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The Caristi–Kirk Fixed Point Theorem from the point of view of ball spaces

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Abstract. We take a fresh look at the important Caristi–Kirk Fixed Point Theorem and link it to the recently developed theory of ball spaces, which provides generic fixed point theorems for contracting functions in a number of applications including, but not limited to, metric spaces. The connection becomes clear from a proof of the Caristi–Kirk Theorem given by J.-P. Penot in 1976. We define Caristi–Kirk ball spaces and use a generic fixed point theorem to reprove the Caristi–Kirk Theorem. Further, we show that a metric space is complete if and only if all of its Caristi–Kirk ball spaces are spherically complete.

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1. Introduction

We consider a metric space (X, d) with a function $f : X \to X$ and ask for the existence of a **fixed point**, that is, a point $x \in X$ such that $f(x) = x$. To simplify notation, we will write fx in place of $f(x)$.

If the metric is an ultrametric, then ultrametric balls can serve well in the proofs of fixed point theorems, such as the Ultrametric Banach's Fixed Point Theorem [14]. This is due to their special property that if two ultrametric balls have nonempty intersection, then they are already comparable by inclusion. In contrast, metric balls in general metric spaces are not usually employed in fixed point theorems.

In the papers [7–9] the notions and tools used for ultrametric spaces have been taken as an inspiration for the development of a unifying approach to fixed point theorems for contracting functions, via the flexible notion of

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ball spaces. It allows applications to various areas, such as ultrametric spaces, topological spaces, ordered abelian groups and fields, partially ordered sets and lattices. It also allows the transfer of ideas and concepts between the various areas. However, while metric spaces can be treated with the same approach, taking metric balls for the formal balls in ball spaces does not lead to shorter or more elegant proofs of existing metric fixed point theorems.

The present paper owes its existence to the discovery that other sets which came up in proofs of the Caristi–Kirk Fixed Point Theorem (discussed below) fit much better to the ball spaces framework. In general, they are not metric balls. We first learnt about the use of these sets, which we will call Caristi–Kirk balls, from the paper [2] by Du. Later we found that already in 1976, Penot ([12, Proposition 2.1]) used these sets to give a short and elegant proof of the Caristi–Kirk Theorem. We will present a modification of this proof in Sect. 2.

In the sequel we give a quick introduction to the idea of ball spaces and present a proof of the Caristi–Kirk Theorem in Sect. 4 which is based on a generic fixed point theorem for ball spaces.

Our paper is meant as an invitation to the interested reader to consider fixed point theory from the point of view of ball spaces. We will be happy if the many open problems originating from the theory of ball spaces will be taken up by other researchers. In particular, it is known that Caristi's Fixed Point Theorem is equivalent to Ekeland's Variational Principle, Takahashi's Nonconvex Minimization Theorem, Danes' Drop Theorem, the Petal Theorem, and the Oettli–Thera Theorem; we refer the reader to [1,11,13,15], to name just a few. It is certainly an interesting question what ball spaces can say about these results and the connections between them, but this is beyond the scope of our present paper.

The Caristi–Kirk Theorem gives a criterion for a fixed point to exist when (X, d) is complete. To formulate it, we need the following notion. A function φ from a metric space (X, d) to $\mathbb{R}$ is called **lower semicontinuous** if for every $y \in X$,

$$\liminf_{x\to y} \varphi(x) \;\geq\; \varphi(y)\,.$$

Theorem 1. *(Caristi–Kirk) Take a complete metric space (X, d) and a lower semicontinuous function $\varphi : X \to \mathbb{R}$ which is bounded from below. If a function $f : X \to X$ satisfies the* **Caristi condition**

$$(\mathbf{CC}) \quad d(x, fx) \;\leq\; \varphi(x) - \varphi(fx),$$

then f has a fixed point on X.

Penot's proof of this theorem is interesting as it works with sets of the form

$$B_x \;:=\; \{y \in X \mid d(x, y) \leq \varphi(x) - \varphi(y)\}\,, \tag{1}$$

for each $x \in X$. Note that in spite of the notation, these sets will in general not be metric balls. We call these sets **Caristi–Kirk balls**.

A **ball space** is a pair $(X, \mathcal{B})$ consisting of a nonempty set X and a nonempty set $\mathcal{B} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ of distinguished nonempty subsets B of X.

The elements B of $\mathcal{B}$ will be called **balls**, in analogy to the case of metric or ultrametric balls.

In analogy to the case of ultrametric spaces, we will call a nonempty collection $\mathcal{N}$ of balls in $\mathcal{B}$ a **nest of balls (in $\mathcal{B}$)** if it is totally ordered by inclusion. We will say that $(X,\mathcal{B})$ is **spherically complete** if the intersection $\bigcap\mathcal{N}$ of each nest of balls in $\mathcal{B}$ is nonempty.

A function f on an arbitrary ball space $(X,\mathcal{B})$ is called **contracting on orbits** if there is a function that associates to every $x\in X$ some ball $B_x\in\mathcal{B}$ such that for all $x\in X$, the following conditions hold:

(SC1) $x\in B_x\,,$

(SC2) $B_{fx}\subseteq B_x\,$, and if $x\neq fx$, then $B_{fx}\subsetneq B_x\,.$

We will say that a nest of balls $\mathcal{N}$ is an **f-nest** if $\mathcal{N}=\{B_x\mid x\in M\}$ for some set $M\subseteq X$ that is closed under f (in other words, with every ball B_x it also contains the ball B_{fx}). The function f will be called **self-contractive** if in addition to (SC1) and (SC2), it satisfies:

(SC3) if $\mathcal{N}$ is an f-nest and if $z\in\bigcap\mathcal{N}$, then $B_z\subseteq\bigcap\mathcal{N}$.

The following fixed point theorem has been proved in [7] (see also [9]), using Zorn's Lemma:

Theorem 2. *Every self-contractive function on a spherically complete ball space has a fixed point.*

Take any function $\varphi:X\to\mathbb{R}$. We define the **ball space induced by** φ to be $(X,\mathcal{B}_\varphi)$ where

$$\mathcal{B}_\varphi\;:=\;\{B_x\mid x\in X\}\,, \tag{2}$$

with B_x defined as in (1). If φ is lower semicontinuous and bounded from below, then we will call $(X,\mathcal{B}_\varphi)$ a **Caristi–Kirk ball space** of (X,d). We wish to show how the Caristi–Kirk Theorem can be deduced from Theorem 2. To this end, we prove in Sect. 4 that a function satisfying the Caristi Condition (CC) is self-contractive in the ball space induced by φ (even if φ is not lower semicontinuous). Then the Caristi–Kirk Theorem will follow from Theorem 2 together with the following result, which we will prove in Sect. 3:

Proposition 3. *Let (X,d) be a metric space. Then the following statements are equivalent:*

(i) The metric space (X,d) is complete.

(ii) Every Caristi–Kirk ball space $(X,\mathcal{B}_\varphi)$ is spherically complete.

(iii) For every continuous function $\varphi\colon X\to\mathbb{R}$ bounded from below, the Caristi–Kirk ball space $(X,\mathcal{B}_\varphi)$ is spherically complete.

Note that it is in general not true that the ball space consisting of all nonempty closed metric balls of a complete metric space is spherically complete. Passing to Caristi–Kirk balls instead remedies this deficiency.

In Sect. 4 we will also show that the Caristi–Kirk Theorem implies the Banach Fixed Point Theorem. More precisely, we prove:

Theorem 4. *Take a metric space (X,d) and assume that for every continuous $\varphi:X\to\mathbb{R}$ bounded from below, its Caristi–Kirk ball space $(X,\mathcal{B}_\varphi)$ is spherically complete. Further, take a function $f:X\to X$ which is*

1) non-expanding, i.e., $d(fx, fy) \leq d(x, y)$ for all $x, y \in X$, and
2) contracting on orbits, i.e., $d(fx, f^2x) \leq Cd(x, fx)$ for all $x, y \in X$, with Lipschitz constant $C < 1$.
Then f has a fixed point on X.

Finally, let us mention that Caristi's original theorem and the Caristi–Kirk Theorem discussed here have been the subject of many papers in the literature. Several of them are listed in the references of, e.g., [2,6]. A recurring question is whether the theorems can be proven without the use of transfinite induction, Zorn's Lemma, or even the axiom of choice (see [10] and the discussion in [3], [4, pages 55–56], [6] together with the literature cited therein). While the first two are avoided in [12] and also in [2,6], the axiom of choice, or at least the axiom of dependent choice, is still present (cf. [6, Section 3]).

In this connection, we should point out that the generic fixed point theorems in the theory of ball spaces are making essential use of Zorn's Lemma. In fact, in this way Zorn's Lemma has provided an elegant replacement of transfinite induction which was used before for the proof of theorems in valuation theory (see [14]).

Another task mentioned in [6] is to avoid defining a partial order in the proof of the Caristi–Kirk Theorem. This is achieved in [2,6] and also in the present paper. As we will point out in Remark 6, the partial order is implicit whenever the Caristi–Kirk balls are used, which are partially ordered by inclusion. However, working with these balls directly is more natural than the detour of defining the partial order explicitly.

2. A modification of Penot's proof of the Caristi–Kirk Theorem

We start by working out the basic properties of the Caristi–Kirk balls B_x that have been defined in (1).

Lemma 5. *Take a metric space (X, d) and any function $\varphi : X \to \mathbb{R}$. Let the sets B_x be defined as in (1). Then the following assertions hold.*

1) *For every $x \in X$, $x \in B_x$.*
2) *If $y \in B_x$, then $B_y \subseteq B_x$; if in addition $x \neq y$, then $B_y \subsetneq B_x$ and $\varphi(y) < \varphi(x)$.*
3) *If $f : X \to X$ is a function for which the Caristi–Kirk condition (CC) holds, then $fx \in B_x$.*
4) *If φ is lower semicontinuous, then all Caristi–Kirk balls B_x are closed in the topology induced by the metric.*

Proof. Assertion 1) holds since $d(x, x) = 0 \leq \varphi(x) - \varphi(x)$, and assertion 3) is obvious.

For the proof of assertion 2), take any $y \in B_x$. Then $\varphi(x) \geq \varphi(y)$ because $d(x, y) \geq 0$. Moreover, $\varphi(x) = \varphi(y)$ can only hold if $x = y$. Hence if $x \neq y$, then $\varphi(y) - \varphi(x) < 0$, which yields that $x \notin B_y$ and hence $B_y \neq B_x$.

Further, if $z \in B_y$, then

$$d(x,z) \;\le\; d(x,y)+d(y,z) \;\le\; \varphi(x)-\varphi(y)+\varphi(y)-\varphi(z) \;=\; \varphi(x)-\varphi(z)\,.$$

Hence $z \in B_x$, so $B_y \subseteq B_x$.

For the proof of assertion 4), observe that the complement $\{y \in X \mid d(x,y)+\varphi(y) > \varphi(x)\}$ of B_x is the preimage of the open subset $(\varphi(x),\infty)$ of $\mathbb{R}$ under the function $d(x,Y)+\varphi(Y)$. Whenever φ is lower semicontinuous, then so is $d(x,Y)+\varphi(Y)$ and this preimage is open in X. $\square$

For the proof of the Caristi–Kirk Theorem, start with any $x_1 \in X$ and construct a sequence $(x_n)_{n\in\mathbb{N}}$ by induction as follows. Suppose that the members x_i are already constructed for $1 \le i \le n$ such that

a) $(\varphi(x_i))_{i\le n}$ is strictly decreasing,

b) $(B_{x_i})_{i\le n}$ is strictly decreasing w.r.t. inclusion.

If B_{x_n} is a singleton, then by parts 1) and 3) of Lemma 5, $B_{x_n} = \{x_n, fx_n\}$ with $x_n = fx_n$. Then we have found a fixed point, and we stop. Otherwise, we choose some $x_{n+1} \in B_{x_n} \setminus \{x_n\}$ such that

$$\varphi(x_{n+1}) \le \inf_{z\in B_{x_n}} \varphi(z) + \frac{1}{n}. \tag{3}$$

Here the infimum exists because we are dealing with a subset of the reals bounded from below.

From Lemma 5 we obtain that $\varphi(x_{n+1}) < \varphi(x_n)$ and $B_{x_{n+1}} \subsetneq B_{x_n}$. So a) and b) hold for $n+1$ in place of n. In this way, if we do not stop at some n having found a fixed point, we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ for which the sequences $(\varphi(x_n))_{n\in\mathbb{N}}$ and $(B_{x_n})_{n\in\mathbb{N}}$ are strictly decreasing.

For every $x \in B_{x_{n+1}}$ we have, using that $B_{x_{n+1}} \subset B_{x_n}$ and (3):

$$\varphi(x) \ge \inf_{z\in B_{x_n}} \varphi(z) \;>\; \varphi(x_{n+1}) - \frac{1}{n}\,, \text{ and}$$

$$d(x,x_{n+1}) \le \varphi(x_{n+1}) - \varphi(x) \;<\; \frac{1}{n}.$$

This shows that the diameter $\sup\{d(x,y) \mid x,y \in B_{x_{n+1}}\}$ of $B_{x_{n+1}}$ is not larger than $\frac{2}{n}$. Therefore, as (X,d) is complete and the sets B_{x_n} are closed by part 4) of Lemma 5, the intersection $\bigcap_{n\in\mathbb{N}} B_{x_n}$ contains exactly one element z. Then $z \in B_{x_n}$ and thus $fz \in B_z \subseteq B_{x_n}$ for all $n \in \mathbb{N}$ by parts 2) and 3) of Lemma 5. Hence $fz \in \bigcap_{n\in\mathbb{N}} B_{x_n} = \{z\}$, showing that $fz = z$. $\square$

Remark 6. In his original proof, Penot uses the partial order $x \le y :\Leftrightarrow d(x,y) \le \varphi(x) - \varphi(y)$. However, this is not necessary, and we have eliminated the explicit use of this partial order. In fact, it is encoded in the partial order of the Caristi–Kirk balls. Indeed, parts 1) and 2) of Lemma 5 show that $x \ge y \Leftrightarrow B_y \subseteq B_x$.

Apart from the fact that the proofs in [2,6] do not explicitly use the partial order, the major difference between these proofs and Penot's original proof as well as the above modification is that Penot shows that the diameters of the sets B_{x_n} converge to 0 and from this deduces without much technical effort that their intersection contains exactly one element which is equal to its image under f.

3. Proof of Proposition 3

First we show that (i) implies (ii).

Assume that the metric space (X, d) is complete, and consider a Caristi–Kirk ball space $(X, \mathcal{B}_\varphi)$ of (X, d). Take a nest $\mathcal{N}$ of balls in $\mathcal{B}_\varphi$. We write $\mathcal{N} = \{B_x \mid x \in M\}$ for some subset $M \subseteq X$. For all $x, y \in M$ we have that $x \in B_y$ or $y \in B_x$ because $\mathcal{N}$ is totally ordered by inclusion. In both cases,

$$d(x, y) \;\leq\; |\varphi(x) - \varphi(y)|\,. \tag{4}$$

Since φ is bounded from below, there exists

$$r \;:=\; \inf_{x \in M} \varphi(x) \in \mathbb{R}\,.$$

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in M such that $\lim_{n\to\infty} \varphi(x_n) = r$. The sequence $(\varphi(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ (as it converges to r), hence (4) implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d). As (X, d) is complete, we obtain that $(x_n)_{n\in\mathbb{N}}$ converges to some $z \in X$. We claim that $z \in \bigcap \mathcal{N}$.

Take any $x \in M$. Since φ is lower semicontinuous,

$$\varphi(z) \;\leq\; \lim_{n\to\infty} \varphi(x_n) \;=\; r\,.$$

For all $n \in \mathbb{N}$ we have $d(x, x_n) \leq |\varphi(x) - \varphi(x_n)|$ by (4). Using the continuity of d, we obtain:

$$\begin{aligned} d(x, z) &= \lim_{n\to\infty} d(x, x_n) \;\leq\; \lim_{n\to\infty} |\varphi(x) - \varphi(x_n)| \;=\; |\varphi(x) - r| \\ &= \varphi(x) - r \;\leq\; \varphi(x) - \varphi(z)\,. \end{aligned}$$

Therefore, $z \in B_x$. As $x \in M$ was arbitrary, we have that $z \in \bigcap \mathcal{N}$, as desired.

It is obvious that (ii) implies (iii).

Finally we show that (iii) implies (i).

Take a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (X, d); we wish to show that it has a limit in X. We may assume that no x_n is a limit of $(x_n)_{n\in\mathbb{N}}$ since otherwise we are done. Define $\psi : X \to \mathbb{R}^{\geq 0}$ by

$$\psi(x) \;:=\; \lim_{n\to\infty} d(x, x_n)$$

for all $x \in X$ and note that this function is continuous.

By induction, we choose a subsequence $(y_k)_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ with $y_k = x_{n_k}$ as follows. We set $n_1 := 1$. If n_k is already chosen, we observe that by assumption, $y_k = x_{n_k}$ is not a limit of $(x_n)_{n\in\mathbb{N}}$ and therefore $\psi(y_k) > 0$. On the other hand, $\lim_{n\to\infty} \psi(x_n) = 0$ since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. It follows that there is some $m > n_k$ such that

$$\frac{1}{2} d(y_k, x_m) \;\leq\; \psi(y_k) \;-\; \psi(x_m)\,. \tag{5}$$

We choose one of such m and set $n_{k+1} := m$. Further, we set

$$\varphi(x) \;:=\; 2\psi(x)\,.$$

Then by construction and inequality (5),

$$d(y_k, y_{k+1}) \;\leq\; \varphi(y_k) \;-\; \varphi(y_{k+1}) \tag{6}$$

for all $k \in \mathbb{N}$, and φ is a continuous function from X to $\mathbb{R}^{\geq 0}$. Hence by assumption, the Caristi–Kirk ball space $(X, \mathcal{B}_\varphi)$ is spherically complete. We will use this to show that $(y_k)_{k\in\mathbb{N}}$ converges to some y in (X, d).

We set

$$\mathcal{N} := \{B_{y_k} \mid k \in \mathbb{N}\}\,.$$

The inequality (6) shows that $y_{k+1} \in B_{y_k}$ and hence $B_{y_{k+1}} \subseteq B_{y_k}$ by part 2) of Lemma 5. This shows that $\mathcal{N}$ is a nest of balls. By spherical completeness, there exists an element $y \in \bigcap \mathcal{N}$. It follows that

$$d(y_k, y) \leq \varphi(y_k) - \varphi(y) \leq \varphi(y_k)$$

for all $k \in \mathbb{N}$. Since $\lim_{k\to\infty} \varphi(y_k) = 0$, this shows that $(y_k)_{k\in\mathbb{N}}$ converges to y in (X, d). Since $(y_k)_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$, the original Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ also converges to y. We have thus proved that the metric space (X, d) is complete. □

Remark 7. The idea for the definition of the function φ is taken from the proof of [5, Theorem 2]. In that Theorem, Kirk states that a metric space must be complete if it satisfies the Caristi–Kirk Theorem. To prove this assertion, he assumes that there is a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (X, d) without a limit in X. He then defines a function $f : X \to X$ by setting $f(x) := x_m$ where m is the smallest natural number such that

$$0 < \frac{1}{2} d(x, x_m) \leq \psi(x) - \psi(x_m)\,.$$

Consequently, f satisfies the Caristi Condition (CC) with respect to $\varphi(x) = 2\psi(x)$. But by construction, f does not have a fixed point.

4. Proofs of Theorem 1 and Theorem 4

Lemma 8. *Take any function $\varphi : X \to \mathbb{R}$ and a function $f : X \to X$ that satisfies condition (CC). Then f is self-contractive in the ball space $(X, \mathcal{B}_\varphi)$.*

If in addition $(X, \mathcal{B}_\varphi)$ is spherically complete, then f admits a fixed point.

Proof. Lemma 5 shows that conditions (SC1) and (SC2) are satisfied.

Take any f-nest $\mathcal{N}$. Then $z \in \bigcap \mathcal{N}$ implies that $z \in B_x$ for all $B_x \in \mathcal{N}$. Therefore, we have $B_z \subseteq B_x$ for all $x \in S$, which shows that $B_z \subseteq \bigcap \mathcal{N}$. Hence, (SC3) holds and we have proven that f is self-contractive.

The last assertion follows from Theorem 2. □

Note that in the proof of the first part of this lemma we have not used that φ is lower semicontinuous and bounded from below. This is only needed to deduce the spherical completeness of $(X, \mathcal{B}_\varphi)$ from the completeness of (X, d).

Proof of Theorem 1:
If the assumptions of the theorem are satisfied, then Proposition 3 shows that $(X, \mathcal{B}_\varphi)$ is spherically complete, and Lemma 8 shows that f admits a fixed point. □

Proof of Theorem 4:
Take a function f on a metric space (X, d) which is non-expanding and contracting on orbits with Lipschitz constant $C < 1$. For each $x \in X$, we define

$$\varphi(x) \;:=\; \frac{d(x, fx)}{1 - C}\,. \tag{7}$$

Since f is contracting on orbits, we find:

$$\varphi(fx) \;=\; \frac{d(fx, f^2x)}{1 - C} \;\leq\; \frac{Cd(x, fx)}{1 - C}\,,$$

whence

$$\varphi(x) \;-\; \varphi(fx) \;\geq\; \frac{d(x, fx)}{1 - C} \;-\; \frac{Cd(x, fx)}{1 - C} \;=\; d(x, fx)\,.$$

This shows that the Caristi Condition (CC) is satisfied. We will now show that φ is continuous. Take arbitrary $x, y \in X$ and assume w.l.o.g. that $\varphi(x) \geq \varphi(y)$. Then we compute, using the fact that f is non-expanding:

$$\begin{aligned}
\varphi(x) \;-\; \varphi(y) &= \frac{1}{1 - C}(d(x, fx) \;-\; d(y, fy)) \\
&\leq \frac{1}{1 - C}(d(x, y) \;+\; d(y, fy) \;+\; d(fy, fx) \;-\; d(y, fy))) \\
&= \frac{1}{1 - C}(d(x, y) \;+\; d(fy, fx)) \;\leq\; \frac{2}{1 - C}d(x, y)\,.
\end{aligned}$$

This implies that φ is continuous. Moreover, it is bounded from below by 0. Hence by assumption, the Caristi–Kirk ball space $(X, \mathcal{B}_\varphi)$ is spherically complete. Since we have shown that f satisfies the Caristi Condition (CC), the existence of a fixed point now follows from Lemma 8. □

References

[1] Araya, Y.: On generalizing Takahashi's nonconvex minimization theorem. Appl. Math. Lett. **22**, 501–504 (2009)

[2] Du, W.-S.: A direct proof of Caristi's fixed point theorem. Appl. Math. Sci. **10**, 2289–2294 (2016)

[3] Jachymski, J.R.: Caristi's fixed point theorem and selections of set-valued contractions. J. Math. Anal. Appl. **227**, 55–67 (1998)

[4] Khamsi, M.A., Kirk, W.A.: An introduction to metric spaces and fixed point theory, Pure and Applied Mathematics (New York). Wiley-Interscience, New York (2001)

[5] Kirk, W.A.: Caristi's fixed point theorem and metric convexity. Colloq. Math. **36**, 81–86 (1976)

[6] Kozlowski, W.M.: A purely metric proof of the Caristi fixed point theorem. Bull. Aust. Math. Soc. **95**, 333–337 (2017)

[7] Kuhlmann, F.-V., Kuhlmann, K.: A common generalization of metric and ultrametric fixed point theorems. Forum Math. **27**, 303–327 (2015); Correction to "A common generalization of metric, ultrametric and topological fixed point theorems". Forum Math. **27**, 329–330 (2015). Alternative corrected version available at: http://math.usask.ca/fvk/GENFPTAL.pdf

[8] Kuhlmann, F.-V., Kuhlmann, K.: Fixed point theorems for spaces with a transitive relation. Fixed Point Theory **18**, 663–672 (2017)

[9] Kuhlmann, F.-V., Kuhlmann, K.: A basic framework for fixed point theorems: ball spaces and spherical completeness **(in preparation)**

[10] Mańka, R.: Some forms of the axiom of choice. Jbuch. Kurt-Gödel-Ges 1988, 24–34

[11] Oettli, W., Théra, M.: Equivalents of Ekeland's principle. Bull. Aust. Math. Soc. **48**, 385–392 (1993)

[12] Penot, J.-P.: Fixed point theorems without convexity, analyse non convexe (Proc. Colloq., Pau, 1977). Bull. Soc. Math. France Mém. **60**, 129–152 (1979)

[13] Penot, J.-P.: The drop theorem, the petal theorem and Ekeland's variational principle. Nonlinear Anal. **10**, 813–822 (1986)

[14] Prieß-Crampe, S.: Der Banachsche Fixpunktsatz für ultrametrische Räume. Results Math. **18**, 178–186 (1990)

[15] Takahashi, W.: Existence theorems generalizing fixed point theorems for multivalued mappings. In: Théra, M.A., Baillon, J.B (eds.) Fixed point theory and applications (Marseille, 1989). Pitman Research Notes in Mathematics Series, vol. 252, pp. 397–406. Longman Scientific & Technical, Harlow (1991)

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