Part II : Homotopic Extension of HTT

Chapter 3. The Univalence Axiom of Voevodsky

Section 3.1: The axiom of function extensionality

Proposition 3.1.1: $A : \mathcal{U}$, $P : A \to \mathcal{U}$

$F, G : \prod_{x:A} P(x)$.

There is a function $\mathrm{happly} : \left(F =_{\prod P(x)} G\right) \to \prod_{x:A} \left(F(x) =_{P(x)} G(x)\right)$

s.t. $\mathrm{happly}(\mathrm{refl}_F) \equiv \lambda(x:A).\mathrm{refl}_{F(x)}$

Proof: $C : \prod_{F,G:\prod_{x:A} P(x)} \prod_{p:F=G} \mathcal{U}$ by $C(F,G,p) \equiv \prod_{x:A} \left(F(x) =_{P(x)} G(x)\right)$

$C(F,F,\mathrm{refl}_F) \equiv \prod_{x:A}\left(F(x) =_{P(x)} F(x)\right)$

Let $c(F) :\equiv \lambda(x:A).\mathrm{refl}_{F(x)} : C(F,F,\mathrm{refl}_F)$ By Ind$_=$

There is $\Phi : \prod_{F,G:\prod_{x:A} P(x)} \prod_{p:F=G} \prod_{x:A}\left(F(x) =_{P(x)} G(x)\right)$.

st $\Phi(F,F,\mathrm{refl}_F) \equiv \lambda(x:A).\mathrm{refl}_{F(x)}$

We define $\mathrm{happly}(p) \equiv \Phi(F,G,p)$

here $\mathrm{happly}(\mathrm{refl}_F) \equiv \lambda(x:A).\mathrm{refl}_{F(x)}$

$\square$

FunExt. Axiom: $A, B : \mathcal{U}$, $P : A \to \mathcal{U}$, $F, G : \prod_{x:A} P(x)$

The function happly is an equivalence. (i.e., $\mathrm{isequiv}(\mathrm{happly})$ is inhabited).

Corollary 3.1.2 By FunExt. happly has quasi-inverse i.e. $\mathrm{funext} \equiv q(\mathrm{happly})$ (not just any quasiinverse)

$$\mathrm{funext} : \left(\prod_{x:A}\left(F(x) =_{P(x)} G(x)\right)\right) \to F = G$$

This we usually call the "function extensionality" function.

Remark 3.1.3 We can break this equivalence into:

(i) An introduction rule for $F =_{\prod_{x:A} P(x)} G$, denoted funext for function-extensionality axiom:

$$\text{funext} : \left(\prod_{x:A} (F(x) =_{P(x)} G(x)) \right) \to F = G$$

(equality of F & G is introduced)

(ii) An elimination rule, which is happly

$$\text{happly} : (F = G) \longrightarrow \left(\prod_{x:A} (F(x) =_{P(x)} G(x)) \right)$$

(equality of F & G is eliminated)

s.t. $\text{happly}(\text{refl}_F) \equiv \lambda(x:A).\text{refl}_{F(x)}$

(iii) The propositional computation rule: If $H : \prod_{x:A} (F(x) =_{P(x)} G(x))$, then

$$\text{happly}(\text{funext}(H), x) = H(x)$$

(iv) The propositional uniqueness principle: If $p : F = G$, then

$$\text{funext}(\lambda(x:A).\ \text{happly}(p, x)) = p$$

Note that these are propositional equalities, while in the inductive defs these are judgmental !!

Explanation of (iii) and (iv): $(\text{happly} \circ \text{funext}) \sim \text{id}_{\prod_{x:A}(F(x) =_P G(x))}$ i.e.,

$$\prod_{H : \prod_{x:A} (F(x) = G(x))} \left((\text{happly} \circ \text{funext})(H) = H \right) \equiv$$

$$\prod_{H : \prod_{x:A} F(x) = G(x)} \left(\text{happly}(\text{funext}(H)) = H \right) \xrightarrow{H : \prod_{x:A}(F(x) = G(x))}$$

$$\text{happly}(\text{funext}(H))(x) = H(x)$$
$$|||$$
$$\text{happly}(\text{funext}(H, x))$$

Similarly $\left(\text{funext} \circ \text{happly}\right) \sim \text{id}_{F=G}$ i.e. $\equiv$

$$\prod_{p:F=G} \left(\text{funext}(\text{happly}(p)) = p\right) \quad \overset{p:F=G}{\Longrightarrow}$$

(iii)' $\boxed{\left(\text{funext}(\text{happly}(p))\right) = p}$ alternative writing of (iii).

since $\text{happly}(p) \equiv \lambda(x:A).\ \text{happly}(p)(x)$

$\equiv \lambda(x:A).\ \text{happly}(p,x)$ by the Uniq $\prod_{x:A} P(x)$

we get (iii) from (iii)'.

Corollary 3.1.4: $F, G, H : \prod_{x:A} P(x)$ $p : F = G$, $q : G = H$

(i) $\text{refl}_F = \text{funext}(x \mapsto \text{refl}_{F(x)})$

(ii) $p^{-1} = \text{funext}(x \mapsto \text{happly}(p,x)^{-1})$

(iii) $p \cdot q = \text{funext}(x \mapsto \text{happly}(p,x) \cdot \text{happly}(q,x))$

Proof: (i) The function $x \mapsto \text{refl}_{F(x)}$ is $\lambda(x:A).\ \text{refl}_{F(x)} \equiv \text{happly}(\text{refl}_F)$ by (ii).

Hence by (iii)' $\text{funext}(x \mapsto \text{refl}_{F(x)}) \equiv \text{funext}(\text{happly}(\text{refl}_F))$

$\equiv \text{refl}_F$

(ii) $C(F, G, p) \equiv \left(p^{-1} = \text{funext}\left(\lambda(x:A).\ \text{happly}(p,x)^{-1}\right)\right)$

$C(F, F, \text{refl}_F) \equiv \left(\text{refl}_F^{-1} = \text{funext}\left(\lambda(x:A).\ \text{happly}(\text{refl}_F, x)^{-1}\right)\right)$

$\equiv \left(\text{refl}_F = \text{funext}\left(\lambda(x:A).\ \text{refl}_{F(x)}^{-1}\right)\right)$

proof of (i) $\equiv \left(\text{refl}_F = \text{funext}\left(\lambda(x:A).\ \text{refl}_{F(x)}\right)\right)$

$\overset{(ii)}{\equiv}$ ~~$\text{refl}_F = \text{funext}(\text{happly}(\text{refl}_F))$~~

// (iii)

Since ~~(iii)~~ $\text{refl}_F = \text{refl}_F$ the type is inhabited, out we use Ind.

(iii) $C(F, G, p) \equiv \left(p \cdot q = \text{funext}\left(x \mapsto \text{happly}(p,x) \cdot \text{happly}(q,x)\right)\right)$

$$C(F,F,\mathrm{refl}_F) \equiv \Big(\mathrm{refl}_F \cdot q = \mathrm{funext}\big(x \mapsto \underbrace{\mathrm{happly}(\mathrm{refl}_F, x)}_{(ii)\ \equiv \mathrm{refl}_{F(x)}} \cdot \mathrm{happly}(q,x)\big)\Big)$$

$$\equiv \Big(q = \mathrm{funext}\big(x \mapsto \mathrm{refl}_{F(x)} \cdot \mathrm{happly}(q,x)\big)\Big)$$

$$\equiv \big(q = \mathrm{funext}(x \mapsto \mathrm{happly}(q,x))\big)$$

$$\cancel{\overset{(iv)}{\equiv}\ q = q} \quad \text{inhabited by } (iv))$$

Hence $\mathrm{refl}_q : C(F,F,\mathrm{refl}_F)$. □

• Remark 3.1.5 (i) ITT ⊬ FunExt (there are some models of ITT which do not validate FunExt) See Streicher's Habil. Thesis 1993 p. 107.

(ii) ITT + UA ⊢ FunExt (Voevodsky) non-trivial proof

(iii) ITT + (HITs) ⊢ FunExt: Higher Inductive Interval ⊢ FunExt.

• Definition 3.1.6: If $A : U$, then $\mathrm{isSet}(A) \equiv \prod_{x,y:A} \prod_{p,q : x =_A y} (p = q)$ (or 0-type)

• Examples 3.1.7 (i) $\mathrm{isSet}(0)$ since $\lambda(x,y : \mathbf{0}.\ \mathrm{Efq}_{\prod_{p,q:x=y}(p=q)}(y)) : \mathrm{isSet}(0)$

Prop. 2.6.8

(ii) $\mathrm{isSet}(1)$: This is based on the preliminary result $(x =_1 y) \simeq 1$, for any $x,y : 1$

Since any two elements of 1 are equal by equivalence, any two elements of $(x =_1 y)$ are equal: $p,q : x =_1 y \Rightarrow f(p) = f(q)$

$\Rightarrow g(f(p)) = g(f(q))$

$\Rightarrow p = q$. □

(iii) $\mathrm{isSet}(\mathbb{N})$: By Corollary 2.6.10 and the fact that 0, 1 are sets

• Definition 3.1.8 $A : U$, A is 1-type if $\prod_{x,y:A} \prod_{p,q : x =_A y} \prod_{r,s : p = q} (r = s)$

• Proposition 3.1.9 If $\mathrm{isSet}(A)$, then A is 1-type

Note: by Univalence there are types which are not sets / and types which are not n-types, for every n. (HoTT p. 103 + Ch. 6, 8)

Proof: Let $F : \text{isSet}(A) \equiv \prod_{x,y:A} \prod_{p\,q : x =_A y} (p = q)$, i.e. $F(x\,y, p, q) : p = q$

Let $G : \prod_{q : x =_A y} (p = q)$ be defined by $G(q) \equiv F(x\,y\,p\,q)$

As we know ~~...~~ $\text{apd}_G : \prod_{r : q = q'} \left(\begin{matrix} q \mapsto p = q \\ r_* \end{matrix} (G(q)) = G(q') \right) \equiv$ Prop. 2.4.4

$$\prod_{r : q = q'} G(q) \cdot r = G(q')$$

i.e. $\text{apd}_G(r) : \left(G(q) \cdot r = G(q') \right)$ (1).

Let $x\,y : A$, $p, q : x =_A y$, $r, s : p = q$; as in the definition of 1-type (...).

(p, q instead of q, q' in (1))

$$G(p) \cdot r = G(q)$$
$$G(p) \cdot s = G(q)$$

$$G(p) \cdot r = G(p) \cdot s \implies r = s$$

- We'll see in the next section that through UA not all types are sets!!

- Proposition 3.1.10 (with Funext): $A : U$, $P : A \to U$, $\forall$ isSet$(P(x))$, for every $x : A$.

Then
$$\text{isSet}\left(\prod_{x:A} P(x) \right)$$

Proof. Let $F, G : \prod_{x:A} P(x)$ and $p, q : F = G$

By (iv) $p = \text{funext}(\lambda(x:A).\ \text{happly}(p, x))$ ~~funext(λ(x:A)...~~
$q = \text{funext}(\lambda(x:A).\ \text{happly}(q, x))$

By the definition of happly: $\text{happly}(p, x) : (F(x) =_{P(x)} G(x))$ $\text{happly}(q, x) : (F(x) =_{P(x)} G(x))$

Since is Set$(P(x))$ and $F(x), G(x) : P(x)$ we get $\text{happly}(p, x) = \text{happly}(q, x)$

By funext $u : \lambda(x:A).\,\mathrm{happly}(p,x) = \lambda(x:A).\,\mathrm{happly}(q,x)$

$$\Downarrow$$

$$\mathrm{ap}_{\mathrm{funext}} : (\lambda(x:A).\,\mathrm{happly}(p,x) = \lambda(x:A).\,\mathrm{happly}(q,x)) \to (p = q)$$

$$\mathrm{ap}_{\mathrm{funext}}(u) : p = q.$$

$\square$

Corollary 3.1.11

$$\frac{A : \mathcal{U} \qquad B : \mathcal{U} \qquad \mathrm{isSet}(B)}{\mathrm{isSet}(A \to B)} \text{ then}$$

(notation as before, A need not be a set, just any type).

Proof : Use Prop. 3.1.10 for $P \equiv B_A$.

Section 3.2 : The Univalence Axiom (of Voevodsky)

$A \simeq_{\mathcal{U}} B \equiv \sum_{f:A\to B} \mathrm{isequiv}(f)$

Proposition 3.2.1 There is a function

$\mathrm{idtoequiv} : (A =_{\mathcal{U}} B) \to (A \simeq_{\mathcal{U}} B)$ such that

$\mathrm{idtoequiv}(\mathrm{refl}_A) \equiv \left(\mathrm{id}_A, w : \mathrm{isequiv}(\mathrm{id}_A)\right)$

(better term for us "eqtoequiv")

Proof 1: Let $C : \prod_{A,B:\mathcal{U}} \prod_{p:A=_{\mathcal{U}}B} \mathcal{U}'$ be defined by

$$C(A,B,p) \equiv (A \simeq_{\mathcal{U}} B)$$

Note that in the formulation of ind $C : \prod_{x,y:A} \prod_{p:x=_A y} \mathcal{U}$ for some $A:\mathcal{U}$

here here we need the $\mathrm{Ind}_=$ for $\mathcal{U}'$.

more accurately $(\mathrm{id}_A, w : \mathrm{isequiv}(\mathrm{id}_A))$

$C(A,A,\mathrm{refl}_A) \equiv (A \simeq_{\mathcal{U}} A)$, hence $\mathrm{id}_A : C(A,A,\mathrm{refl}_A)$

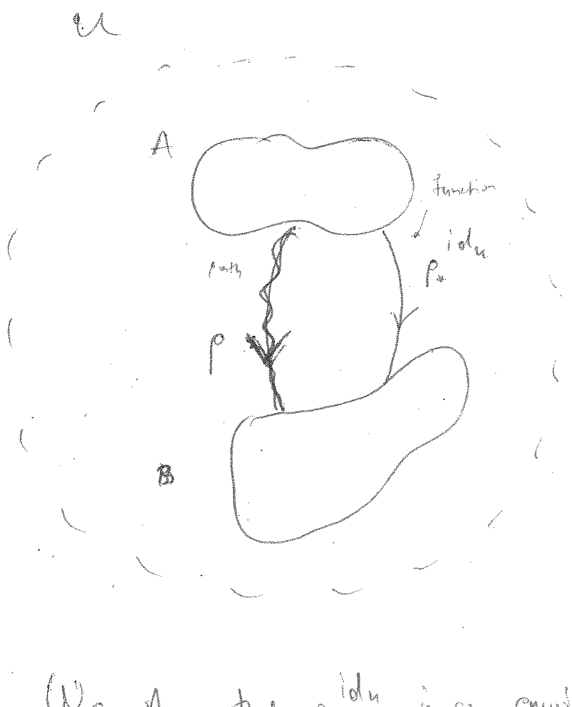
By ind of $\mathcal{U}'$ there is $\Phi : \prod_{A,B:\mathcal{U}} \prod_{p:A=_{\mathcal{U}}B} (A \simeq_{\mathcal{U}} B)$ s.t. $\Phi(A,A,\mathrm{refl}_A) \equiv \mathrm{id}_A$.

Define $\mathrm{idtoequiv}(p) \equiv \Phi(A,B,p)$, for every $p : (A =_{\mathcal{U}} B)$.

Proof 2: (more informative) Let $\mathrm{id}_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}$ i.e., $\mathrm{id}_{\mathcal{U}}(A) \equiv A$, for every $A:\mathcal{U}$

If $p : A =_{\mathcal{U}} B$, then $p_*^{\mathrm{id}_{\mathcal{U}}} : \mathrm{id}_{\mathcal{U}}(A) \to \mathrm{id}_{\mathcal{U}}(B)$ i.e.,

$$p_*^{\mathrm{id}_{\mathcal{U}}} : A \to B$$



One would have also thought of

$P : \mathcal{U} \to \mathcal{U}'$

$P \equiv \lambda(A:\mathcal{U}).A$

to be in accordance with the standard treatment of the definition of type families:

$A:\mathcal{U}, \quad P : A \to \mathcal{U}$

It is expected that this naturally generated function of type $A \to B$ to be the required equivalence (Theorem of Naturality)

We show that $p_*^{\mathrm{id}_{\mathcal{U}}}$ is an equivalence: Let $D : \prod_{A,B:\mathcal{U}} \prod_{p:A=_{\mathcal{U}}B} \mathcal{U}'$ be defined by

$\mathcal{D}(A,B,p) \equiv \mathrm{isequiv}(p_*^{\mathrm{id}_\mathcal{U}})$. Hence

$$\mathcal{D}(A,A,\mathrm{refl}_A) \equiv \mathrm{isequiv}((\mathrm{refl}_A)_*^{\mathrm{id}_\mathcal{U}})$$
$$\equiv \mathrm{isequiv}(\mathrm{id}_{\mathrm{id}_\mathcal{U}(A)})$$
$$\equiv \mathrm{isequiv}(\mathrm{id}_A)$$

(Examples 2.6.2 (ii))

and we have already shown that this type is inhabited $\underbrace{((\mathrm{id}_A; \mathrm{refl}_A), (\mathrm{id}_A, \mathrm{refl}_A))}_{\overset{\equiv}{\mathrm{eqv}_A}} : \mathrm{isequiv}(\mathrm{id}_A)$

By $\mathrm{Ind}_=$ in $\mathcal{U}$ there is

$$\Psi : \prod_{A,B:\mathcal{U}} \prod_{p:A=_\mathcal{U}B} \mathrm{isequiv}(p_*^{\mathrm{id}_\mathcal{U}}) \quad \text{s.t.}$$

$$\Psi(A,A,\mathrm{refl}_A) \equiv \mathrm{eqv}_A$$

We define $\boxed{\mathrm{idtoequiv}(p) \equiv p_*^{\mathrm{id}_\mathcal{U}}}$; for every $p : A =_\mathcal{U} B$.

By $\Psi(A,B,p) : \mathrm{isequiv}(p_*^{\mathrm{id}_\mathcal{U}})$ we get that $\mathrm{idequiv} : (A =_\mathcal{U} B) \to (A \simeq_\mathcal{U} B)$

$\boxed{p \mapsto (p_*^{\mathrm{id}_\mathcal{U}}, \Psi(A,B,p))}$

Moreover, $\mathrm{idequiv}(\mathrm{refl}_A) \equiv (\mathrm{refl}_A)_*^{\mathrm{id}_\mathcal{U}} \equiv \mathrm{id}_{\mathrm{id}_\mathcal{U}(A)} \equiv \mathrm{id}_A$ ∎

Remark 3.2.2 - The two functions defined in proofs 1 and 2 are ~~[illegible]~~

$$\Phi(A,B,p) : A \simeq_\mathcal{U} B \equiv \sum_{f:A\to B} \mathrm{isequiv}(f)$$
$$p_*^{\mathrm{id}_\mathcal{U}} : A \simeq_\mathcal{U} B$$

~~that any two terms of the type $A \simeq_\mathcal{U} B$ are equal~~

$$\Phi : \prod_{A,B:\mathcal{U}} \prod_{p:A=_\mathcal{U}B} A \simeq_\mathcal{U} B$$

$$\Phi' \equiv \lambda(A,B:\mathcal{U}, p : A =_\mathcal{U} B).\, p_*^{\mathrm{id}_\mathcal{U}}$$

s.t. $\Phi(A,A,\mathrm{refl}_A) \equiv \Phi'(A,A,\mathrm{refl}_A) \equiv \mathrm{id}_A$.

By the uniqueness (or just by path-induction) of extension we get:

$$\underset{\mathrm{idequiv}(p)}{\overset{\equiv}{\Phi(A,B,p)}} = \underset{p_*^{\mathrm{id}_\mathcal{U}}}{\overset{\equiv}{\Phi'(A,B,p)}}$$

i.e., the two functions $\overset{\mathrm{idequiv}}{}$ defined in Proof 1 and 2 are pointwise equal $\overset{\mathrm{Funext}}{\Longrightarrow}$ equal functions.

$\sum_{f:A\to B} \mathrm{isequiv}(f) \equiv A \simeq_u B$ (*) On $\mathrm{Expl}(\mathrm{ii})$

$(f, u) : (A \simeq_u B) \quad (f, u') : (A \simeq_u B)$

$$E : \prod_{u, u' : \mathrm{isequiv}(f)} (u = u').$$

By 2.7.2 (*): $$\left((f, u) =_{A \simeq_u B} (f, u')\right) \simeq \sum_{p : pr_1((f,u)) = pr_1((f,u'))} \left(p_*^{f \mapsto \mathrm{isequiv}(f)} (pr_2((f,u))) = pr_2((f,u'))\right)$$

$$= \sum_{p : f = f} \left(p_*^{f \mapsto \mathrm{isequiv}(f)} (u) = u'\right)$$

Let $(\mathrm{refl}_f, E(u, u')) :$

$$\Big(\underbrace{(\mathrm{refl}_f)_*^{f \mapsto \mathrm{isequiv}(f)} (u)}_{\mathrm{id}_{\mathrm{isequiv}(f)}(u) = u'} = u'\Big) = (u = u')$$

Hence since the $\sum$ type is inhabited, the equivalent type is also inhabited

$$\boxed{(f, u) =_{A \simeq_u B} (f, u').}$$

2.7.2. $A : \mathcal{U}$, $P : A \to \mathcal{U}$, $w, w' : \sum_{x:A} P(x)$. Then

$$(w = w') \simeq \sum_{p : pr_1(w) =_A pr_1(w')} \left(p_*^P (pr_2(w)) =_{P(pr_1(w'))} pr_2(w')\right)$$

- **Axiom of Univalence** (UA, Voevodsky): The function idtoequiv: $(A =_u B) \to (A \simeq_u B)$ is an equivalence, hence

$$(A =_u B) \simeq (A \simeq_u B)$$

- Corollary 3.23 This function has a quasiinverse, $\psi(\text{idtoequiv}) \equiv ua : (A \simeq_u B) \to (A =_u B)$

- Remark 3.24: We can break UA into:

(I) An introduction rule $ua : (A \simeq_u B) \to (A =_u B)$

(II) The elimination rule $\text{idtoequiv} : (A =_u B) \to (A \simeq_u B)$

$$\text{idtoequiv}(p) \equiv p_*^{id_u}$$

s.t. $\text{idtoequiv}(\text{refl}_A) \equiv id_A$

(III) The propositional computation rule: $\text{idtoequiv}(ua(f))x) = f(x)$

(IV) The propositional uniqueness principle: $p = ua(\text{idtoequiv}(p))$

- Explanation of (III), (IV): $(\text{idtoequiv} \circ ua) \sim id_{A \simeq_u B} \equiv$

(a)

$$\prod_{\substack{w : A \simeq_u B \\ \shortparallel \\ w : \sum_{f : A \to B} \text{isequiv}(f)}} \left(\text{idtoequiv}(ua(w)) = w\right)$$

let $(f : A \to B,\ u : \text{isequiv}(f)) : A \simeq_u B$ a canonical element

$$\text{idtoequiv}(ua((f,u))) = (f,u)$$

$$\Downarrow$$

$$pr_1(\qquad\qquad) = pr_1((f,u)) \equiv f$$

In the HoTT-book there is an identification of the two equalities. One should write

$$\text{idtoequiv}' : A =_u B \to (A \to B)$$

$$\text{idtoequiv}'(p) \equiv pr_1(\text{idtoequiv}(p))$$

and the accurate writing of (III) is $\text{idtoequiv}'(ua((f,u))) = f$ (III)' $\Rightarrow$
$\text{idtoequiv}'(ua((f,u)), x) = f(x)$ (III)''

Moreover $(ua \circ idequiv) \sim id_{A =_u B}$ $\equiv$

$$\prod_{p : A =_u B} ua(idtoequiv(p)) = p$$ and (IV) follows automatically.

(So the wording of all these in the HoTT-book is a bit ~~slo~~ sloppy)

• **Proposition 3.2.5** (i) $refl_A = ua(id_A)$

(ii) $ua(f) * ua(g) = ua(g \circ f)$

(iii) $ua(f^{-1}) = ua(f)^{-1}$

(In complete analogy to Prop. 3.1.4.)

In both sites we omitt the second component of the inputs.

Explanation (b) Note that $\boxed{ua(f) \text{ is a simplification for } ua((f, u))}$ since all proofs u: isequiv(f) are equal and then the pairs $(f, u), (f, u')$ are equal in $\sum_{f : A \to B} isequiv(f)$ (not proved this come, but it is evident). Hence $ua((f, u)) = ua((f, u'))$

(see white page)

$$\begin{array}{c} ua((f,u)) = ua((f,u')) \\ ||| \\ ua(f) \end{array}$$

So it is not a simplification rather a natural notation.

Proof of Prop. 3.2.5 (i) $ua(id_A) = ua(idtoequiv(refl_A)) \overset{(IV)}{=} refl_A$

(ii) $ua(g \circ f) = ua(idtoequiv(ua(g)) \circ idtoequiv(ua(f)))$

$= ua(ua(g)_*^{id_u} \circ ua(f)_*^{id_u})$

$(p_*^P \circ q_*^P)(u) = (q \cdot p)_*^P(u)$ $\overset{\text{funext}}{=} ua(((ua(f) * ua(g))_*^{id_u})$

$= ua(idtoequiv(ua(f) * ua(g)))$

$\overset{(IV)}{=} ua(f) * ua(g)$

(iii) $ua(f \circ f^{-1}) = ua(f^{-1}) * ua(f)$

$|||$

$ua(id_B) = refl_B$

$ua(f^{-1} \circ f) = ua(f) \circ ua(f^{-1})$ (not needed)

$|||$

$ua(refl_A) = refl_A$

$$ua(f^{-1}) * ua(f) = refl_B = ua(f)^{-1} * ua(f)$$

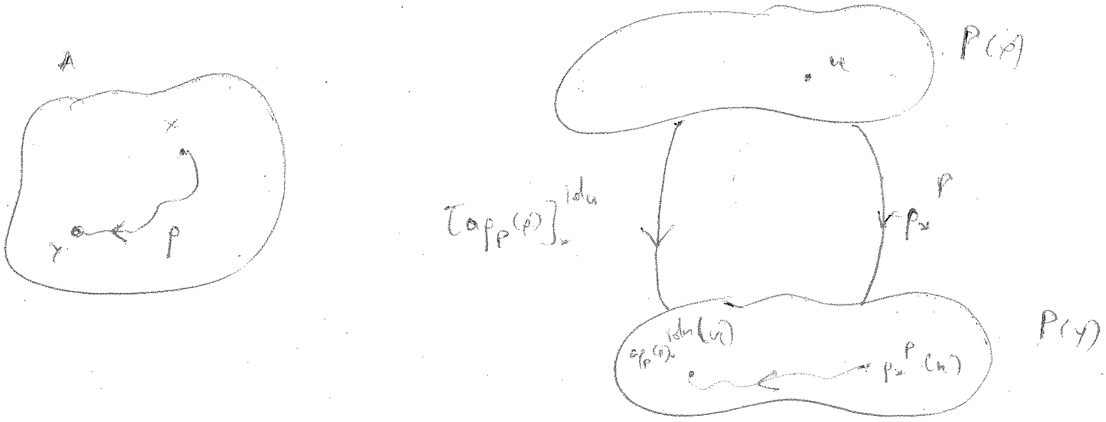
$$\downarrow$$

$$ua(f^{-1}) = ua(f)^{-1}$$

Proposition 3.2.6 : $A : \mathcal{U}$, $P : A \to \mathcal{U}$, $x, y : A$, $p : x =_A y$, $u : P(x)$. Then

$$p_*^P(u) = [ap_P(p)]_*^{id_\mathcal{U}}(u)$$
$$\equiv \left[idtoequiv\,(ap_P(p))\right](u) \qquad \text{(by def of idtoequiv)}$$

Proof 1 :



$$P : A \to \mathcal{U}$$

$$ap_P : (x =_A y) \longrightarrow (P(x) =_\mathcal{U} P(y)) \xrightarrow{idtoequiv} P(x) \simeq_\mathcal{U} P(y)$$
$$p \longmapsto ap_P(p) \longmapsto [ap_P(p)]_*^{id_\mathcal{U}}$$

If $C(x,y,p) \equiv \left(p_*^P(u) = [ap_P(p)]_*^{id_\mathcal{U}}(u)\right)$, then

$$refl_u : \quad C(x,x,refl_x) \equiv \left((refl_x)_*^P(u) = [ap_P(refl_x)]_*^{id_\mathcal{U}}(u)\right)$$
$$\equiv \left(id_{P(x)}(u) = (refl_{P(x)})_*^{id_\mathcal{U}}(u)\right)$$
$$\equiv \left(u = id_{id_\mathcal{U}(P(x))}(u)\right)$$
$$\equiv u = id_{P(x)}(u))$$
$$\equiv u = u$$, and then we path-induction

$\square$

Proof 2 : By Prop. 2.4.6 , $p_*^{P \circ f}(u) =_{P(f(y))} (ap_f(p))_*^P(u)$ $\quad f : A \to B$, $P : B \to \mathcal{U}$

take $P \equiv id_\mathcal{U}$ $f \equiv P$

$$p_*^{id_\mathcal{U} \circ P}(u) = p_*^P(u) = (ap_P(p))_*^{id_\mathcal{U}}(u)$$

$\square$

• **Corollary 3.2.7**: $A:\mathcal{U}$, $P: A \to \mathcal{U}$, $x:A$, $p: x =_A x$, $u: P(x)$

If $\mathsf{ap}_P(p) = \mathsf{ua}(f)$ for some $f: P(x) \to P(x)$ s.t. $\mathsf{isequiv}(f)$,

then $\boxed{p_*^P(u) = f(u)}$

Proof: By Prop. 3.2.6

$$p_*^P(u) = \left[\mathsf{idtoequiv}\left(\mathsf{ap}_P(p)\right)\right](u)$$
$$= \left[\mathsf{idtoequiv}\left(\mathsf{ua}(f)\right)\right](u)$$
$$= \cancel{\ldots}$$
$$\overset{(III)}{=} f(u).$$

$$\left(\mathsf{ap}_P(p) = \mathsf{ua}(f) \xrightarrow{\mathsf{ap}_{\mathsf{idtoequiv}}} \mathsf{idtoequiv}(\cdot) = \mathsf{idtoequiv}(\cdot)\right)$$

By FunExt: $p_*^P = f$. □

(This will be used in the proof: $\mathsf{loop}_*^{\mathsf{code}} = \mathsf{succ}_{\mathbb{Z}}$)

• **Proposition 3.2.8** $\mathcal{U}$ is not a set (i.e. $\mathsf{isSet}(\mathcal{U}) \to \mathbf{0}$ is inhabited)

Proof: Suppose $\mathsf{isSet}(\mathcal{U}) \equiv \prod_{A,B:\mathcal{U}} \prod_{p,q:A=_{\mathcal{U}}B} (p=q)$ is inhabited.

By Rec_2 there is $f: 2 \to 2$ s.t. $f(0_2) \equiv 1_2$, $f(1_2) \equiv 0_2$

By Ind_2 $\prod_{x:2}\left(\underbrace{(f \circ f)(x) = id_2(x)}\right) \equiv f \circ f \sim id_2$

$P(0_2)$: $f(f(0_2)) \equiv f(1_2) \equiv 0_2 = 0_2$

$P(1_2)$: $f(f(1_2)) \equiv f(0_2) \equiv 1_2 = 1_2$

i.e., $(f, u) : 2 \simeq 2$

Hence $\mathsf{ua}(f) : 2 =_{\mathcal{U}} 2$. Of course, $\mathsf{refl}_2 : 2 =_{\mathcal{U}} 2$

The hypothesis $\mathsf{isSet}(\mathcal{U})$ implies $\mathsf{ua}(f) = \mathsf{refl}_2$.

i.e., by Prop. 3.2.5(ii) $\mathsf{ua}(f) = \mathsf{ua}(id_2) \Rightarrow$

$$\mathsf{idtoequiv}(\mathsf{ua}(f))(0) = \mathsf{idtoequiv}(\mathsf{ua}(id_2))(0) \Rightarrow$$

(III) $f(0)$ ~~$(f = id_2) \Rightarrow$~~ $id(0)$

$1_2 =_? 0_2$.

But then we take an inhabitant of $\mathbf{0}$, i.e. we proved $\mathsf{isSet}(\mathcal{U}) \to \mathbf{0}$. □

• This fact is used in the proof of loop $\neq$ refl$_{base}$ (!)

[Incompatibility of UA with Class. logic !!]

<u>Theorem 3.2.9</u>: TFTU: $\left(\prod_{A:\mathcal{U}} (\neg\neg A \to A)\right) \to 0$

<u>Proof (Coquand)</u>: Let $F : \prod_{A:\mathcal{U}} (\neg\neg A \to A)$

Let $e^{:2\to 2}$ as in the proof of Prop. 3.28 and $p \equiv ua(e) : (2 =_{\mathcal{U}} 2)$

$F(2) : (\neg\neg 2 \to 2)$

Since $apd_F : \prod_{q : A = B} \left(q_*^{Q} (F(A)) = F(B) \right)$, hence

$$apd_F(p) : \left(p_*^{Q} (F(2)) = F(2) \right)$$

where $Q(A) \equiv \neg\neg A \to A$

$\equiv Q_1(A) \to Q_2(A)$

$Q_1(A) \equiv \neg\neg A$ and $Q_2(A) \equiv A$, i.e., $Q_2 \equiv id_{\mathcal{U}}$

By definition

$$happly : \left(p_*^{Q} (F(2)) = F(2) \right) \to \prod_{u : \neg\neg 2} \left(p_*^{Q} (F(2)) \right)(u) = \left(F(2)\right)(u)$$

hence $$\left(happly\left(apd_F(p)\right)\right)(u) : \left(\left(p_*^{Q} (F(2))\right)(u) = \left(F(2)\right)(u) \right) \quad (1)$$

By Prop. 2.9.5: $\left(p_*^{Q_1 \to Q_2}(f)\right)(u) = p_*^{Q_2}\left(f\left(\left(p^{-1}\right)_*^{Q_1} (u) \right)\right)$

Hence

$$\left(p_*^{\overset{Q}{\equiv}\,Q_1 \to Q_2} (F(2))\right)(u) = p_*^{id_{\mathcal{U}}}\left(F(2)\left(\left(p^{-1}\right)_*^{A \mapsto \neg\neg A} (u) \right)\right) \quad (2)$$

<u>Claim 1</u>: $\prod_{u,v : \neg\neg 2} \prod_{z : \neg 2} (u(z) = v(z))$

proof of Claim 1: If $u, v : \neg\neg 2$, then for any $z : \neg 2$ $u(z) =_0 v(z)$, since trivially any two elements of 0 are equal. □

Hence, by FunExt $\prod_{u,v:\neg\neg 2} (u = v)$.

Since $\rho : 2 =_u 2$, $\rho^{-1} : 2 =_u 2$ and

$$\left(\rho^{-1}\right)_*^{A \mapsto \neg\neg A} : \neg\neg 2 \to \neg\neg 2,$$

we get $\left(\rho^{-1}\right)_*^{A \mapsto \neg\neg A}(u) : \neg\neg 2$, hence by Claim 1 and FunExt

$$\left(\rho^{-1}\right)_*^{A \mapsto \neg\neg A}(u) = u \qquad (3)$$

Because of (3), (2) we get $\left(\rho_*^{d}(F(2))\right)(u) = \rho_*^{id_u}\left(F(2)(u)\right)$ (4)

and with (1) we get a term of type

$$\rho_*^{id_u}\left(F(2)(u)\right) = \left(F(2)\right)(u) \qquad (5)$$

$$|||$$

$$idtoeqv\left(\cancel{F(2)}\,\rho\right)\left(F(2)(u)\right)$$

$$|||$$

$$idtoeqv\left(ua(e)\right)\left(F(2)(u)\right) \overset{(III)}{=\!=} e\left(F(2)(u)\right)$$

i.e., there is $t : e\left(F(2)(u)\right) = F(2)(u)$ (6)

Claim 2 $\quad \eta : \prod_{x:2} \neg\left(e(x) =_2 x\right) \quad : \quad P(0_2) : \neg\left(e(0_2) =_2 0\right) \equiv \neg(1=0)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad P(1_2) : \neg\left(e(1) =_2 1\right) \equiv \neg(0=1)$

So, we define $\lambda : \prod_{A:U}(\neg\neg A \to A) \longrightarrow 0$ by

$$\lambda(F) \equiv \eta\left(F(2)(u), t\right), \quad \text{for every } F : \prod_{A:U}(\neg\neg A \to A)$$

□

Corollary 3.2.10 (EXERCISE) TFTH

$$\left(\prod_{A:\mathcal{U}} A + (\neg A) \right) \to 0$$

Proof: It suffices to show that $\left(\prod_{A:\mathcal{U}} A + (\neg A) \right) \to \left(\prod_{A:\mathcal{U}} (\neg\neg A \to A) \right)$

is inhabited and then use Theorem 3.2.9

If $A:\mathcal{U}$ and $G : \prod_{A:\mathcal{U}} A + (\neg A)$, then $G(A) : A + (\neg A)$

Claim $\left(A + (\neg A) \right) \to (\neg\neg A \to A)$ is inhabited

Proof by Rec_+ : $g_\ell : A \to (\neg\neg A \to A)$

$g_\ell(a) \equiv \lambda(x : \neg\neg A) . a$, if $a : A$

$g_r : (\neg A) \to (\neg\neg A \to A)$ is defined by

$g_r(y)(u) \equiv Efq_A(u(y))$, $y : \neg A \equiv A \to 0$
$u : \neg\neg A \equiv ((A \to 0) \to 0)$

where $u(y) : 0$

$Ef_{\neg A} : 0 \to A$

hence $Ef_{\neg A}(u(y)) : A$.

hence there is $f : A + (\neg A) \to (\neg\neg A \to A)$

We define $F : \prod_{A:\mathcal{U}} (\neg\neg A \to A)$

$F \equiv \lambda(A:\mathcal{U}) . f(G(A))$

$G \mapsto F$ is the required inhabitant $\square$

• $E_a \equiv \sum_{x:A} (a =_A x)$.

$\Lambda_a : \prod_{u:E_a} \left((a, \mathrm{refl}_a) =_{E_a} u \right)$ st. $\Lambda_a\left((a,\mathrm{refl}_a)\right) \equiv \mathrm{refl}_{(a,\mathrm{refl}_a)}$

Take (a, p) where $p : (a =_A a)$.

$$\boxed{\Lambda_a\left((a,p)\right) : \left((a, \mathrm{refl}_a) =_{E_a} (a,p) \right)}$$

$\Downarrow$

$\mathrm{ap}_{pr_2}(\Lambda_a(a,p)) : pr_2\left((a,\mathrm{refl}_a)\right) = pr_2\left((a,p)\right)$

$\qquad\qquad \equiv \qquad\qquad \equiv$

$\qquad\qquad \mathrm{refl}_a \qquad\qquad p$

non-dependent pr_2 is not a function: It's a dependent one!!

?

I.e., $\mathrm{refl}_a = p$, for every $p : (a =_A a)$.

$pr_2\left((a,\mathrm{refl}_a)\right) \equiv \mathrm{refl}_a : (a = a) \equiv P(pr_1((a,\mathrm{refl}_a))) \equiv P(a)$

$pr_2\left((a,p)\right) \equiv p$

$pr_1 : E_a \to A$

$P(x) \equiv a = x$

$P \circ pr_1$

$u \mapsto P(pr_1(u))$

$(x, p) = (a, \mathrm{refl}_a)$

$pr_2 : \prod_{u:E_a} P(pr_1(u))$

$\mathrm{apd}_{pr_2}\left(\Lambda_a((a,p))\right) : \left[\Lambda_a((a,p))\right]_* \left(pr_2(a,p)\right)$

$= pr_2\left((a,\mathrm{refl}_a)\right)$

Let $(a,p) = (a,\mathrm{refl}_a)$

$\Lambda_a(a,p)$:

GOOD

$\equiv \left(\Lambda_a((a,p))\right)_*^{P \circ pr_1} (p) = \mathrm{refl}_a$

By Prop. 2.4.6

$= \cancel{\mathrm{ap}_{pr_1}(p)} \left(\mathrm{ap}_{pr_1}\left(\Lambda_a\left((a,p)\right)\right)\right)_*^{x \mapsto a = x} (p) = p \cdot \mathrm{ap}_{pr_1}(\Lambda_a((a,p)))$

pr_2 $\prod_{z \in C_x} x =_? pr_1(z)$

$\mathrm{apd}_{pr_2}(\Lambda_a((a,1))) = [\Lambda_a((a,1))]_*^{z \mapsto a = pr_1(z)} (a = pr_1(a,1)) =$

$\alpha =_A pr_1(\alpha, \mathrm{refl}_A)$

$= [\Lambda_a((a,1))]_*^{z \mapsto a = pr_1(z)} (\alpha = \alpha) = (\alpha = \alpha)$

(i) $\boxed{(0 =_{\mathbb{N}} 0) \simeq 1}$ Pag. 26

$f : (0 =_{\mathbb{N}} 0) \to 1$ $f \equiv \lambda(p : 0 = 0).\, 0_1$

$g : 1 \to (0 =_{\mathbb{N}} 0)$ Let $\mathrm{refl}_{0_{\mathbb{N}}} : 0 =_{\mathbb{N}} 0$.

$g(0_1) \equiv \mathrm{refl}_{0_{\mathbb{N}}}$ by Rec 1.

$\prod_{z:1} \big(\underbrace{f(g(z)) = z}_{P(z)}\big)$. refl_{0_1} : $P(0_1) \equiv f(g(0_1)) = 0_1$

$\equiv f(\mathrm{refl}_{0_{\mathbb{N}}}) = 0_1$

$\equiv 0_1 = 0_1$

$\prod_{p : 0 =_{\mathbb{N}} 0} \big(g(f(p)) = p\big)$ $g(f(\mathrm{refl}_{0_{\mathbb{N}}})) \equiv g(0_1) \equiv \mathrm{refl}_{0_{\mathbb{N}}}$.

$\prod_{p : 0 = 0} \big(\mathrm{refl}_{0_{\mathbb{N}}} = p\big)$ We can't use the same method i.e. path-induction as in theorem of $(x = y) \simeq 1$.

$E_0 \equiv \sum_{x : \mathbb{N}} (0 =_{\mathbb{N}} x)$

By UP($=$) $(0_{\mathbb{N}}, p) =_{E_0} (0_{\mathbb{N}}, \mathrm{refl}_{0_{\mathbb{N}}})$, for any $(0, p) : E_0$

Hence $pr_2 : \cancel{f}(0, p) = pr_2(0, \mathrm{refl}_{0_{\mathbb{N}}})$

$p = \mathrm{refl}_{0_{\mathbb{N}}}$ WRONG

Hence we get an inhabitant of $\prod_{p : 0 = 0} \big(g(f(p)) = p\big)$ as follows

$p = \mathrm{refl}_{0_{\mathbb{N}}} = g\,f(p)$.

(devil's-mode) Can I avoid it ??

(ii) $\boxed{\left(0 =_{\mathbb{N}} \mathrm{succ}(x)\right) \simeq 0}$ (and similarly $(\mathrm{succ}(x) = 0) \simeq 0$)

We have ~~0~~ $\mathrm{Efq}_{0=\mathrm{succ}(x)} : 0 \to (0 = \mathrm{succ}(x))$

We have shown $e : (0 = \mathrm{succ}(x)) \to 0$.

Hence $\prod_{z:0} \left(e(\mathrm{Efq}(z)) = z\right)$: trivially by Ind_0

$(F :) \prod_{p : 0 = \mathrm{succ}(x)} \left(\mathrm{Efq}(e(p)) = p\right)$

$F(p) \equiv \mathrm{Efq}_{\mathrm{Efq}(e(p)) = p}(e(p))$

(iii) $\boxed{\left(\mathrm{succ}(x) = \mathrm{succ}(y)\right) \simeq (x = y)}$

$\mathrm{ap}_{\mathrm{succ}} : x = y \to \mathrm{succ}(x) = \mathrm{succ}(y)$

we have proven $\mathrm{succ}(x) = \mathrm{succ}(y) \to x = y$. Clearly these are equal.

Corollary 2.6.10

Hence $\boxed{\prod_{x,y:\mathbb{N}} (x = y \simeq 0) + (x = y \simeq 1)}$ by $\mathrm{Ind}_{\mathbb{N}}$ (maybe double induction)