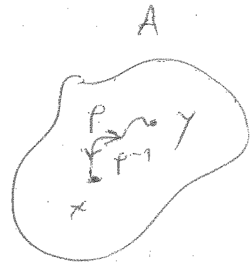
Chapter 2: ~~[illegible]~~ Applications of Path-induction

Two inequalities

Section 2.1: $0 \neq 1$ in 2 and in $\mathbb{N}$.

Section 2.2: Properties of equality types. (or Operations on paths)



Proposition 2.2.1: There is a function $^{-1} : (x =_A y) \to (y =_A x)$ $\quad A : \mathcal{U}, \; x, y : A$.

$$p \mapsto p^{-1}$$

s.t. $(\mathrm{refl}_x)^{-1} \equiv \mathrm{refl}_x$, for every $x : A$.

Proof: Let $C : \prod_{x,y:A} \prod_{p : x =_A y} \mathcal{U}$ be defined by $C(x, y, p) \equiv (y =_A x)$.

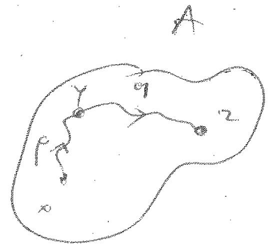
hence $C(x, x, \mathrm{refl}_x) \equiv (x =_A x)$.

Let $\mathrm{refl}_x : (x =_A x)$, for every $x : A$.

By $\mathrm{Ind}_=$, there is $F : \prod_{x,y:A} \prod_{p : x =_A y} (y =_A x)$ s.t. $F(x, x, \mathrm{refl}_x) \equiv \mathrm{refl}_x$.

We define $\quad ^{-1} :\equiv \lambda(p : x =_A y).\, F(x, y, p)$

here $\quad ^{-1}(\mathrm{refl}_x) :\equiv \mathrm{refl}_x^{-1} \equiv F(x, x, \mathrm{refl}_x) \equiv \mathrm{refl}_x$. $\square$



Proposition 2.2.2 If $A : \mathcal{U}$, $x, y, z : A$, there is a function

$$* : (x =_A y) \to (y =_A z) \to (x =_A z)$$

$$(p, q) \mapsto p * q \quad \text{, the concatenation of } p, q$$

such that $\mathrm{refl}_x * q \equiv q$, for every $q : (x =_A z)$

Proof: Define $\quad C_z(x, y, p) \equiv \big((y =_A z) \to (x =_A z)\big)$, here

$$C_z(x, x, \mathrm{refl}_x) \equiv (x =_A z) \to (x =_A z)$$

Let $\quad c_z \equiv \mathrm{id}_{(x =_A z)}$

By $\mathrm{Ind}_=$ there is $F_z : \prod_{x,y:A} \prod_{p : x =_A y} \big((y =_A z) \to (x =_A z)\big)$ s.t.

$$F_z(x, x, \mathrm{refl}_x) \equiv \mathrm{id}_{x =_A z}$$

We define $\quad * \equiv \lambda(p : x =_A y, q : y =_A z).\, F_z(x, y, p, q)$

Hence $\mathrm{refl}_x * q \equiv F_z(x, x, \mathrm{refl}_x, q) \equiv \mathrm{id}_{x =_A z}(q) \equiv q$. ∎

<u>Corollary 2.2.3</u>. If $x : A$, $\mathrm{refl}_x * \mathrm{refl}_x \equiv \mathrm{refl}_x$

(i.e. we get the condition of Lemma 2.1.1 in HoTT-book)

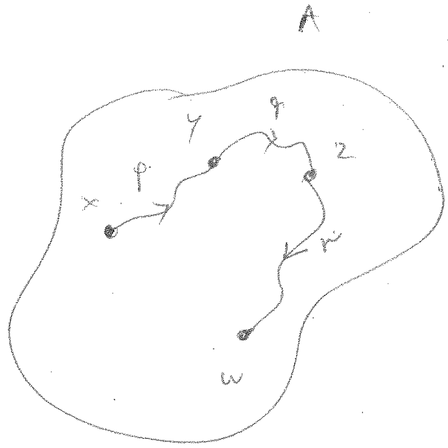
<u>Proof</u>: By Prop. 2.2.2 for $x, z \equiv x$, $\mathrm{refl}_x * \underbrace{\mathrm{refl}_x}_{q} \equiv \mathrm{refl}_x$.

<u>Proposition 2.2.4</u> (i) $p * \mathrm{refl}_y = p$, if $A : \mathcal{U}$ $x, y, z, w : A$ $p : x =_A y$, $q : y =_A z$, $r : z =_A w$

(ii) $p * p^{-1} = \mathrm{refl}_x$ and $p^{-1} * p = \mathrm{refl}_y$

(iii) $(p^{-1})^{-1} = p$

(iv) $(p * q) * r = p * (q * r)$



<u>Proof</u> (i) $C(x, y, p) \equiv (p * \mathrm{refl}_y = p)$

$C(x, x, \mathrm{refl}_x) \equiv \mathrm{refl}_x * \mathrm{refl}_x = \mathrm{refl}_x$

(Cor. 2.2.3) $\equiv \mathrm{refl}_x = \mathrm{refl}_x$

Take $c \equiv \mathrm{refl}_{\mathrm{refl}_x}$.

Hence, there is $F_{(i)} : \prod_{x,y:A} \prod_{p : x =_A y} (p * \mathrm{refl}_y = p)$ s.t. $F_{(i)}(x, x, \mathrm{refl}_x) \equiv \mathrm{refl}_{\mathrm{refl}_x}$

i.e., $F_{(i)}(x, y, p) : (p * \mathrm{refl}_y = p)$.

Note the difference between 2.2.4.(i) and the comp. rule in Prop. 2.2.2.

(ii) $C(x, y, p) \equiv ((p * p^{-1}) = \mathrm{refl}_x)$

$C(x, x, \mathrm{refl}_x) \equiv ((\mathrm{refl}_x * \mathrm{refl}_x^{-1}) = \mathrm{refl}_x)$

Prop. 2.2.1 $\equiv (\mathrm{refl}_x * \mathrm{refl}_x) = \mathrm{refl}_x$

Cor. 2.2.3 $\equiv \mathrm{refl}_x = \mathrm{refl}_x$

and take $c \equiv \mathrm{refl}_{\mathrm{refl}_x}$

Similarly:

$C(x, y, p) \equiv ((p^{-1} * p) = \mathrm{refl}_y)$

$C(x, x, \mathrm{refl}_x) \equiv (\mathrm{refl}_x^{-1} * \mathrm{refl}_x) = \mathrm{refl}_x$

$\equiv \mathrm{refl}_x = \mathrm{refl}_x$.

(iii) $C(x, y, p) \equiv (p^{-1})^{-1} = p$, $C(x, x, \mathrm{refl}_x) \equiv ((\mathrm{refl}_x^{-1})^{-1} = \mathrm{refl}_x) \equiv (\mathrm{refl}_x^{-1} = \mathrm{refl}_x) \equiv (\mathrm{refl}_x = \mathrm{refl}_x)$

take again $c \equiv \mathrm{refl}_{\mathrm{refl}_x}$.

(iv) $$C(x,y,p) \equiv \prod_{q:y=_A z} \prod_{r:z=_A w} \left((p*q)*r = p*(q*r)\right)$$

$$C(x,x,refl_x) \equiv \prod_{q:x=_A z} \prod_{r:z=_A w} \left((refl_x * q)*r = refl_x * (q*r)\right)$$

$$\overset{\text{(Prop. 2.2.2, 2 times)}}{\equiv} \prod_{q:x=_A z} \prod_{r:z=_A w} (q*r = q*r)$$

define $c(x) \equiv \lambda(q:x=_A z,\ r:z=_A w).\ refl_{q*r}$. $\square$

<u>Proposition 2.2.5.</u> If $A:\mathcal{U}$, $x,y,z:A$, $p:x=_A y$, $q:y=_A z$, then

<u>(Exercise)</u> $$(p*q)^{-1} = q^{-1} * p^{-1}$$

<u>Proof</u>: $$C(x,y,p) \equiv \prod_{q:y=_A z} \left((p*q)^{-1} = (q^{-1}*p^{-1})\right)$$

$$C(x,x,refl_x) \equiv \prod_{q:x=_A z} (refl_x * q)^{-1} = (q^{-1} * refl_x^{-1})$$

$$\overset{\text{Prop. 2.2.2}}{=} \prod_{q:x=_A z} q^{-1} = (q^{-1} * refl_x)$$

We prove that $q^{-1} = q^{-1} * refl_x$ by Prop. 2.2.4 (i)

So $c \equiv \lambda(q:x=_A z).\ F_{(i)}(z,x,q^{-1})$, where $F_{(i)}$ is defined in the proof of 2.2.4 (i).

Then use Ind$_=$. $\square$

* Recall that $ap_{f_{x,y}} : (x=_A y) \to (f(x) =_B f(y))$ st. $ap_f(refl_x) \equiv refl_{f(x)}$

<u>Proposition 2.2.6</u>: $A, B, C : \mathcal{U}$, $f:A\to B$, $g:B\to C$, $x,y,z:A$ (Exercise).

$p:x=_A y$, $q:y=_A z$

(i) $ap_{f_{x,z}}(p*q) = ap_{f_{x,y}}(p) * ap_{f_{y,z}}(q)$

(ii) $ap_{f_{y,x}}(p^{-1}) = (ap_f(p))^{-1}$

(iii) $ap_{g\circ f}(p) = ap_g(ap_f(p))$

(iv) $ap_{id_A}(p) = p$

Proof (i)

$$C(x,y,p) :\equiv \prod_{z:A}\ \prod_{q:y=_A z} \big(ap_f(p\cdot q) = ap_f(p)\cdot ap_f(q)\big)$$

$$C(x,x,refl_x) \equiv \prod_{z:A}\ \prod_{q:x=_A z} \big(ap_f(refl_x\cdot q) = ap_f(refl_x)\cdot ap_f(q)\big)$$

$$\equiv \prod_{z:A}\ \prod_{q:x=_A z} \big(ap_f(q) = refl_{f(x)}\cdot ap_f(q)\big)$$

$$\equiv \prod_{z:A}\ \prod_{q:x=_A z} \big(ap_f(q) = ap_f(q)\big)$$

We define $c :\equiv \lambda(z:A, q:x=_A z).\, refl_{ap_f(q)}$ and use $Ind_=$.

(ii)

$$C(x,y,p) \equiv \big(ap_f(p^{-1}) = (ap_f(p))^{-1}\big)$$

$$C(x,x,refl_x) \equiv \big(ap_f(refl_x^{-1}) = (ap_f(refl_x))^{-1}\big)$$

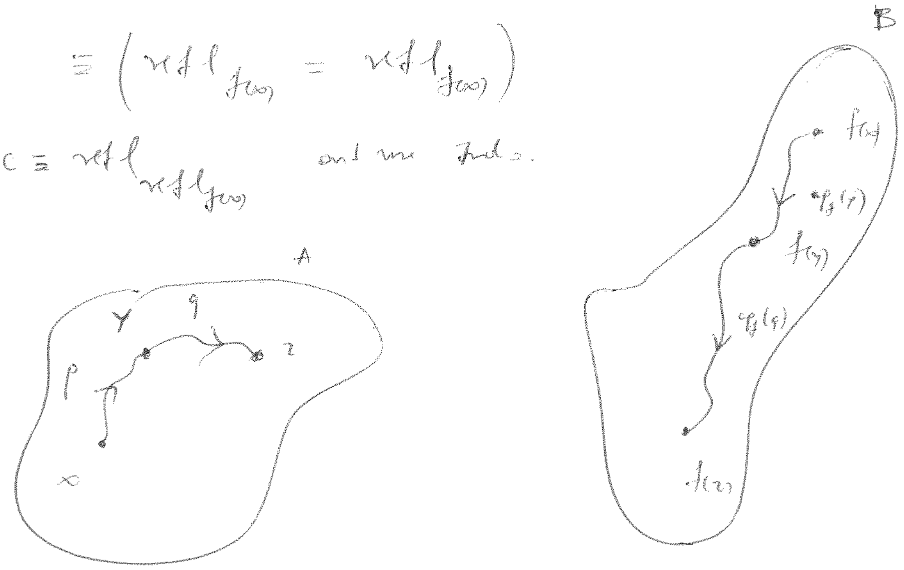
Def. 2.2.1 $\equiv ap_f(refl_x) = refl_{f(x)}^{-1}$

$$\equiv \big(refl_{f(x)} = refl_{f(x)}^{-1}\big)$$

$$\equiv \big(refl_{f(x)} = refl_{f(x)}\big)$$

take $c :\equiv refl_{refl_{f(x)}}$ and use $Ind_=$.

For (i) the figure is



(iii)

$$C(x,y,p) \equiv \big(ap_{g\circ f}(p) = ap_g(ap_f(p))\big)$$

$$C(x,x,refl_x) \equiv ap_{g\circ f}(refl_x) = ap_g(ap_f(refl_x))$$

$$\equiv \big(refl_{g f(x)} = ap_g(refl_{f(x)})\big)$$

$$\equiv refl_{g(f(x))} = refl_{g(f(x))}$$

Take $c :\equiv refl_{refl_{g(f(x))}}$ and apply $Ind_=$

(iv)

$$C(x,y,p) \equiv \big(ap_{id_A}(p) = p\big)$$

$C(x,x,refl_x) \equiv \big(ap_{id_A}(refl_x) = refl_x\big) \equiv \big(refl_{id_A(x)} = refl_x\big) \equiv \big(refl_x = refl_x\big)$. Take $c :\equiv refl_{refl_x}$

□

Proposition 2.27. Let $A, B : \mathcal{U}$, $b : B$

(fx). $f : A \to B$ $f \equiv \overline{b}_A$, the constant map b.

If $x, y : A$ ~~[illegible]~~, then $\mathrm{ap}_f(p) = \mathrm{refl}_b$, for every $p : x =_A y$.

i.e. $\boxed{f \text{ constant} \Rightarrow \mathrm{ap}_f \text{ is } =\text{-constant}}$

Proof: Let $C : \prod_{y:A} \prod_{p : x =_A y} \mathcal{U}$ be defined by

$$C(y, p) \equiv (\mathrm{ap}_f(p) = \mathrm{refl}_b)$$

$$\begin{aligned} C(y, \mathrm{refl}_y) &\equiv (\mathrm{ap}_f(\mathrm{refl}_y) = \mathrm{refl}_b) \\ &\equiv (\mathrm{refl}_{f(y)} = \mathrm{refl}_b) \\ &\equiv (\mathrm{refl}_b = \mathrm{refl}_b) \end{aligned}$$

Let $c \equiv \mathrm{refl}_{\mathrm{refl}_b} : C(y, \mathrm{refl}_y)$

By based-Ind, there is $F : \prod_{y:A} \prod_{p : x =_A y} C(y, p)$

st. $F(y, \mathrm{refl}_y) \equiv \mathrm{refl}_{\mathrm{refl}_b}$

i.e. $F(y, p) : (\mathrm{ap}_f(p) = \mathrm{refl}_b)$.

$\square$

Section 2.3 Application of dependent functions

Proposition 2.3.1 If $F : \prod_{x:A} P(x)$ and $x,y:A$, there is a map $\mathrm{apd}_F : \prod_{p : x =_A y} \left(p^P_*(F(x)) =_{P(y)} F(y) \right)$

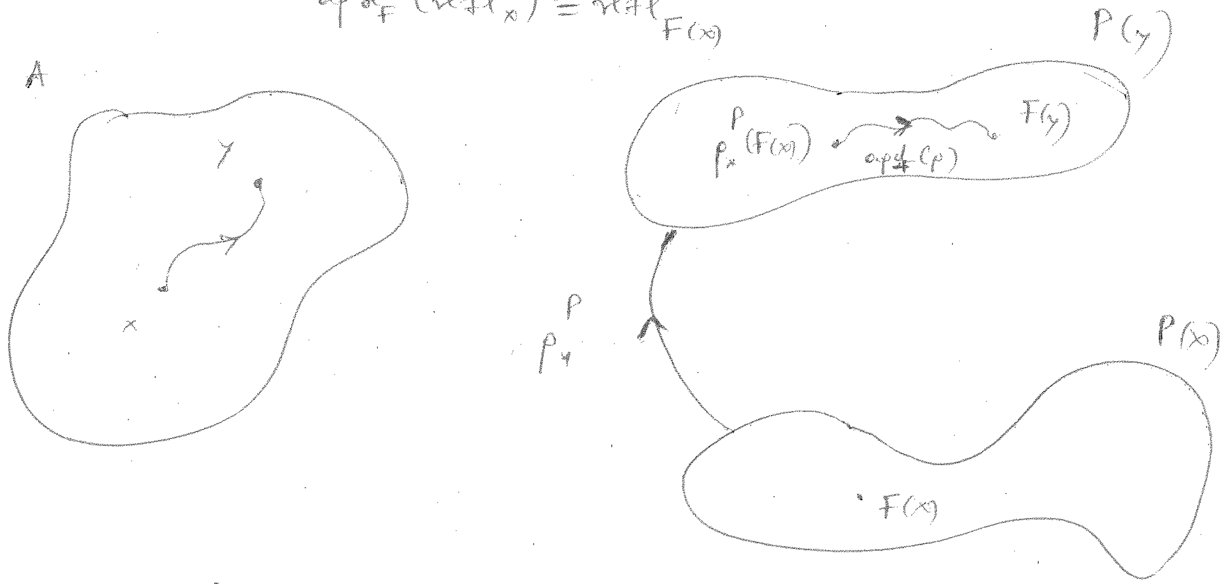
such that $\mathrm{apd}_F(\mathrm{refl}_x) : \left(\mathrm{refl}^P_{x*}(F(x)) = F(x) \right) \equiv$

$\left(\mathrm{id}_{P(x)}(F(x)) = F(x) \right) \equiv$

$F(x) = F(x)$

$$\mathrm{apd}_F(\mathrm{refl}_x) \equiv \mathrm{refl}_{F(x)}$$

Proof:



$$C(x,y,p) \equiv \left(p^P_*(F(x)) =_{P(y)} F(y) \right)$$

$$C(x,x,\mathrm{refl}_x) \equiv \left(F(x) = F(x) \right)$$

If $c(x) \equiv \mathrm{refl}_{F(x)}$, then we are done. $\square$

Proposition 2.3.2: $P : A \to \mathcal{U}$ $\quad P(x) \equiv B$, for every $x:A$

$x, y : A$, $p : x =_A y$

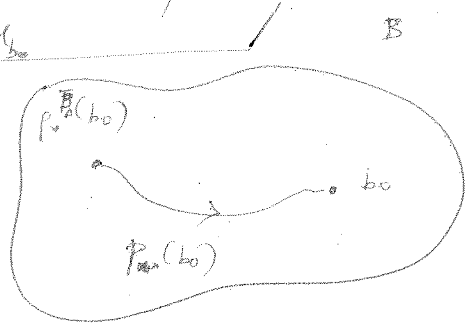
$b_0 : B$

There is a path $P_{x*}(b_0) : \left(p^{\bar{B}_A}_*(b_0) = b_0 \right)$

$P_A(\mathrm{refl}_{x*}(b_0)) \equiv \mathrm{refl}_{b_0}$

where $p^{\bar{B}_A}_* : P(x) \to P(y) \equiv B \to B$

$b_0 \mapsto p^P_*(b_0) : B$



Proof: $C(x,y,p) \equiv \left(p_*^{\bar{B}_A}(b_0) = b_0\right)$

$$C(x,x,\text{refl}_x) \equiv \left(\left(\text{refl}_x\right)_*^{\bar{B}_A}(b_0) = b_0\right)$$

$$\equiv \left(\text{id}_{\bar{B}_A(x)}(b_0) = b_0\right)$$

$$\equiv \text{id}_B(b_0) = b_0$$

$$\equiv b_0 = b_0$$

let $\text{refl}_{b_0} : C(x,x,\text{refl}_x)$

By ind, there is $F : \prod_{x,y:A} \prod_{p:x=_A y} \left(p_*^{\bar{B}_A}(b_0) = b_0\right)$ st

$$F(x,x,\text{refl}_x) \equiv \text{refl}_{b_0}$$

hence $p_*(b_0) \equiv F(x,y,p)$

Moreover $\text{refl}_{x*}^{\bar{B}}(b_0) \equiv \text{refl}_{b_0} \equiv F(x,x,\text{refl}_x)$ □

Proposition 2.3.3 $A:U$, $x,y:A$, $f:A \to B$, $p:x=_A y$

f can be understood as an element of $\prod_{x:A} B$

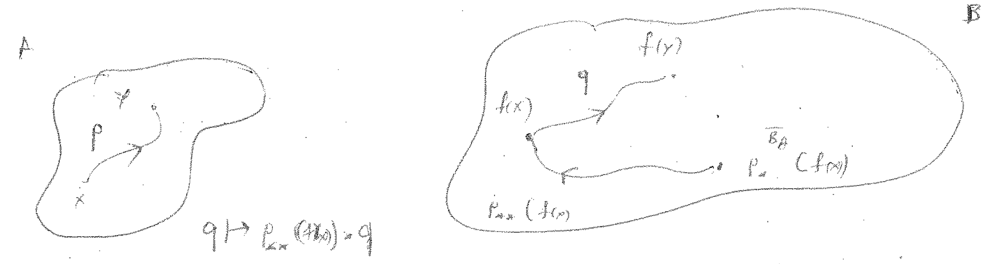
hence $\text{apd}_f : \prod_{p:x=_A y} \left(p_*^{\bar{B}_A}(f(x)) =_B f(y)\right)$

and of course $\text{ap}_f : (x =_A y) \to (f(x) =_B f(y))$

There are functions $d : \left(f(x) =_B f(y)\right) \to \left(p_*^{\bar{B}_A}(f(x)) =_B f(y)\right)$ st $d \circ h = \text{id}$, $h \circ d = \text{id}$

$$h : \left(p_*^{\bar{B}_A}(f(x)) =_B f(y)\right) \to \left(f(x) =_B f(y)\right)$$

Proof:



Definition of d: $q \mapsto p_{**}(f(x)) \cdot q$

Basic lemma in §2.3

$\circledast$ $r = r' \to r * p = r' * p$ Induct. on p.

$C(y, y, \mathrm{refl}_y)$. $r * \mathrm{refl}_y = r = r' = r' * \mathrm{refl}_y$.

Ex. (used in many proofs): $r, r' : x =_A y$ $p : y =_A z$. (similarly for the dual. $r = r' \to p * r = p * r'$)

$r = r' \to r * p = r' * p$.

Proof: $C : \prod_{y, z : A} \prod_{p : y =_A z} \mathcal{U}$ $C(y, z, p) \equiv (r * p = r' * p)$. Let $e_0 : r = r'$

$C(y, y, \mathrm{refl}_y) \equiv (r * \mathrm{refl}_y = r' * \mathrm{refl}_y)$

$$\begin{array}{ccc} r * \mathrm{refl}_y & = & r' * \mathrm{refl}_y \\ e \parallel & & \parallel e' \\ r & \underset{e_0}{=} & r' \end{array}$$

i.e., $e * e_0 * (e')^{-1} : r * \mathrm{refl}_y = r' * \mathrm{refl}_y$

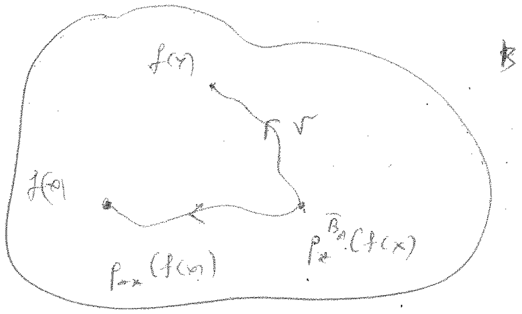
where e, e' are given from Prop. 2.2.4. (i).

By induc. There is $F : \prod_{y, z : A} \prod_{p : y =_A z} C(y, z, p)$ st

$F(y, y, \mathrm{refl}_y) \equiv e * e_0 * (e')^{-1}$

i.e., $F(y, z, p) : r * p = r' * p$.

Definition of h:



$$r \mapsto p_{**}(f(x))^{-1} * r$$

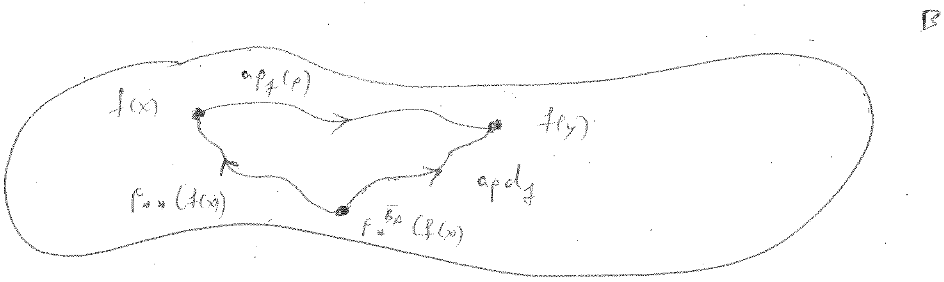
one of the basic lemmas (previous white page)

$$\left.\begin{aligned} d(h(r)) &\equiv d\left(p_{**}(f(x))^{-1} * r\right) \\ &\equiv p_{**}(f(x)) * \left(p_{**}(f(x))^{-1} * r\right) \\ &\equiv \left(p_{**}(f(x)) * p_{**}(f(x))^{-1}\right) * r \\ &\equiv refl_{p_*^{\overline{B}_A}(f(x))} * r \\ &\equiv r \end{aligned}\right\} \quad d(h(r)) = r$$

$$\left.\begin{aligned} h(d(q)) &\equiv h\left(p_{**}(f(x)) * q\right) \\ &\equiv p_{**}(f(x))^{-1} * \left(p_{**}(f(x)) * q\right) \\ &= (\qquad\qquad) * q \\ &= refl_{f(x)} * q \\ &\equiv q \end{aligned}\right\} \quad h(d(q)) = q$$

<u>Proposition 2.3.4:</u> If $f : A \to B$, $p : x =_A y$, then (Relation between apd_f and ap_f)

$$apd_f(p) = p_{**}(f(x)) * ap_f(p)$$



Proof: $C(x,y,p) \equiv \left(\mathrm{apd}_f(p) = p_{**}(f(x)) * \mathrm{ap}_f(p) \right)$

$p_*^{B_A}(f(x)) =_B f(y)$

$$C(x,x,\mathrm{refl}_x) \equiv \left(\mathrm{apd}_f(\mathrm{refl}_x) = \mathrm{refl}_{x*}(f(x)) * \mathrm{ap}_f(\mathrm{refl}_x) \right)$$

$$\equiv \left(\mathrm{refl}_{f(x)} = \underbrace{\mathrm{refl}_{f(x)} * \mathrm{refl}_{f(x)}}_{\equiv} \right)$$

$$\equiv \mathrm{refl}_{f(x)} = \mathrm{refl}_{f(x)}$$

here $\mathrm{refl}_{\mathrm{refl}_{f(x)}} : C(x,x,\mathrm{refl}_x)$

And by Ind$_=$ there is $F : \prod_{x,y:A} \prod_{p:x=_A y} \left(\mathrm{apd}_f(p) = p_{**}(f(x)) * \mathrm{ap}_f(p) \right)$

st. $F(x,x,\mathrm{refl}_x) = \mathrm{refl}_{\mathrm{refl}_{f(x)}}$ □

Section 2.9 : Calculation of Transport

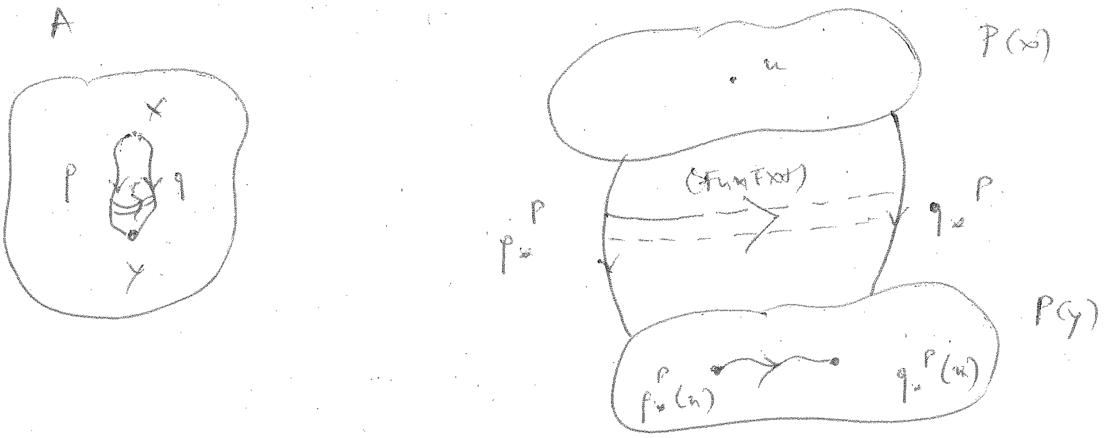
Proposition 2.9.1 : $A : U$, $x, y : A$, $p, q : x =_A y$, $r : p = q$, $P : A \to U$

Then for the mapping $p_*^P : P(x) \to P(y)$

$q_*^P : P(x) \to P(y)$

we have that

$$\prod_{u : P(x)} \left(p_*^P(u) =_{P(y)} q_*^P(u) \right)$$



Proof : Define the type family $Q : (x =_A y) \to U$, for $p : x =_A y$, $s : x =_A y$ parameter

$$Q(s) \equiv \prod_{u : P(x)} \left(p_*^P(u) =_{P(y)} s_*^P(u) \right)$$

Since $r : p = q$ then $r_*^Q : Q(p) \to Q(q)$

i.e. $r_*^Q : \left(\prod_{u : P(x)} \left(p_*^P(u) =_{P(y)} p_*^P(u) \right) \right) \to \left(\prod_{u : P(x)} \left(p_*^P(u) =_{P(y)} q_*^P(u) \right) \right)$

Let $F \equiv \lambda(u : P(x)).\ refl_{p_*^P(u)} \quad : \quad Q(p)$

Then $r_*^Q(F) : Q(q)$, i.e. r_*^Q is the required inhabitant. ∎

Remark 2.9.2.

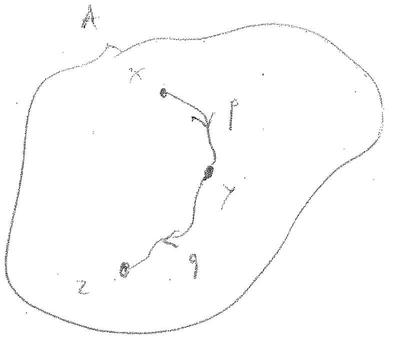
(a) So, when we want to calculate $p_*^P(u)$ and $p = q$, we can freely write

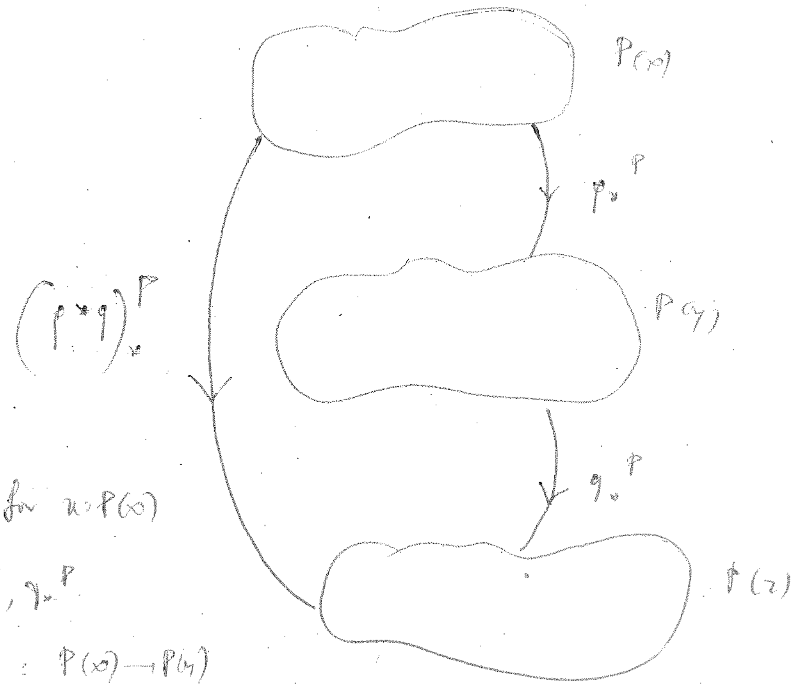
$$p_*^P(u) = q_*^P(u)$$

(b) With Funext we get $p_*^P =_{P(x) \to P(y)} q_*^P$ and the right diagram becomes also commutative.

Section 2.9 Calculation of transports

- Let
$P : A \to \mathcal{U}$
$x, y, z : A$
$p : x =_A y$
$q : y =_A z$





Problem:

We want to calculate $(p * q)_*^P(u)$, for $u : P(x)$

from the given data (p,q) i.e. given p_*^P, q_*^P

The data indicate that $q_* \circ p_* : P(x) \to P(z)$

is of the same type as the object we have. By the Theorem of Nothing, "the data suffice" if there is nothing in between

we guess: $(p*q)_*^P(u) = (q_*^P \circ p_*^P)(u)$. As expected this is the case. We explain that we need to express everything pointwisely only for $u : P(x)$

Proposition 2.9.1: $A : \mathcal{U}$, $x,y,z : A$, $p : x =_A y$, $q : y =_A z$. Then

$$(p * q)_*^P(u) = (q_*^P \circ p_*^P)(u)$$

Proof: Define $C(x,y,p) :\equiv ((p*q)_*^P(u) = (q_*^P \circ p_*^P)(u))$

$$C(x,x,\mathrm{refl}_x) \equiv ((\mathrm{refl}_x * q)_*^P(u) = q_*^P(\mathrm{refl}_x)_*^P(u))$$

$\mathrm{refl}_x * q \equiv q$

$$\equiv (q_*^P(u) = (q_*^P \circ id_{P(x)})(u))$$

$$\equiv q_*^P(u) = q_*^P(u)$$

!! We'd like to have the lemma, if we wanted to show $(p*q)_*^P = q_*^P \circ p_*^P$

Lemma 2.9.2: $f : A \to B$, $id_A : A \to A$, $id_A :\equiv \lambda(x:A).x$. Then

$$f \circ id_A =_{A \to B} f$$

Proof:

$$f \circ id_A \equiv \lambda(x:A).\ f(id_A(x)) \qquad (id_A(x) \equiv x)$$

$$\equiv \lambda(x:A).\ f(x)$$

$$\equiv f \qquad \text{by the def of } A \to B$$

• We'd like to have $a \equiv_A b$, then $a =_A b$

Since $\mathrm{refl}_a : (a =_A a)$ then $\mathrm{refl}_a : (a = b)$ ⟵

It seems that refl_x pertains to $x : A$ both in canonical form. We want to avoid!

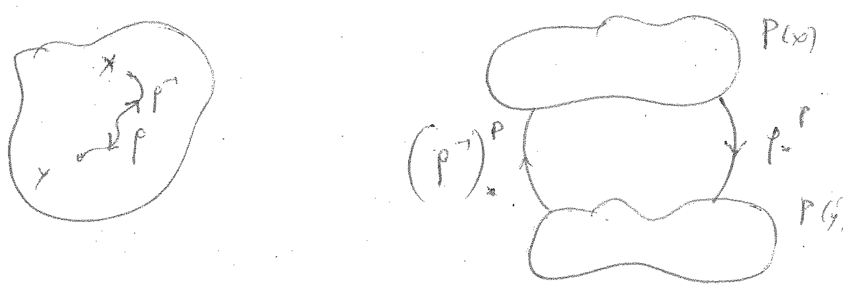
So, we can't say that we have $\boxed{\text{proof of Lemma 2.4.2}}$

So, take $\mathrm{refl}_{q_*^P(u)} : C(x, x, \mathrm{refl}_x)$

By induction there is $F : \prod_{x,y:A} \prod_{p:x=_A y} C(x,y,p)$ s.t. $F(x, x, \mathrm{refl}_x) \equiv \mathrm{refl}_{q_*^P(u)}$

i.e., $F(x,y,p)$ is the required inhabitant. □

<u>Corollary 2.4.3:</u>



$p_*^P : P(x) \to P(y)$ $\quad (p^{-1})_*^P : P(y) \to P(x)$ $\quad u : P(x)$, $w : P(y)$

By Prop. 2.4.2 $\quad (p * p^{-1})_*^P(u) = (p^{-1})_*^P\left(p_*^P(u)\right)$

$p * p^{-1} = \mathrm{refl}_x$ $\quad$ Prop. 2.4.1 $\parallel$

$$(\mathrm{refl}_x)_*^P(u) \equiv \mathrm{id}_{P(x)}(u) \equiv u \qquad (1)$$

Similarly $\quad (p^{-1} * p)_*^P(w) = p_*^P\left((p^{-1})_*^P(w)\right)$

$p^{-1} * p = \mathrm{refl}_y$ $\quad$ Prop. 2.4.1 $\parallel$

$$(\mathrm{refl}_y)_*^P(w) = \mathrm{id}_{P(y)}(w) \equiv w \qquad (2)$$

i.e., $\prod_{u:P(x)} \left((p * p^{-1})_*^P(u) = u\right)$ and $\prod_{u:P(x)} \left((p^{-1})_*^P \circ p_*^P\right)(u) = u$

and $\prod_{w:P(y)} \left(p_*^P \circ (p^{-1})_*^P\right)(w) = w$

i.e., $\left((p^{-1})_*^P, s\right) : \underbrace{\sum_{g:P(y)\to P(x)} (f \circ g \sim \mathrm{id}_{P(y)})}_{}$ with $\mathrm{rinv}(f) \;|||$ above, and $\left((p^{-1})_*^P, t\right) : \sum_{g:P(y)\to P(x)} g \circ f \sim \mathrm{id}_{P(x)}$ with $\mathrm{linv}(f) \;|||$ above

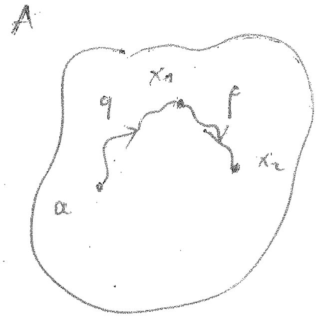
- How to calculate more concrete transports: If $A : \mathcal{U}$, $a : A$, let $x_1, x_2 : A$, $p : x_1 =_A x_2$.

Let $P : A \to \mathcal{U}$

$$P(x) \equiv (a =_A x)$$

$$p_*^P : P(x_1) \to P(x_2) \equiv (a =_A x_1) \to (a =_A x_2)$$

Let $q : a =_A x_1$: we want to determine $p_*^P(q) : (a =_A x_2)$



By the data given $q * p : (a =_A x_2)$ (the only candidate)

Hence by Nat. Thesis we expect

$$p_*^P(q) = q * p$$

Proposition 2.4.4: Let p, q as above. Then $p_*^P(q) = q * p$ or

$$p_*^{x \mapsto a =_A x}(q) = q * p$$

Proof: $C(x_1, x_2, p) \equiv \left(p_*^P(q) = q * p \right)$

$$C(x_1, x_1, \mathrm{refl}_{x_1}) \equiv \left((\mathrm{refl}_{x_1})_*^P (q) = q * \mathrm{refl}_{x_1} \right)$$

$$\equiv \mathrm{id}_{P(x_1)}(q) = q * \mathrm{refl}_{x_1}$$

$$\equiv q = q * \mathrm{refl}_{x_1}$$

By Prop. 2.2.5 (i) $C(x, x, \mathrm{refl}_x)$ is inhabited (be careful, you can't write $\Xi q = q$) and by $\mathrm{Ind}_=$ you get an inhabitant. □

- Exercise: Let all data as above. $A : \mathcal{U}$, $a : A$, $x_1, x_2 : A$, $p : x_1 =_A x_2$

(i) Predict $p_*^{x \mapsto x =_A a}(q)$, $q : x_1 =_A a$

and $p_*^{x \mapsto x =_A x}(q)$, $q : x_1 =_A x_1$

(ii) Prove your predictions.

Proposition 2.9.5 $A : \mathcal{U}$

$P : A \to \mathcal{U}$, $Q : A \to \mathcal{U}$

$P \to Q : A \to \mathcal{U}$ is defined by $(P \to Q)(x) \equiv P(x) \to Q(x)$

$x, y : A$, $p : x =_A y$

$p_*^P : P(x) \to P(y)$ $\qquad$ $p_*^Q : Q(x) \to Q(y)$

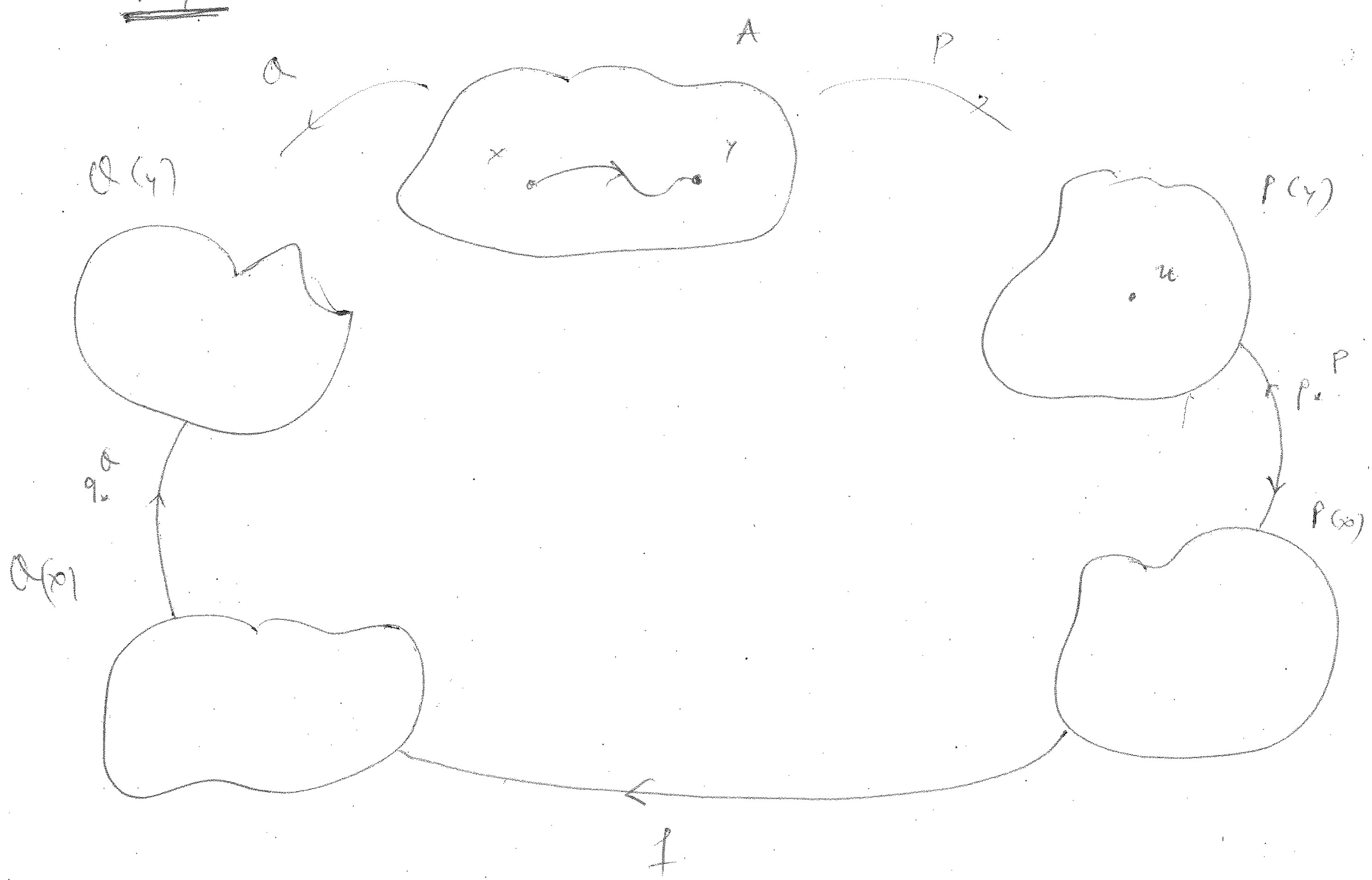
$p_*^{P \to Q} : \big(P(x) \to Q(x)\big) \to \big(P(y) \to Q(y)\big)$

Let $f : P(x) \to Q(x)$

We want to calculate $p_*^{P \to Q}(f) : P(y) \to Q(y)$

i.e., given $u : P(y)$ we want to calculate $\big(p_*^{P \to Q}(f)\big)(u) : Q(y)$

__Analysis__ of the data in order to find such a possible inhabitant of $Q(y)$ from them:



The data form the following candidates $u \mapsto (p_*^P)^{-1}(u) = (p^{-1})_*^P(u)$ Coral 2.4.3

$\mapsto f\big((p^{-1})_*^P(u)\big)$

$\mapsto p_*^Q\big(f((p^{-1})_*^P(u))\big)$.

the verification

$$\left(p_*^{P \to Q}(f)\right)(u) = p_*^{Q}\left(f\left((p^{-1})_*^{P}(u)\right)\right) \qquad (\mathrm{refl}_x)^{-1} \equiv \mathrm{refl}_x$$

<u>Proof.</u>

$$C(x, y, p) \equiv \text{—''—}$$

$$C(x, x, \mathrm{refl}_x) \equiv \left(\left(\mathrm{refl}_x\right)_*^{P \to Q}(f)\right)(u) = \left(\mathrm{refl}_x\right)_*^{Q}\left(f\left(\left(\mathrm{refl}_x\right)^{-1}\right)_*^{P}(u)\right)$$

$$\equiv \left(\mathrm{id}_{(P \to Q)(x)}(f)\right)(u) = \mathrm{id}_{Q(x)}\left(f\left(\mathrm{refl}_x\right)_*^{P}(u)\right)$$

$$\equiv f(u) = f\left(\mathrm{id}_{P(x)}(u)\right)$$

$$\equiv \left(f(u) = f(u)\right)$$

Take $\mathrm{refl}_{f(u)} : C(x, x, \mathrm{refl}_x)$ and apply $\mathrm{Ind}_=$.

- <u>Exercises (continued)</u> 1, 3, 4.

1. $A, B : \mathcal{U} \quad p : a =_A a' \quad f, g : A \to B \quad q : f(a) =_B g(a)$

$Q : A \to \mathcal{U}$ defined by $Q(x) \equiv f(x) =_B g(x)$

Hence $p_*^{Q} : \left(f(a) =_B g(a)\right) \longrightarrow \left(f(a') =_B g(a')\right)$

Predict and prove the formula for $p_*^{Q}(q) : f(a') =_B g(a')$

2. $A, B : \mathcal{U} \quad , p : x =_A y \quad , P : B \to \mathcal{U} \quad , f : A \to B$

It will be needed for univalence, so I do it.

$P \circ f : A \longrightarrow \mathcal{U}$ is defined by $(P \circ f)(x) \equiv P(f(x))$

Hence $p_*^{P \circ f} : P(f(x)) \longrightarrow P(f(y))$

Let $u : P(f(x))$.

Predict and prove the formula for $p_*^{P \circ f}(u)$

3. $A : \mathcal{U} \quad P, Q : A \to \mathcal{U} \quad p : x =_A y$

$F : \prod_{x : A}\left(P(x) \to Q(x)\right) \qquad p_*^{Q} : Q(x) \to Q(y)$

If $u : P(x)$ then predict and calculate $p_*^{Q}\left(F(x, u)\right)$

Proposition 2.4.6 : $A, B : \mathcal{U}$ (will use it in the UA-section).

$f : A \to B$

$P : B \to \mathcal{U}$

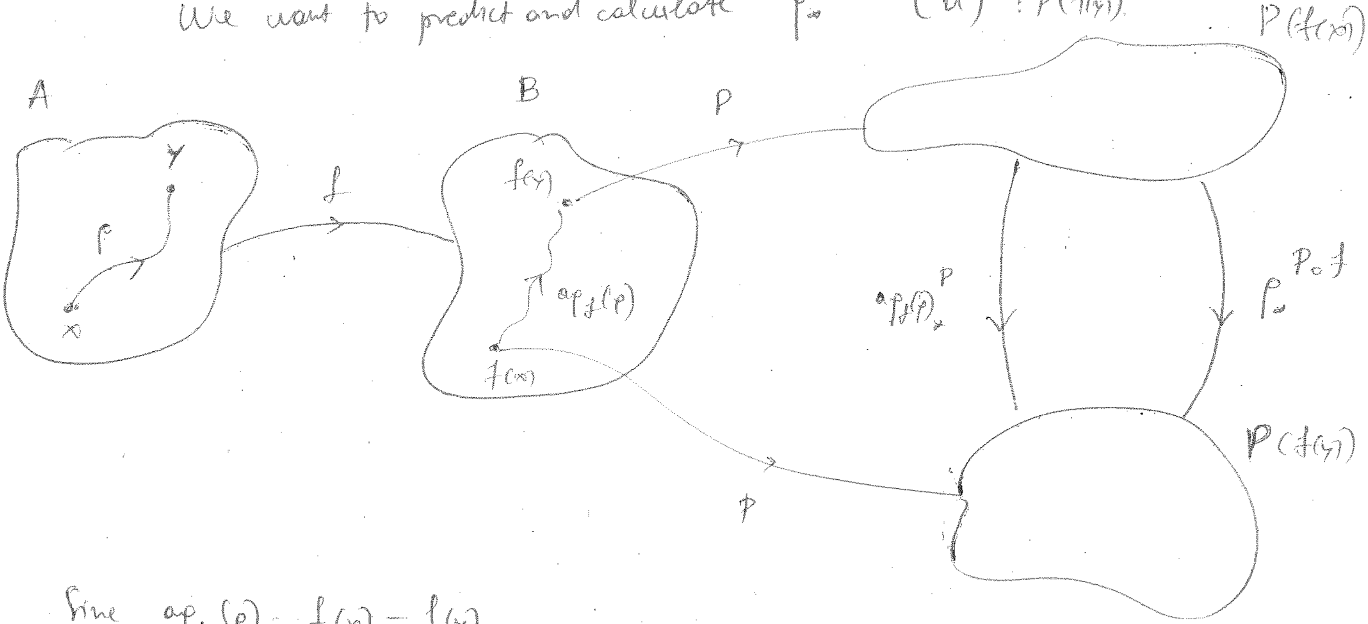
$P \circ f : A \to \mathcal{U}$ is defined by $(P \circ f)(x) \equiv P(f(x))$, for every $x : A$

$x, y : A \quad p : x =_A y$

$p_*^{P \circ f} : P(f(x)) \longrightarrow P(f(y))$

Let $u : P(f(x))$

We want to predict and calculate $p_*^{P \circ f}(u) : P(f(y))$



Since $ap_f(p) : f(x) =_B f(y)$,

$ap_f(p)_*^P : P(f(x)) \longrightarrow P(f(y))$

i.e. $(ap_f(p))_*^P(u) : P(f(y))$

i.e. $\boxed{p_*^{P \circ f}(u) =_{P(f(y))} (ap_f(p))_*^P(u)}$

Proof : $C(x, y, p) \equiv \left(p_*^{P \circ f}(u) = ap_f(p)_*^P(u) \right)$

refl_u : $C(x, x, \mathrm{refl}_x) \equiv \left((\mathrm{refl}_x)_*^{P \circ f}(u) = (ap_f(\mathrm{refl}_x))_*^P(u) \right)$

$\equiv \left(id_{P(f(x))}(u) = (\mathrm{refl}_{f(x)})_*^P(u) \right)$

$\equiv u =_{P(f(x))} id_{P(f(x))}(u)$

$\equiv u = u$

and we are done. $\square$

2.11.2 : $A : U$

$a : A \quad , x_1 x_2 : A$

$p : x_1 =_A x_2$

of x pitman ms 8.1.6

(i) $p_*^{x \mapsto (a=x)}(q) = q \cdot p \quad , \text{ if } q : a =_A x_1$

(ii) $p_*^{x \mapsto (x=a)}(q) = p^{-1} \cdot q \quad , \text{ if } q : x_1 =_A a$

(iii) $p_*^{x \mapsto (x=x)}(q) = p^{-1} \cdot q \cdot p \quad , \text{ if } q : x_1 =_A x_1$

Proof $\quad P : A \to U \qquad P(x) = (a =_A x)$

$p_*^P \quad P(x_1) \to P(x_2) \quad \text{i.e.} \quad p_*^P : \underset{q}{(a = x_1)} \to (a = x_2)$

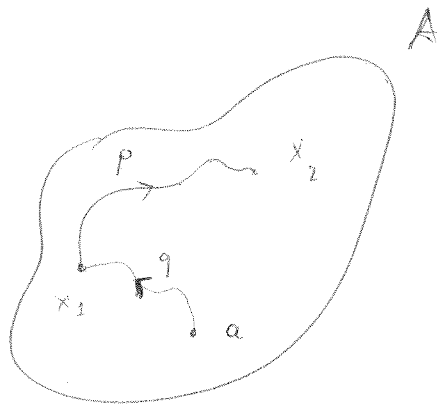
$C(x_1, x_2, p) = \left(p_*^P(q) = q \cdot p \right)$

$C(x_1, x_1, \mathrm{refl}_{x_1}) = \left((\mathrm{refl}_{x_1})_*^P (q) = q \cdot \mathrm{refl}_{x_1} \right)$

$= \left(\mathrm{id}_{P(x_1)}(q) = q \cdot \mathrm{refl}_{x_1} \right)$

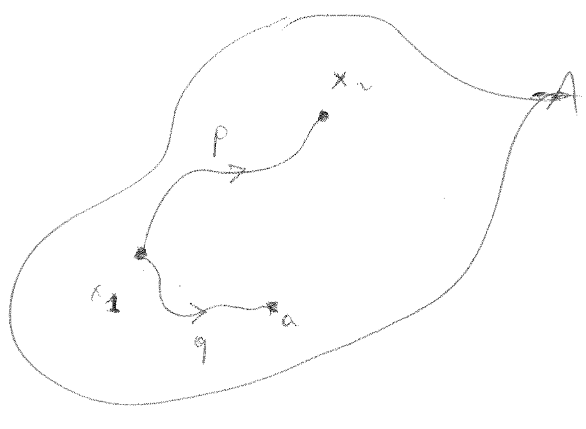
$= \quad q_1 = q_1$

$c(x) := \mathrm{refl}_{q_1} \qquad \square$



Why (i) is expected : Clearly $q \cdot p : (a =_A x_2)$

-//- (ii) -//-



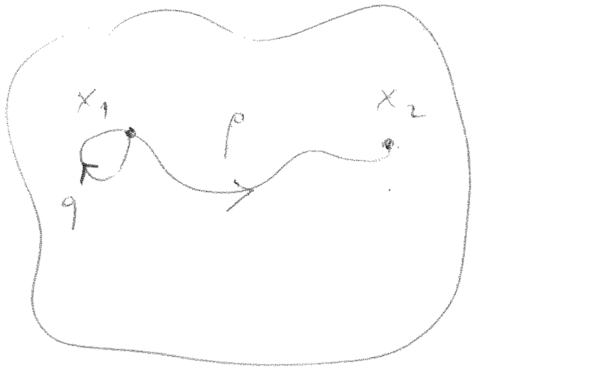
$p^{-1} \cdot q : x_2 =_A a$

How we can guess the required equality from the behavior of $refl_x$

$P : A \longrightarrow U$ $\qquad$ $p_*^P : P(x_1) \to P(x_2)$ i.e. $p_*^P : (x_1 =_A x_1) \longrightarrow (x_2 =_A x_2)$

$P(x) = (x =_A x)$ $\qquad$ $p : x_1 =_A x_2$

$C(x_1, x_2, p) =$



$refl_{x_1} : P(x_1) \to P(x_1)$

From the nature of P

better

: you want to create a loop at x_2

hence $p^{-1} \cdot q \cdot p$

i.e. $\boxed{\text{Given } p \text{ and } q : P(x_1) \text{ create } r : P(x_2) \text{ in the most natural way.}}$

$p_*^P : P(x_1) \to P(x_2)$

"q is transported along p"

i.e. in many cases we can guess the behavior of

$$\boxed{p_*^P(q)}$$

Is the SAME for ALL CASES !!!

Th 2.11.5 : $p : x =_A y$, $q : x =_A x$, $r : y =_A y$

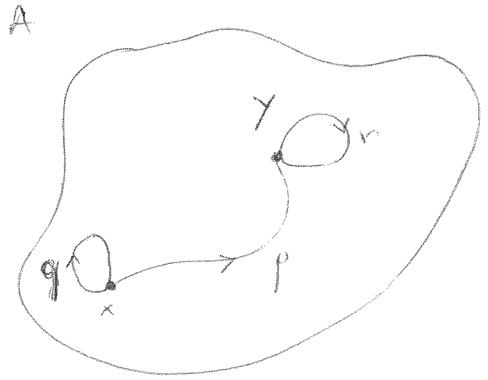
$$\left(p_*^{x\mapsto x=_A x}(q) = r\right)$$

$$p_*^{x\mapsto (x=x)} : (x=x) \to (y=y)$$

$$p_*^{x\mapsto x=x}(q) : (y=y)$$

By 2.11.2 $\quad p_*^{x\mapsto (x=x)}(q) = p^{-1}\cdot q\cdot p$

A



$$\left(p_*^{x\mapsto x=_A x}(q) = r\right) \simeq \left(p^{-1}\cdot q\cdot p = r\right)$$
$$\simeq \left(q\cdot p = p\cdot r\right)$$

Here, it's "natural" to find the $\simeq$.

Proof.
$$C(x,y,p) \equiv \prod_{q:x=x}\prod_{r:y=y}\left(p_*^{x\mapsto x=x}(q)=r\right) \simeq \left(q\cdot p = p\cdot r\right)$$

$$C(x,x,\mathrm{refl}_x) \equiv \prod_{q,r:x=x}\left(\mathrm{refl}_*^{x\mapsto x=x}(q)=r\right) \simeq \left(q\cdot \mathrm{refl}_x = \mathrm{refl}_x\cdot r\right)$$

$$\equiv \prod_{q,r:x=_A x}\left(\underbrace{\mathrm{refl}_*^{x\mapsto x=x}(q)=r}_{\mathrm{refl}_x^{-1}\cdot q\cdot \mathrm{refl}_x = r}\right) \simeq (q=r)$$

$$\equiv \prod_{q,r:x=x}(q=r) \simeq (q=r)$$

$$c(x) \equiv \lambda(q,r:x=x).\overset{\text{is equiv}}{\left(\mathrm{id}_{q=r}\right)}$$

and then path-induction.

Ch. 2.11.3. $f, g : A \to B$

$p : a =_A a'$

$q : f(a) =_B g(a)$

$Q \quad A \to U$

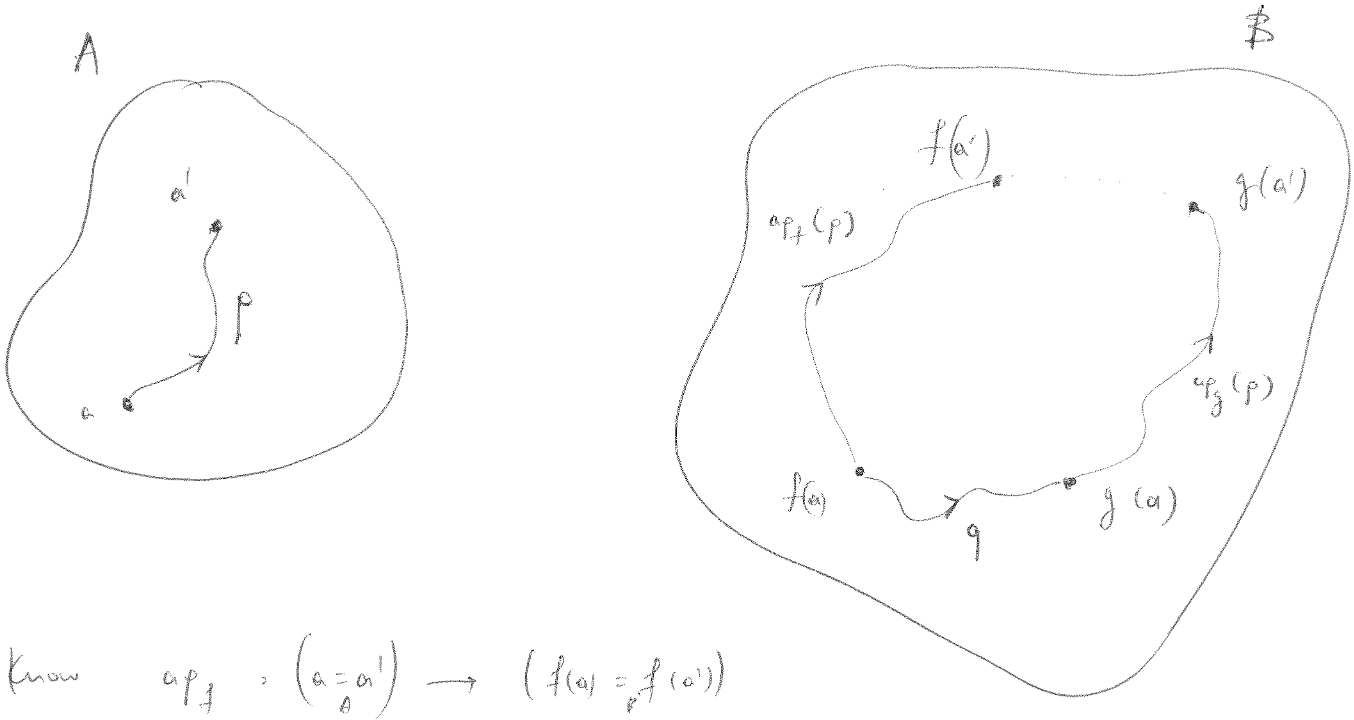
$Q(x) \equiv (f(x) =_B g(x))$

$p_*^Q : Q(a) \to Q(a')$

$p_*^Q : (f(a) =_B g(a)) \to (f(a') =_B g(a'))$

Generalization of $x \mapsto x =_A x$, $f = g = id_A$

We want to calculate $p_*^Q(q) : (f(a') =_B g(a'))$



We know $ap_f : (a =_A a') \to (f(a) =_B f(a'))$

$ap_g : (a =_A a') \to (g(a) =_B g(a'))$

With these data we want to define a path in $f(a') =_B g(a')$

Of course, by NATURALITY THESIS

$$\boxed{p_*^Q(q) = [ap_f(p)]^{-1} * q * ap_g(p)}$$

Proof. $C(x, y, p) \equiv \prod_{q : (f(x) = g(x))} \left(p_*^Q(q) = ap_f(p)^{-1} * q * ap_g(p)\right)$

$C(x, x, refl_x) \equiv \prod_{q : (f(x) = g(x))} \left((refl_x)_*^Q(q) = \underbrace{ap_f(refl_x)^{-1} * q * ap_g(refl_x)}_{refl_{f(x)}^{-1} * q * refl_{g(x)} = refl_{f(x)} * q * refl_{g(x)} = q}\right)$

$(refl_x)_*^Q(q) \equiv id_{Q(x)}(q) \equiv q$

Hence $c(x) \equiv refl_q$

L. 2.3.11

$P : A \to U$

$Q : A \to U$

$$f : \prod_{x:A} \big(P(x) \to Q(x) \big)$$

$p : x =_A y$

$u : P(x)$

$p_*^P : P(x) \to P(y)$

$p_*^Q : Q(x) \to Q(y)$

$p_*^Q(f(x,u)) : Q(y)$

$\uparrow$

$f(x,u) : Q(x)$, $\quad f(y, p_*^P(u)) : Q(y)$

$$\boxed{p_*^Q\big(f(x,u)\big) = f\big(y, p_*^P(u)\big)}$$

Proof: $\quad C_{f,x,u}(x,y,p) \equiv p_*^Q\big(f(x,u)\big) = f\big(y, p_*^P(u)\big)$

$$C(x,x,\mathrm{refl}_x) \equiv (\mathrm{refl}_x)_*^Q\big(f(x,u)\big) = f\big(x, (\mathrm{refl}_x)_*^P(u)\big)$$

$$\equiv \mathrm{id}_{Q(x)}\big(f(x,u)\big) = f\big(x, \mathrm{id}_{P(x)}(u)\big)$$

$$\equiv f(x,u) = f(x,u)$$

$$c(x) \equiv \mathrm{refl}_{f(x,u)}$$

$\square$

See next page for the intuitive explanation of the definition.

Il sottoscritto, infine, si impegna a fornire qualsiasi informazione inerente l'insegnamento (disponibilità orarie per le lezioni, orari di ricevimento, riferimenti personali o altro) ed a produrre qualsiasi documentazione richiesta nei tempi e nei modi segnalati successivamente dalla Direzione Didattica e Servizi agli Studenti - U.O. Didattica Scienze e Ingegneria.

Allega alla presente fotocopia di un documento di riconoscimento in corso di validità.

Il sottoscritto

DICHIARA

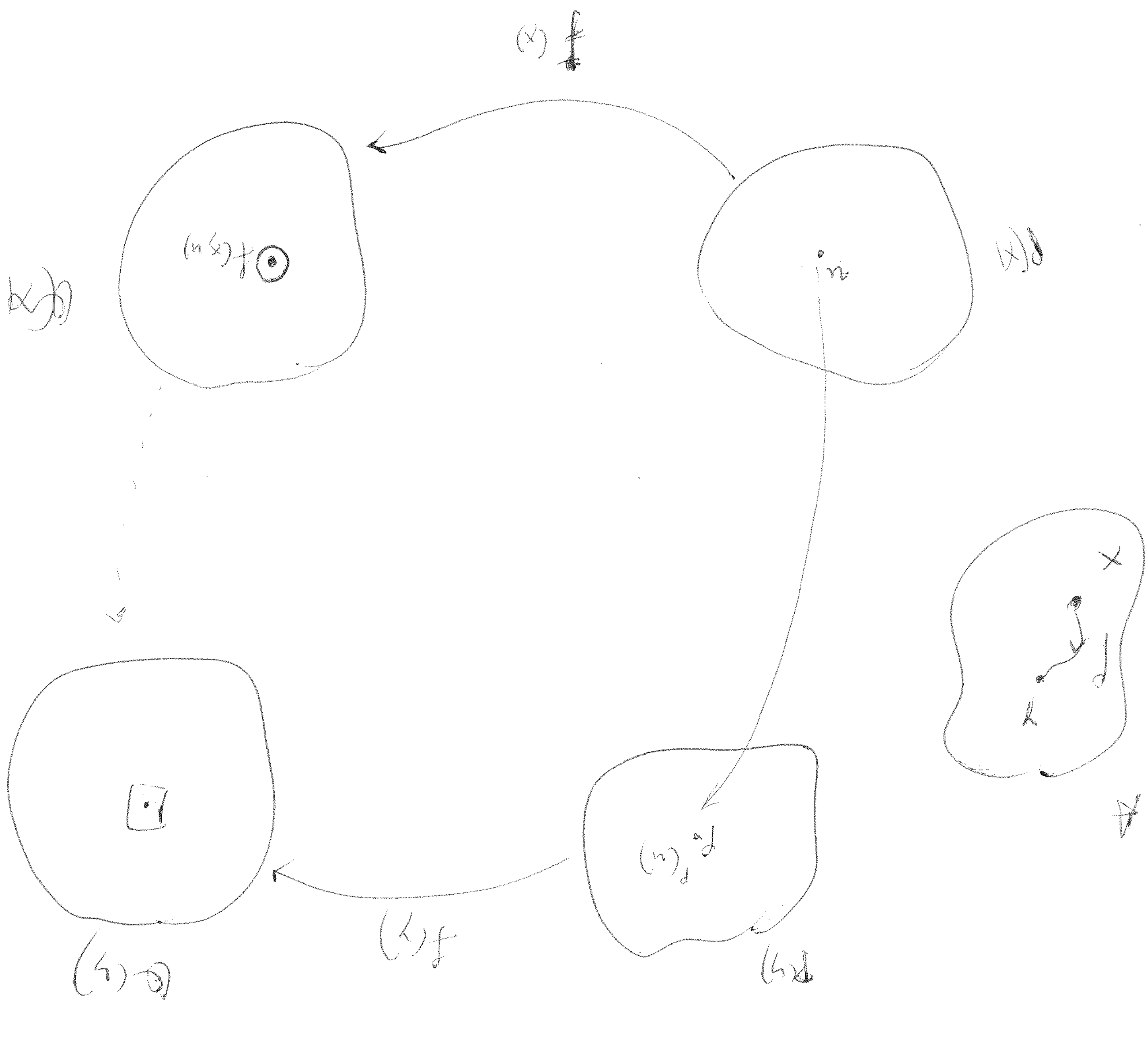
altresì di essere informato che i dati personali raccolti saranno trattati anche con strumenti informatici esclusivamente nell'ambito del procedimento per il quale la presente richiesta viene presentata e che il trattamento è disciplinato dal decreto Legislativo 30.06.2003, n 196 "Codice *in materia di protezione dei dati personali*", pubblicato nella Gazzetta Ufficiale del 29 luglio 2003, Serie generale n. 174, Supplemento ordinario n. 123/L.

In fede

Verona,____________________

— we want to define $\rho_0^A(\cdot\, f(x,u))$ ad map to ... —

We do it as if it was a commutative diagram indeed. i.e. we go from u to $P(y)$ through $\rho_0^P(u)$ ... and then we use $f(y)$.



Exercise (4).

$P : X \to \mathcal{U}$ $\qquad$ $p_*^P : P(x_1) \to P(x_2)$

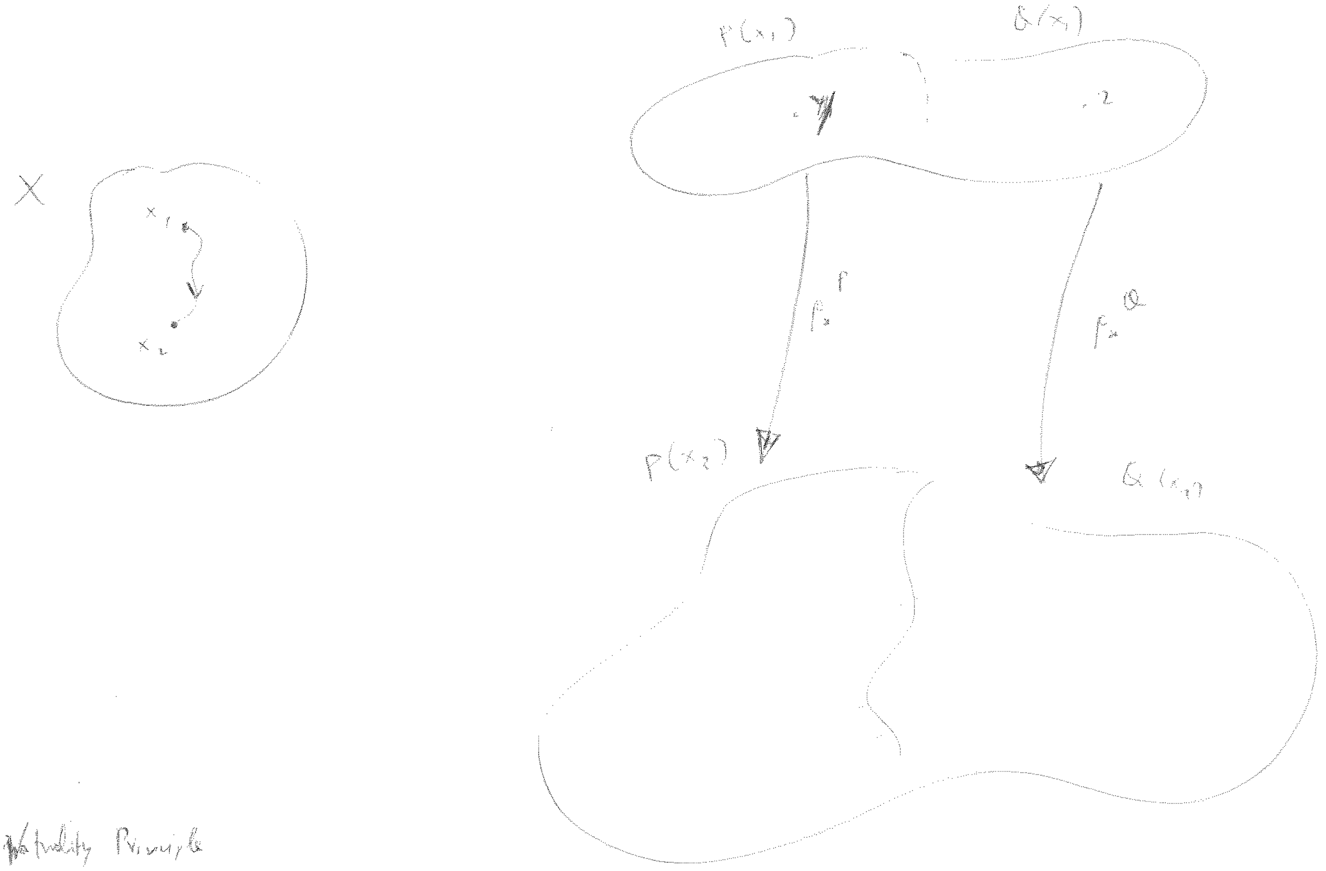
$Q : X \to \mathcal{U}$ $\qquad$ $p_*^Q : Q(x_1) \to Q(x_2)$

entry §2.12

$P + Q : X \to \mathcal{U}$

$(P+Q)(x) = P(x) + Q(x)$

$p : x_1 =_X x_2 \qquad p_*^{P+Q} : (P(x_1) + Q(x_1)) \to (P(x_2) + Q(x_2))$



By Naturality Principle

$$\boxed{\begin{aligned} p_*^{P+Q}(\mathrm{inl}(y)) &= \mathrm{inl}(p_*^P(y)), \quad y : P(x_1) \\ p_*^{P+Q}(\mathrm{inr}(z)) &= \mathrm{inr}(p_*^Q(z)), \quad z : Q(x_1) \end{aligned}}$$

Proof

$C : \prod_{x_1, x_2 : X} \prod_{p : x_1 = x_2} \mathcal{U} \qquad C(x_1, x_2, p) :\equiv p_*^{P+Q}(\mathrm{inl}(y)) = \mathrm{inl}(p_*^P(y)) \qquad y : P(x)$

$C(x, x, \mathrm{refl}_x) \equiv (\mathrm{refl}_x)_*^{P+Q}(\mathrm{inl}(y)) = \mathrm{inl}((\mathrm{refl}_x)_*^P(y))$

$(\mathrm{refl}_x)_*^{P+Q} = \mathrm{id}_{(P+Q)(x)} = \mathrm{id}_{P(x)+Q(x)}$ $\qquad \equiv \mathrm{id}_{P(x)+Q(x)}(\mathrm{inl}(y)) = \mathrm{inl}(\mathrm{id}_{P(x)}(y))$

$(\mathrm{refl}_x)_*^P = \mathrm{id}_{P(x)}$ $\qquad \equiv \mathrm{inl}(y) = \mathrm{inl}(y)$

$c(x) :\equiv \mathrm{refl}_{\mathrm{inl}(y)}$, and path ind. $\square$

Section 2.5 : Homotopies

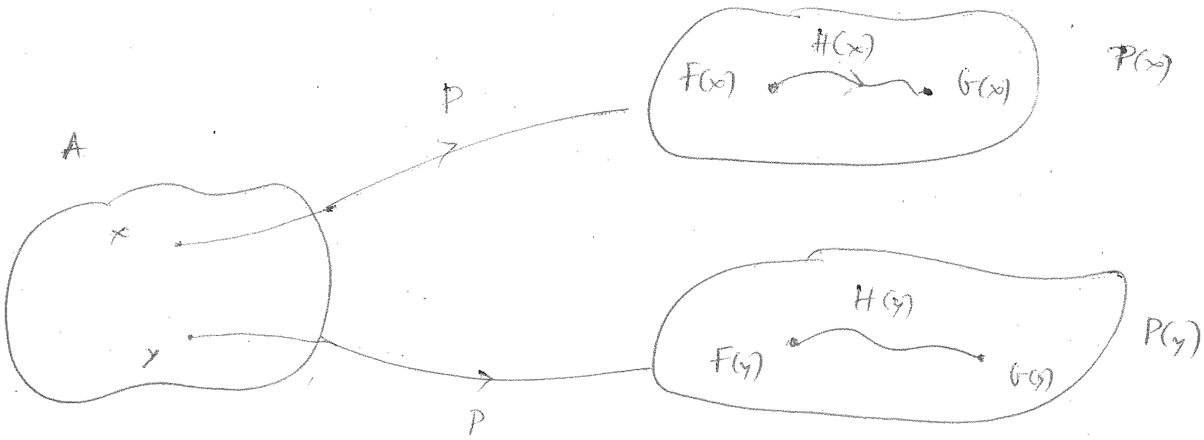
Definition 2.5.1 $A : \mathcal{U}$

$P : A \to \mathcal{U}$

$F, G : \prod_{x:A} P(x)$

We define $F \sim G :\equiv \prod_{x:A} \left(F(x) =_{P(x)} G(x)\right)$

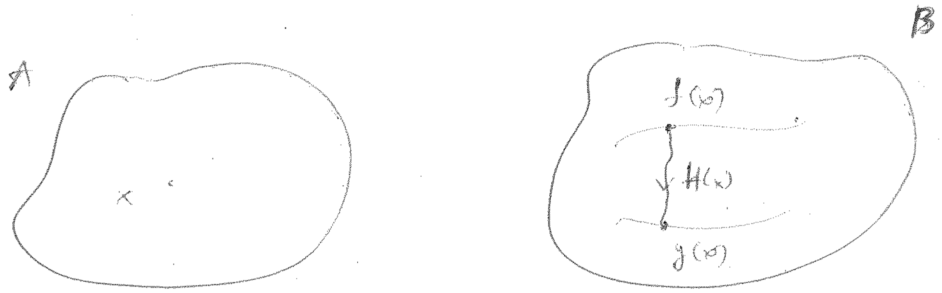
If $H : F \sim G$, is called a homotopy from F to G.



As a special case if $f, g : A \to B$,

$$f \sim g \equiv \prod_{x:A} \left(f(x) =_B f(y)\right)$$

and $H : f \sim g$ is a homotopy from f to g.



This is the type theoretic analogue of $H : X \times [0,1] \xrightarrow{\text{cont.}} Y$ s.t. $H(x,0) = f(x)$, $H(x,1) = g(x)$.

Proposition 2.5.2 : TFAE : if $A, B : \mathcal{U}$,

(i) $\prod_{f:A\to A} (f \sim f)$

(ii) $\prod_{f,g:A\to B} (f \sim g) \to (g \sim f)$

(iii) $\prod_{f,g,h:A\to B} (f\sim g)\to(g\sim h)\to(f\sim h)$.

<u>Proof</u>: (i) $f\sim f \equiv \prod_{x:A}\left(f(x)=_B f(x)\right)$

Define $H_f \equiv \lambda(x:A).\ \mathsf{refl}_{f(x)}$

Define $H \equiv \lambda(f:A\to A).\ h_f \quad : \prod_{f:A\to A}(f\sim f)$.

(ii) If $H: f\sim g$ then $H(x): f(x)=_B g(x)$, $x:A$.

Then $H(x)^{-1}: g(x)=_B f(x)$, $x:A$

We define $H^{-1}\equiv \lambda(x:A).\ H(x)^{-1}$,

hence $H^{-1}: g\sim f$, and the required inhabitant is

$\lambda(f,g:A\to B).\ {}^{-1}$

where ${}^{-1}:(f\sim g)\to(g\sim f)$ is defined by $H\mapsto H^{-1}$, $H:f\sim g$.

(iii) Let $F(x): f(x)=_B g(x)$ $\quad G(x): g(x)=_B h(x)$.

We define $H(x)\equiv F(x)\ast G(x)$, $x:A$. Clearly, $H: f\sim h$.

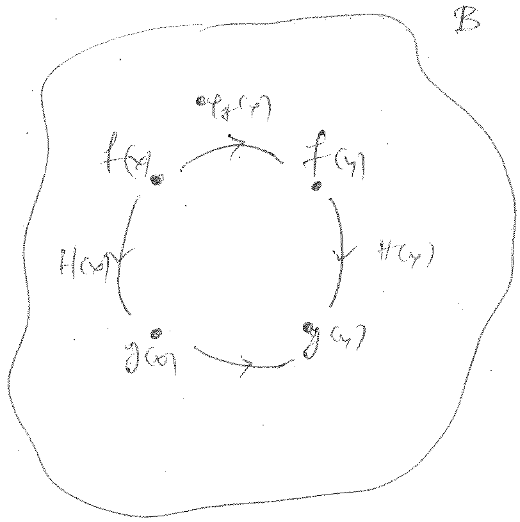
The required inhabitant is $\lambda(f,g,h:A\to B).\ \ast$,

where $\ast:(f\sim g)\to(g\sim h)\to(f\sim h)$ $\quad \ast(F,G)\equiv H \equiv F\ast G$. □

<u>Proposition 2.5.3</u> (Naturality of homotopies). $A,B:\mathcal{U}$, $f,g:A\to B$, $H:f\sim g$, $p: x=_A y$. Then

$$H(x)\ast ap_g(p) = ap_f(p)\ast H(y).$$

$$\begin{tikzcd}
f(x) \arrow[r, equal, "ap_f(p)"] \arrow[d, equal, "H(x)"'] & f(y) \arrow[d, equal, "H(y)"] \\
g(x) \arrow[r, equal, "ap_g(p)"'] & g(y)
\end{tikzcd}$$



Proof: $C((x,y,p)) \equiv H(x) * ap_g(p) = ap_f(p) * H(y)$

$$C(x,x,refl_x) \equiv H(x) * ap_g(refl_x) = ap_f(refl_x) * H(x)$$

$$\equiv H(x) * refl_{g(x)} = refl_{f(x)} * H(x)$$

$$\equiv H(x) * refl_{g(x)} = H(x) \qquad \text{Prop 2.2 (since } refl_x * g \equiv g)$$

and by Prop. 2.2.4.(i) $C(x,x,refl_x)$ is inhabited. Then we are done. □

Remark 2.5.4: If A, B are seen as categories then f, g are seen as functors (preservation of paths) and Proposition 2.5.3 expresses corresponding to homotopies being natural transformations.

Corollary 2.5.5: Let $H : f \sim id_A$, for some $A : \mathcal{U}$ and $f : A \to A$. If $x : A$, then

$$H(f(x)) = ap_f(H(x))$$

Proof: Consider Prop 2.5.3 for $(x,y,p) \equiv (f(x), x, H(x))$ and $(f,g) \equiv (f, id_A)$. Then the diagram becomes

$$\begin{tikzcd} ff(x) \arrow[r, equal, "ap_f(H(x))"] \arrow[d, equal, "H(f(x))"'] & f(x) \arrow[d, equal, "H(x)"] \\ f(x) \arrow[r, equal, "ap_{id_A}(H(x))"'] & x \end{tikzcd}$$

$$ap_{id_A}(H(x)) = H(x)$$

By basic lemma in §2.3 $H(f(x)) * ap_{id_A}(H(x)) = H(f(x)) * H(x)$

$$\| \text{ Prop. 2.5.3}$$

$$ap_f(H(x)) * H(x)$$

Hence $\left(ap_f(H(x)) * H(x)\right) * (H(x))^{-1} = \left(H(f(x)) * H(x)\right) * (H(x))^{-1} \Rightarrow$

$$ap_f(H(x)) * (H(x) * H(x)^{-1}) = H(f(x)) * (H(x) * H(x)^{-1}) \Rightarrow$$

$$ap_f(H(x)) * refl_{f(x)} = H(f(x)) * refl_{f(x)} \Rightarrow$$

$$ap_f(H(x)) = H(f(x)).$$ □

Proposition 2.5.2 $A, B, C : \mathcal{U}$,

$f, g : A \to B$,

$h : B \to C$ and $e : C \to A$

(EXERCISE)

(i) $f \sim g \Rightarrow h \circ f \sim h \circ g$ $\left(\text{or} \prod_{f,g : A \to B} \prod_{h : B \to C} (f \sim g) \to (h \circ f \sim h \circ g) \right)$

(ii) $f \sim g \to f \circ e \sim g \circ e$ $\left(\prod_{e : C \to A} \right)$

Proof: (i) $A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{h} C$ Let $H : f \sim g$ i.e., $H(x) : f(x) =_B g(x)$, $x : A$.

$ap_{h,x} : \left(f(x) =_B g(x) \right) \longrightarrow \left(h(f(x)) =_C h(g(x)) \right)$

We define $h \circ H \equiv \lambda(x : A).\ ap_{h,x}(H(x))$ $: \prod_{x:A} \left((h \circ f)(x) =_C (h \circ g)(x) \right)$

(ii) $C \xrightarrow{e} A \overset{f}{\underset{g}{\rightrightarrows}} B$

Define $H \circ e \equiv \lambda(z : C).\ H(e(z))$ $: \prod_{z:C} \left(f(e(z)) =_B g(e(z)) \right)$. $\square$

Section 2.6 Equivalences

Definition 2.6.1: If $A, B : \mathcal{U}$ and $f : A \to B$ we define the type:

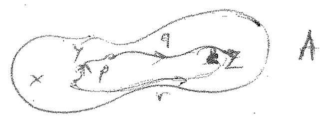
$$\mathrm{qinv}(f) \equiv \sum_{g : B \to A} \Big(\big((f \circ g) \sim id_B \big) \times \big((g \circ f) \sim id_A \big) \Big)$$

and we say that $pr_1(\mathrm{qinv}(f))$ is a <u>quasi-inverse</u> of f. Better a quasi-inverse of f is an inhabitant of $\mathrm{qinv}(f)$ i.e. a triple (g, H, G) $g : B \to A$, $H : (f \circ g) \sim id_B$, and $G : (g \circ f) \sim id_A$.

- **Examples 2.6.2** (i) $id_A : A \to A$, then $(id_A, H, G) : \mathrm{qinv}(id_A)$

  where $H : id_A \circ id_A \sim id_A \equiv \prod_{x : A} (x =_A x)$ $(H \equiv \mathrm{refl}_A) \equiv G)$

  (ii) $p : x =_A y$, $z : A$



  Define $f : (y =_A z) \to (x =_A z)$ $f(q) = p * q$

  We have $(g, H, G) : \mathrm{qinv}(f)$, where $g : (x =_A z) \to (y =_A z)$

  $g(r) = p^{-1} * r$

  $f(g(r)) = p * g(r) = p * (p^{-1} * r) = (p * p^{-1}) * r = r$

  $g(f(q)) = p^{-1} * f(q) = p^{-1} * (p * q) = (p^{-1} * p) * q = q$. The homotopies now follow easily.

  (iii) $p : x =_A y$, $P : A \to \mathcal{U}$ $p_*^P : P(x) \to P(y)$ has as quasinverse the $(p^{-1})_*^P : P(y) \to P(x)$. (See Corollary 2.4.3).

- **Remark 2.6.3** - There may be many unequal inhabitants of $\mathrm{qinv}(f)$ (EXAMPLE??)

- **Definition 2.6.4:** $A, B : \mathcal{U}$ $f : A \to B$.

$$\mathrm{isequiv}(f) :\equiv \left(\sum_{g : B \to A} (f \circ g) \sim id_B \right) \times \left(\sum_{h : B \to A} (h \circ f) \sim id_A \right)$$

Proposition 26.5: $A, B : \mathcal{U}$, $f : A \to B$.

(i) $\mathrm{qinv}(f) \to \mathrm{isequiv}(f)$

(ii) $\mathrm{isequiv}(f) \to \mathrm{qinv}(f)$

(iii) $\prod_{e_1, e_2 : \mathrm{isequiv}(f)} (e_1 = e_2)$. (i.e., is Prop $(\mathrm{isequiv}(f))$)

Proof: (i) We define $\varphi : \mathrm{qinv}(f) \to \mathrm{isequiv}(f)$ by

$$\varphi(g, H, G) \equiv ((g, H), (g, G)) \quad , \text{ for every } (g, H, G)$$

(ii) We want to define $\psi : \mathrm{isequiv}(f) \to \mathrm{qinv}(f)$

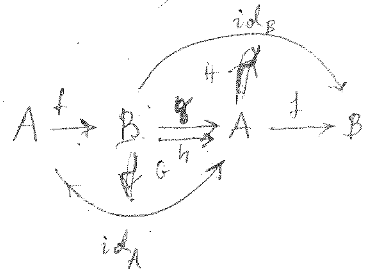
Let $g : B \to A$ and $H : (f \circ g) \sim id_B \equiv \prod_{y:B} (f(g(y)) =_B y)$ (1)

Let $h : B \to A$ and $G : (h \circ f) \sim id_A \equiv \prod_{x:A} (h(f(x)) =_A x)$ (2)

We need to find $(e : B \to A, H', G')$ where

$$H' : (f \circ e) \sim id_B \equiv \prod_{y:B} (f(e(y)) =_B y)$$

$$G' : (e \circ f) \sim id_A \equiv \prod_{x:A} (e(f(x)) =_A x)$$



We define $e \equiv h \circ f \circ id_A \circ g$

$$\begin{aligned} f(e(y)) &\equiv f(h(f(id_A(g(y))))) \\ &\equiv f(h(f(g(y)))) \\ &\overset{(2)}{=} f(g(y)) \\ &\overset{(1)}{=} y \end{aligned}$$

by successive application of path-preservation

$$\begin{aligned} e(f(x)) &\equiv h(f(id_A(g(f(x))))) \\ &\equiv h(f(g(f(x)))) \\ &\overset{(1)}{=} h(f(x)) \\ &\overset{(2)}{=} x \end{aligned}$$

(iii) postponed (?)

Definition 2.6.6: If $A, B : \mathcal{U}$, $\quad A \simeq_{\mathcal{U}} B \equiv \sum_{f: A \to B} \mathrm{isequiv}(f)$

Hence to show that $A \simeq_{\mathcal{U}} B$ we need to find $f: A \to B$ and prove isequiv(f). $A \simeq_{\mathcal{U}} B$ corresponds to homotopic equivalent topol. spaces

Proposition 2.6.7 $\quad A, B, C : \mathcal{U}$.

(i) $A \simeq_{\mathcal{U}} A$

(ii) $A \simeq_{\mathcal{U}} B \to B \simeq_{\mathcal{U}} A$

(iii) $A \simeq_{\mathcal{U}} B \to B \simeq_{\mathcal{U}} C \to A \simeq_{\mathcal{U}} C$

Proof: (i) $id_A : A \to A$ and qinv(id_A) $(\to$ isequiv$(id_A))$

(ii) Let $f : A \to B$ and $p :$ isequiv(f). Hence $\psi(p)$: qinv(f). Ie, there is $g : B \to A$ and $H : f \circ g \sim id_B$ and $G : g \circ f \sim id_A$.

We denote $f^{-1} \equiv g$. We just need to show isequiv(f^{-1})

Since qinv(f^{-1}), we get isequiv(f^{-1}) :

(f, G, H) : qinv(f^{-1}), then $\varphi(f, G, H)$: isequiv(f^{-1}).

(iii) Let $f : A \to B$ and p : isequiv(f) $\quad (f \circ f^{-1} \sim id_B \;,\; f^{-1} \circ f \sim id_A)$

$g : B \to C$ and q : isequiv(g) $\quad (g \circ g^{-1} \sim id_C \;,\; g^{-1} \circ g \sim id_B)$

Clearly $g \circ f : A \to C$ and we show isequiv$(g \circ f)$

Expected $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

$$(g \circ f)((f^{-1} \circ g^{-1})(z)) \equiv g(\underbrace{f(f^{-1}}(g^{-1}(z))))$$
$$= g(g^{-1}(z))$$
$$= z$$

$$(f^{-1} \circ g^{-1})(g \circ f)(x) = f^{-1}(g^{-1}(g(f(x))))$$
$$= f^{-1}(f(x))$$
$$= x.$$

or
$$(g \circ f) \circ (f^{-1} \circ g^{-1}) \sim g \circ ((f \circ f^{-1}) \circ g^{-1})$$
$$\sim g \circ (id_B \circ g^{-1})$$
$$\sim (g \circ id_B) \circ g^{-1}$$
$$\sim g \circ g^{-1}$$
$$\sim id_C$$

using Prop. 2.5.2 repeatedly.

Proposition 2.6.8 If $x, y : 1$, then $(x =_1 y) \simeq 1$

Proof: $f : (x =_1 y) \to 1$ $\quad f \equiv \lambda(p : x =_1 y).\, 0_1$

$g : 1 \to (x =_1 y)$ is to be defined as follows:

$$\cancel{\prod_{x,y:1} \prod_{p : x =_1 y}}$$

$$\cancel{\prod_{x:1} \prod_{p : 0 =_1 x} \sum_{g : 1 \to 0 =_1 x} (g(0_1) \equiv p)}$$

$$\cancel{C(0, x, \mathrm{refl}) \equiv \sum_{g : 1 \to 0 = 0} (g(0_1) = \mathrm{refl}_0)}$$

By $\mathrm{UP}(1)$

$\exists F : \prod_{x:1} (0 =_1 x)$ $\quad$ let $F(x) : 0 =_1 x \to F(x)^{-1} : x =_1 0$

$F(y) : 0 =_1 y$

Hence $F(x)^{-1} \ast F(y) : x =_1 y$

Define $g \equiv \lambda(z : 1)\, (F(x)^{-1} \ast F(y))$

- $\prod_{z:1} (f(\underbrace{g(z)) = z}_{P(z)})$: $P(0_1) \equiv f(g(0_1)) = 0_1$

  $\equiv 0_1 = 0_1$ hence

  $\mathrm{refl}_{0_1} : P(0_1)$

  By ind$_1$ the type is inhabited

- $\prod_{p : x =_1 y} (g(f(p)) = p)$: $g(f(p)) \equiv g(0_1)$

$C(x, y, p) \equiv (g(f(p)) = p)$

$C(x, x, \mathrm{refl}_x) \equiv (g(f(\mathrm{refl}_x)) = \mathrm{refl}_x)$ $\quad F(x)^{-1} \cdot F(x) = \mathrm{refl}_x$

$\equiv (g(0_1) = \mathrm{refl}_x)$

$\equiv (F(x)^{-1} \ast F(x)) = \mathrm{refl}_x$

By Prop. 2.4 it is inhabited. ∎