\mathrm{code}'}(\mathrm{ap}_g(\mathrm{loop})) \\ &= \mathrm{ap}_{\mathrm{code}'}(\mathrm{loop}') \\ &= \mathrm{ua}_{\mathbb{A}}(\mathrm{succ}_{\mathbb{Z}}) \end{aligned}$$

(use ap_g twice)

$\square$

(Remark : This is sloppily written in the HoTT-book.)

Lemma 9.3.7 $\mathrm{loop}_*^{\mathrm{code}} : \mathrm{code}(\mathrm{base}) \to \mathrm{code}(\mathrm{base}) \equiv \mathbb{Z} \to \mathbb{Z}$

$(\mathrm{loop}^{-1})_*^{\mathrm{code}}$

$$\boxed{\begin{aligned} &\text{(i)} \quad \mathrm{loop}_*^{\mathrm{code}}(x) = \mathrm{succ}_{\mathbb{Z}}(x) \\ &\text{(ii)} \quad (\mathrm{loop}^{-1})_*^{\mathrm{code}}(x) = \mathrm{pred}_{\mathbb{Z}}(x) \end{aligned}}$$

$x : \mathbb{Z}$

Proof : (i) By Cor 3.2.7 ($A : \mathcal{U}$, $P : A \to \mathcal{U}$, $p : x =_A x$, $u : P(x)$, $f : P(x) \to P(x)$ is equiv(f), $\mathrm{ap}_P(p) = \mathrm{ua}(f)$, then $p_*^P(u) = f(u)$)

$\mathrm{loop}_*^{\mathrm{code}}(x) = \mathrm{succ}_{\mathbb{Z}}(x) \qquad \mathrm{ap}_{\mathrm{code}}(\mathrm{loop}) = \mathrm{ua}(\mathrm{succ}_{\mathbb{Z}})$

(ii) By Corol. 2.4.3 $(p^{-1})^P_*: P(y) \to P(x)$ is the inverse of p^P_*

Hence $(\mathrm{loop}^{-1})^{\mathrm{code}}_*$ is the inverse of $\mathrm{succ}_{\mathbb{Z}}$ i.e. $\mathrm{pred}_{\mathbb{Z}}$. $\square$

__Definition 4.3.8__ $\mathrm{encode} : \prod_{x:S^1} \left((\mathrm{base} =_{S^1} x) \longrightarrow \mathrm{code}(x) \right)$

$$\boxed{\mathrm{encode}(x, p) \equiv p^{\mathrm{code}}_*(0_{\mathbb{Z}})}$$

where

$p : \mathrm{base} =_{S^1} x$

$p^{\mathrm{code}}_* : \mathrm{code}(\mathrm{base}) \to \mathrm{code}(x) \equiv \mathbb{Z} \to \mathrm{code}(x)$

hence $p^{\mathrm{code}}_*(0_{\mathbb{Z}}) : \mathrm{code}(x)$

__Definition 4.3.9__ There is a dep function $\boxed{\mathrm{decode} : \prod_{x:S^1} \left(\mathrm{code}(x) \to (\mathrm{base} =_{S^1} x) \right)}$

__Proposition__

$P : S^1 \to U$

Proof: $P(x) \equiv \overset{Q_1(x)}{\mathrm{code}(x)} \to \overset{Q_2(x)}{(\mathrm{base} =_{S^1} x)}$

$P(\mathrm{base}) \equiv \mathbb{Z} \to (\mathrm{base} =_{S^1} \mathrm{base})$

We take $\mathrm{loop}^{-} : \mathbb{Z} \to (\mathrm{base} =_{S^1} \mathrm{base})$

We also need, in order to apply Ind_{S^1}, some

$$\ell : \left(\mathrm{loop}^P_*(\mathrm{loop}^{-}) =_{P(\mathrm{base})} \mathrm{loop}^{-} \right)$$

But,

$$\mathrm{loop}^P_*(\mathrm{loop}^{-}) \equiv \mathrm{loop}^{Q_1 \to Q_2}_*(\mathrm{loop}^{-})$$

$$= \mathrm{loop}^{Q_2}_* \circ (\mathrm{loop}^{-}) \circ (\mathrm{loop}^{-1})^{Q_1}_* \qquad \text{• Prop. 2.4.5 } p^{P \to Q}_*(f) = p^{Q}_* \circ f \circ (p^{-1})^P_*$$

$$= \mathrm{loop}^{x \mapsto (\mathrm{base} =_{S^1} x)}_* \circ \left((\mathrm{loop}^{-}) \circ \mathrm{pred}_{\mathbb{Z}} \right) \qquad \text{• Lemma 4.3.7}$$

$$= \left(\mathrm{loop}^{-} \circ \mathrm{pred}_{\mathbb{Z}} \right) \cdot \mathrm{loop} \qquad \text{• Ex. } p^{x \mapsto a = x}_*(q) = q \cdot p$$

$$= \left(\lambda(x:\mathbb{Z}).\ \mathrm{loop}^{\mathrm{pred}(x)} \right) \cdot \mathrm{loop}$$

$$\overset{(*)}{=} \lambda(x:\mathbb{Z}).\ \mathrm{loop}^{-}(x)$$

$$= \mathrm{loop}^{-}$$

Here we use (a) $\prod_{x:\mathbb{Z}} \left(\mathrm{loop}^{\mathrm{pred}(x)} * \mathrm{loop} = \mathrm{loop}^{x} \right)$

(b) function-extensionality.

For (a) we prove it inductively: $\underline{(\mathbb{Z}\text{-}x)}$.

$0_{\mathbb{Z}}$: $\quad \mathrm{loop}^{\mathrm{pred}(0_{\mathbb{Z}})} * \mathrm{loop} = \mathrm{loop}^{\mathrm{neg}(0_{\mathbb{N}})} * \mathrm{loop}$

$$= \mathrm{loop}^{-1} * \mathrm{loop}$$

$$= \mathrm{refl}_{\mathrm{base}}$$

$$= \mathrm{loop}^{0_{\mathbb{Z}}}$$

pos(n)-case: $\quad \mathrm{loop}^{\mathrm{pred}(\mathrm{pos}(n))} * \mathrm{loop} = \mathrm{loop}^{\mathrm{pos}(n)}$ is to be shared

and for that we use Ind$_{\mathbb{N}}$:

$n=0$: $\quad \mathrm{loop}^{\mathrm{pred}(\mathrm{pos}(0_{\mathbb{N}}))} * \mathrm{loop} = \mathrm{loop}^{0_{\mathbb{Z}}} * \mathrm{loop}$

$$= \mathrm{refl}_{\mathrm{base}} * \mathrm{loop}$$

$$= \mathrm{loop}$$

$$= \mathrm{loop}^{\mathrm{pos}(0)}$$

succ(n)-case: $\quad \mathrm{loop}^{\mathrm{pred}(\mathrm{pos}(\underbrace{\mathrm{succ}_{\mathbb{N}}(n)}))} * \mathrm{loop} =$

$$\mathrm{loop}^{\mathrm{pos}(n)} * \mathrm{loop} = \mathrm{loop}^{\mathrm{pos}(\underbrace{\mathrm{succ}_{\mathbb{N}}(n)})}$$

Similarly for the neg(n)-case.

loop^{-}

$\equiv$

By Ind$_{S^1}$ $\boxed{\begin{array}{l} \mathrm{decode}(\mathrm{base}) \equiv \mathrm{loop}^{-} \\ \mathrm{apd}_{\mathrm{decode}}(\mathrm{loop}) = \ell \end{array}}$ refl wrt and $\mathrm{apd}_{\mathrm{decode}} : \prod_{p:\mathrm{base}=_{S^1}\mathrm{base}} p_*(\mathrm{decode}(\mathrm{base})) = \mathrm{decode}(b)$

Lemma 4.3.10 Let $x : S^1$, $p : \mathrm{base} =_{S^1} x$. Then

$$\boxed{\mathrm{decode}\left(x, \mathrm{encode}(x, p)\right) = p}$$

Proof:

$$\mathrm{decode}\left(x, \mathrm{encode}(x, p)\right) \equiv \mathrm{decode}\left(x, p_*^{\mathrm{code}}(0_{\mathbb{Z}})\right) \equiv \mathrm{decode}(x)\left(p_*^{\mathrm{code}}(0_{\mathbb{Z}})\right)$$

$$= p_*^{P}(\mathrm{loop}^{-})\left(p_*^{\mathrm{code}}(0_{\mathbb{Z}})\right)$$

since $\mathrm{apd}_{\mathrm{decode}}(p) : \left(p_*^{P}(\mathrm{loop}^{-}) = \mathrm{decode}(x)\right)$

$$(P :\equiv a_1 \to a_2)$$

$$= p_*^{x \mapsto \mathrm{base} = x}\left(\mathrm{loop}^{-}\left((p^{-1})_*^{\mathrm{code}}\left(p_*^{\mathrm{code}}(0_{\mathbb{Z}})\right)\right)\right)$$

$$= p_*^{x \mapsto \mathrm{base} = x}\left(\mathrm{loop}^{-}\left((p \cdot p^{-1})_*^{\mathrm{code}}(0_{\mathbb{Z}})\right)\right)$$

$$= p_*^{x \mapsto \mathrm{base} = x}\left(\mathrm{loop}^{-}\left((\mathrm{refl}_{\mathrm{base}})_*^{\mathrm{code}}(0_{\mathbb{Z}})\right)\right)$$

$$= p_*^{x \mapsto \mathrm{base} = x}\left(\mathrm{loop}^{-}\left(\mathrm{id}_{\mathrm{code}(\mathrm{base})}(0_{\mathbb{Z}})\right)\right)$$

$$= p_*^{x \mapsto \mathrm{base} = x}\left(\mathrm{loop}^{-}\left(0_{\mathbb{Z}}\right)\right)$$

$$\equiv p_*^{x \mapsto \mathrm{base} = x}\left(\mathrm{refl}_{\mathrm{base}}\right)$$

$$= \mathrm{refl}_{\mathrm{base}} \cdot p$$

$$= p.$$

No induction on S^1 is needed here. □

Lemma 4.3.11: Let $x : S^1$, $c : \mathrm{code}(x)$. Then

$$\boxed{\mathrm{encode}\left(x, \mathrm{decode}(x, c)\right) = c}$$

$\cancel{\mathrm{encode}\left(x, \mathrm{decode}(x, c)\right) = c}$

Proof : We show $\prod_{x:S^1}\Big(\underbrace{\prod_{c:\mathrm{code}(x)}\big(\mathrm{encode}(x,\mathrm{decode}(x,c))=c\big)}_{P(x)}\Big)$

We need to find $a_0 : P(\mathrm{base}) \equiv \prod_{c:\mathrm{code}(\mathrm{base})}\big(\mathrm{encode}(\mathrm{base},\mathrm{decode}(\mathrm{base},c))=c\big)$

$$\equiv \prod_{x:\mathbb{Z}} \mathrm{encode}\big(\mathrm{base},\mathrm{decode}(\mathrm{base},x)\big)=x$$

~~Since decode(base) = loop ... by ... decode ... (loop~~

$$\equiv \prod_{x:\mathbb{Z}} \mathrm{encode}\big(\mathrm{base},\mathrm{decode}(\mathrm{base})(x)\big)=x$$

$\mathrm{decode}(\mathrm{base}) \equiv \mathrm{loop}^{-}$

$$\equiv \prod_{x:\mathbb{Z}}\big(\mathrm{encode}(\mathrm{base},\mathrm{loop}^{x})=x\big)$$

$$\equiv \prod_{x:\mathbb{Z}}\Big[\big(\mathrm{encode}(\mathrm{base})\big)(\mathrm{loop}^{x})=x\Big]$$

$k \equiv 0_{\mathbb{Z}}$: $\big(\mathrm{encode}(\mathrm{base})\big)\big(\mathrm{loop}^{0_{\mathbb{Z}}}\big) \equiv \mathrm{encode}(\mathrm{base},\mathrm{refl}_{\mathrm{base}})$

$$\equiv (\mathrm{refl}_{\mathrm{base}})^{\mathrm{code}}_{*}(0_{\mathbb{Z}})$$

$$\equiv \mathrm{id}_{\mathrm{code}(\mathrm{base})}(0_{\mathbb{Z}})$$

$$\equiv \mathrm{id}_{\mathbb{Z}}(0_{\mathbb{Z}})$$

$$\equiv 0_{\mathbb{Z}}.$$

$k \equiv \mathrm{pos}(n)$ Ind on $\mathbb{N}$.

$n = 0_{\mathbb{N}}$: $\mathrm{encode}(\mathrm{base},\mathrm{loop}^{\mathrm{pos}(0)}) \equiv \mathrm{encode}(\mathrm{base},\mathrm{loop}$

$$\equiv \mathrm{loop}_*^{\mathrm{code}}(0_{\mathbb{Z}})$$

$$\overset{L.4.37}{=} \mathrm{succ}_{\mathbb{Z}}(0_{\mathbb{Z}})$$

$$\equiv \mathrm{pos}(0_{\mathbb{N}})$$

$$\mathrm{succ}_{\mathbb{N}}(n) : \quad \mathrm{encode}\left(\mathrm{base}, \mathrm{loop}^{\underline{\mathrm{pos}(\mathrm{succ}_{\mathbb{N}}(n))}}\right) \equiv \mathrm{encode}\left(\mathrm{base}, \mathrm{loop}^{\mathrm{pos}(n)} \cdot \mathrm{loop}\right)$$

$$\equiv \left(\mathrm{loop}^{\mathrm{pos}(n)} \cdot \mathrm{loop}\right)_*^{\mathrm{code}}(0_{\mathbb{Z}})$$

$$= \mathrm{loop}_*^{\mathrm{code}}\left(\left(\mathrm{loop}^{\mathrm{pos}(n)}\right)_*^{\mathrm{code}}(0_{\mathbb{Z}})\right)$$

$$\overset{I.H}{\equiv} \mathrm{loop}_*^{\mathrm{code}}(\mathrm{pos}(n))$$

$$\overset{L.4.37}{=} \mathrm{succ}_{\mathbb{Z}}(\mathrm{pos}(n))$$

$$\underset{\mathrm{succ}_{\mathbb{Z}}}{\overset{\mathrm{def}}{\equiv}} \underline{\mathrm{pos}(\mathrm{succ}_{\mathbb{N}}(n))}$$

Similarly for

$$u = \mathrm{neg}(n) \quad \begin{cases} n = 0 \\ \mathrm{succ}(n) \end{cases}$$

So, there is such $a_0 : P(\mathrm{base})$

To apply Ind_{S^1} we need $\mathrm{l\hat{o}}\ \mathrm{loop}_*^P(a_0) = a_0$

But a_0 ~~and~~ $\mathrm{loop}_*^P(a_0)$ are term of type $\prod_{x:\mathbb{Z}}\left(\mathrm{encode}(\mathrm{base}, \mathrm{loop}^x) = x\right)$

$$a_0(x),\ \mathrm{loop}_*^P(a_0)(x) : \mathrm{encode}(\mathrm{base}, \mathrm{loop}^x) = x$$

Since $\mathbb{Z}$ is a set: $a_0(x) = \mathrm{loop}_*^P(a_0)(x)$.

By Funextensionality $a_0 = \mathrm{loop}_*^P(a_0)$. So, we can use Ind_{S^1} to complete the proof

$\square$

Theorem 4.3.12 $\prod_{x:S^1} \left[\left(\mathrm{base} =_{S^1} x \right) \simeq \mathrm{code}(x) \right]$

Proof: Use for $x:S^1$ lemmas 4.3.10, 4.3.11

Corollary 4.3.13: $\boxed{\Omega(S^1, \mathrm{base}) \simeq \mathbb{Z}}$

Proof: From Th. 4.3.12 for $x \equiv \mathrm{base}$.

· With an appropriate def of the fundamental group

$$\pi_1(S^1) = \mathbb{Z}$$

where $\pi_1(S^1) \equiv \| \Omega(S^1, \mathrm{base}) \|_0 = \| \mathbb{Z} \|_0 = \mathbb{Z}$

(and $\|A\|_0$ is the ~~...~~ (set) truncation of A) the closest set to A

in HoTT.

· Hypothetical things (Canonicity property)

Semantics of HoTT.