APPENDIX 8

Toolbox

10. Limits and Colimits, Products and Equalizers

Limit constructions are a very important tool in category theory. We will introduce the basic facts on limits and colimits in this section.

Deønition 8.10.1. A diagramscheme $\mathcal{D}$ is a small category (i. e. the class of objects is a set). Let $\mathcal{C}$ be an arbitrary category. A diagram in $\mathcal{C}$ over the diagramscheme $\mathcal{D}$ is a covariant functor $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$.

Example 8.10.2. (for diagramschemes)

1. The empty category $\mathcal{D}$.
2. The category with precisely one object D and precisely one morphism 1_D.
3. The category with two objects D_1, D_2 and one morphism $f : D_1 \longrightarrow D_2$ (apart from the two identities).
4. The category with two objects D_1, D_2 and two morphisms $f, g : D_1 \longrightarrow D_2$ between them.
5. The category with a family of objects $(D_i | i \in I)$ and the associated identities.
6. The category with four objects $D_1, \ldots, D_4$ and morphisms f, g, h, k such that the diagram

$$\begin{tikzcd} D_1 \arrow[r, "f"] \arrow[d, "g"'] & D_2 \arrow[d, "k"] \\ D_3 \arrow[r, "h"'] & D_4 \end{tikzcd}$$

   commutes, i. e. $kf = hg$.

Deønition 8.10.3. Let $\mathcal{D}$ be a diagramscheme and $\mathcal{C}$ a category. Each object $C \in \mathcal{C}$ deønes a constant diagram $\mathcal{K}_C : \mathcal{D} \longrightarrow \mathcal{C}$ with $\mathcal{K}_C(D) := C$ for all $D \in \mathcal{D}$ and $\mathcal{K}(f) := 1_C$ for all morphisms in $\mathcal{D}$. Each morphism $f : C \longrightarrow C'$ in $\mathcal{C}$ deønes a constant natural transformation $\mathcal{K}_f : \mathcal{K}_C \longrightarrow \mathcal{K}_{C'}$ with $\mathcal{K}_f(D) = f$. This deønes a constant functor $\mathcal{K} : \mathcal{C} \longrightarrow \mathrm{Funct}(\mathcal{D}, \mathcal{C})$ from the category $\mathcal{C}$ into the category of diagrams $\mathrm{Funct}(\mathcal{D}, \mathcal{C})$.

Let $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ be a diagram. An object C together with a natural transformation $\pi : \mathcal{K}_C \longrightarrow \mathcal{F}$ is called a limit or a projective limit of the diagram $\mathcal{F}$ with the projection π if for each object $C' \in \mathcal{C}$ and for each natural transformation $\varphi : \mathcal{K}_{C'} \longrightarrow \mathcal{F}$ there is a unique morphism $f : C' \longrightarrow C$ such that

$$\begin{tikzcd} \mathcal{K}_{C'} \arrow[d, "\mathcal{K}_f"'] \arrow[dr, "\varphi"] & \\ \mathcal{K}_C \arrow[r, "\pi"'] & \mathcal{F} \end{tikzcd}$$

commutes, i. e. the diagrams

$$\begin{tikzcd} C \arrow[r, "\pi_i"] \arrow[dr, "\pi_j"'] & \mathcal{F}(D_i) \arrow[d, "\mathcal{F}(g)"] \\ & \mathcal{F}(D_j) \end{tikzcd}$$

commute for all morphisms $g : D_i \longrightarrow D_j$ in $\mathcal{D}$ (π is a natural transformation) and the diagrams

$$\begin{tikzcd} C' \arrow[d, "f"'] \arrow[dr, "\varphi_i"] & \\ C \arrow[r, "\pi_i"'] & \mathcal{F}(D_i). \end{tikzcd}$$

commute for all objects D_i in $\mathcal{D}$.

A category $\mathcal{C}$ has limits for diagrams over a diagramscheme $\mathcal{D}$ if for each diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ over $\mathcal{D}$ there is a limit in $\mathcal{C}$. A category $\mathcal{C}$ is called complete if each diagram in $\mathcal{C}$ has a limit.

Example 8.10.4. 1. Let $\mathcal{D}$ be a diagramscheme consisting of two objects D_1, D_2 and the identities. A diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ is deøned by giving two objects C_1 and C_2 in $\mathcal{C}$. An object $C_1 \times C_2$ together with two morphisms $\pi_1 : C_1 \times C_2 \longrightarrow C_1$ and $\pi_2 : C_1 \times C_2 \longrightarrow C_2$ is called a product of the two objects if $C_1 \times C_2, \pi : \mathcal{K}_{C_1 \times C_2} \longrightarrow \mathcal{F}$ is a limit, i. e. if for each object C' in $\mathcal{C}$ and for any two morphisms $\varphi_1 : C' \longrightarrow C_1$ and $\varphi_2 : C' \longrightarrow C_2$ there is a unique morphism $f : C' \longrightarrow C_1 \times C_2$ such that

$$\begin{tikzcd} & C' \arrow[dl, "\varphi_1"'] \arrow[d, "f"] \arrow[dr, "\varphi_2"] & \\ C_1 & C_1 \times C_2 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & C_2 \end{tikzcd}$$

commutes. The two morphisms $\pi_1 : C_1 \times C_2 \longrightarrow C_1$ and $\pi_2 : C_1 \times C_2 \longrightarrow C_2$ are called the projections from the product to the two factors.

2. Let $\mathcal{D}$ a diagramscheme consisting of a ønite (non empty) set of objects $D_1, \ldots, D_n$ and the associated identities. A limit of a diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ is called a ønite product of the objects $C_1 := \mathcal{F}(D_1), \ldots, C_n := \mathcal{F}(D_n)$ and is denoted by $C_1 \times \ldots \times C_n = \prod_{i=1}^n C_i$.

3. A limit over a discrete diagram (i. e. $\mathcal{D}$ has only the identities as morphisms) is called product of the $C_i := \mathcal{F}(D_i)$, $i \in I$ and is denoted by $\prod_I C_i$.

4. Let $\mathcal{D}$ be the empty diagramscheme and $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ the (only possible) empty diagram. The limit $C, \pi : \mathcal{K}_C \longrightarrow \mathcal{F}$ of $\mathcal{F}$ is called the ønal object. It has the property that for each object C' in $\mathcal{C}$ (the uniquely determined natural transformation $\varphi : \mathcal{K}_{C'} \longrightarrow \mathcal{F}$ does not have to be mentioned) there is a unique morphism $f : C' \longrightarrow C$. In

Set the one-point set is a ønal object. In Ab , Gr , Vec the zero group 0 is a ønal object.

5. Let $\mathcal{D}$ be the diagramscheme from 8.8.2 4. with two objects and two morphisms (diœerent from the two identities). A diagram over $\mathcal{D}$ consists of two objects C_1 and C_2 and two morphisms $g, h : C_1 \longrightarrow C_2$. The limit of such a diagram is called Equalizer of the two morphisms and is given by an object $\mathrm{Ker}(g, h)$ and a morphism $\pi_1 : \mathrm{Ker}(g, h) \longrightarrow C_1$. The second morphism to C_2 arises from the composition $\pi_2 = g\pi_1 = h\pi_1$. The equalizer has the following universal property. For each object C' and each morphism $\varphi_1 : C' \longrightarrow C_1$ with $g\varphi_1 = h\varphi_1 (= \varphi_2)$ there is a unique morphism $f : C' \longrightarrow \mathrm{Ker}(g, h)$ with $\pi_1 f = \varphi_1$ (and thus $\pi_2 f = \varphi_2$, i. e. the diagram

$$\begin{tikzcd} & C' \arrow[dl, "f"'] \arrow[d, "\varphi_1"] & \\ \mathrm{Ker}(g,h) \arrow[r, "\pi_1"] & C_1 \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C_2 \end{tikzcd}$$

commutes.

Problem 8.10.1. 1. Let $\mathcal{F} : \mathcal{D} \longrightarrow$ Set be a discrete diagram. Show that the cartesian product over $\mathcal{F}$ coincides with the categorical product.

2. Let $\mathcal{D}$ be a pair of morphisms as in 8.8.4 5. and let $\mathcal{F} : \mathcal{D} \longrightarrow$ Set be a diagram. Show that the set $\{x \in \mathcal{F}(D_1) | \mathcal{F}(f)(x) = \mathcal{F}(g)(x)\}$ with the inclusion map into $\mathcal{F}(D_1)$ is an equalizer of $\mathcal{F} : \mathcal{D} \longrightarrow$ Set .

3. Let $\mathcal{F} : \mathcal{D} \longrightarrow$ Set be a diagram. Show that the set

$$\{(x_D | D \in \mathrm{Ob}\, D, x_D \in \mathcal{F}(D)) | \forall (f : D \longrightarrow D') \in \mathcal{D} : \mathcal{F}(f)(x_D) = x_{D'}\}$$

with the projections into the single of the families is the limit of $\mathcal{F}$.

Deønition 8.10.5. Let $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ be a diagram. An object C and a natural transformation $\iota : \mathcal{F} \longrightarrow \mathcal{K}_C$ is called colimit or inductive limit of the diagram $\mathcal{F}$ with the injection ι if for each object $C' \in \mathcal{C}$ and for each natural transformation $\varphi : \mathcal{F} \longrightarrow \mathcal{K}_{C'}$ there is a unique morphism $f : C \longrightarrow C'$ such that

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \mathcal{K}_C \arrow[d, "\mathcal{K}_f"] \\ & \mathcal{K}_{C'} \end{tikzcd}$$

commutes, i. e. the diagram

$$\begin{tikzcd} \mathcal{F}(D_i) \arrow[d, "\mathcal{F}(g)"'] \arrow[dr, "\iota_i"] & \\ \mathcal{F}(D_j) \arrow[r, "\iota_j"'] & C \end{tikzcd}$$

commutes for all morphisms $g : D_i \longrightarrow D_j$ in $\mathcal{D}$ (ι is a natural transformation) and the diagram

$$\begin{tikzcd} \mathcal{F}(D_i) \arrow[r, "\iota_i"] \arrow[dr, "\varphi_i"'] & C \arrow[d, "f"] \\ & C'. \end{tikzcd}$$

commutes for all objects D_i in $\mathcal{D}$.

The special colimits that can be formed over the diagrams as in Example 8.8.4 are called coproduct, initial object, resp. coequalizer.

Example 8.10.6. In Vec the object 0 is an initial object. In $\mathbb{K}$-Alg the object $\mathbb{K}$ is an initial object. In Geom the one-element functor $A \mapsto \{*\}$ is a ønal object. In $\mathbb{K}$-Alg the object $\{a \in A | f(a) = g(a)\}$ is the equalizer of the two algebra homomorphisms $f : A \longrightarrow B$ and $g : A \longrightarrow B$. In $\mathbb{K}$Alg the cartesian (set of pairs) and the categorical products coincide.

Remark 8.10.7. A colimit of a diagram $\mathcal{C}$ is a limit of the corresponding (dual) diagram in the dual category $\mathcal{C}^{op}$. Thus theorems about limits in arbitrary categories automatically also produce (dual) theorems about colimits. However, observe that theorems about limits in a particular category (for example the category of vector spaces) translate only into theorems about colimits in the dual category, which most often is not too useful.

Proposition 8.10.8. Limits and colimits of diagrams are unique up to isomorphism.

Proof. Let $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ be a diagram and let C, π and $\tilde{C}, \tilde{\pi}$ be limits of $\mathcal{F}$. Then there are unique morphisms $f : \tilde{C} \longrightarrow C$ and $g : C \longrightarrow \tilde{C}$ with $\pi\mathcal{K}_f = \tilde{\pi}$ and $\tilde{\pi}\mathcal{K}_g = \pi$. This implies $\pi\mathcal{K}_{1_C} = \pi\, \mathrm{id}_{\mathcal{K}_C} = \pi = \tilde{\pi}\mathcal{K}_g = \pi\mathcal{K}_f\mathcal{K}_g = \pi\mathcal{K}_{fg}$ and analogously $\tilde{\pi}\mathcal{K}_{1_{\tilde{C}}} = \tilde{\pi}\mathcal{K}_{gf}$. Because of the uniqueness this implies $1_C = fg$ and $1_{\tilde{C}} = gf$. $\square$

Remark 8.10.9. Now that we have the uniqueness of the limit resp. colimit (up to isomorphism) we can introduce a uniøed notation. The limit of a diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ will be denoted by $\varprojlim(\mathcal{F})$, the colimit by $\varinjlim(\mathcal{F})$.

Theorem 8.10.10. If $\mathcal{C}$ has arbitrary products and equalizers then $\mathcal{C}$ has arbitrary limits, i. e. $\mathcal{C}$ is complete.

Proof. Let $\mathcal{D}$ be a diagramscheme and $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ a diagram. First we form the products $\prod_{D \in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D)$ and $\prod_{f \in \mathrm{Mor}\mathcal{D}} \mathcal{F}(\mathrm{Codom}(f))$ where $\mathrm{Codom}(f)$ is the codomain (range) of the morphism $f : D' \longrightarrow D''$ in $\mathcal{D}$ so in this case $\mathrm{Codom}(f) = D''$. We deøne for each morphism $f : D' \longrightarrow D''$ two morphisms as follows

$$p_f := \pi_{\mathcal{F}(D'')} : \prod_{D \in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D) \longrightarrow \mathcal{F}(D'') = \mathcal{F}(\mathrm{Codom}(f))$$

and

$$q_f := \mathcal{F}(f)\pi_{\mathcal{F}(D')} : \prod_{D \in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D) \longrightarrow \mathcal{F}(D') \longrightarrow \mathcal{F}(D'') = \mathcal{F}(\mathrm{Codom}(f)).$$

These two families of morphisms induce two morphisms into the corresponding product

$$p, q : \prod_{D \in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D) \longrightarrow \prod_{f \in \mathrm{Mor}\mathcal{D}} \mathcal{F}(\mathrm{Codom}(f))$$

with $\pi_f q = q_f$ and $\pi_f p = p_f$. Now we show that the equalizer of these two morphisms

$$\mathrm{Ker}(p,q) \xrightarrow{\ \psi\ } \prod_{D \in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{f \in \mathrm{Mor}\mathcal{D}} \mathcal{F}(\mathrm{Codom}(f))$$

is the limit of the diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$. We have $p\psi = q\psi$. The morphism $\rho(D) := \pi_{\mathcal{F}(D)}\psi : \mathrm{Ker}(p,q) \longrightarrow \prod_{D \in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D) \longrightarrow \mathcal{F}(D)$ defines a family of morphisms for $D \in \mathrm{Ob}\,\mathcal{D}$. If $f : D' \longrightarrow D''$ is in $\mathcal{D}$ then the diagram

$$\begin{tikzcd} \mathrm{Ker}(p,q) \arrow[r, "\rho(D')"] \arrow[dr, "\rho(D'')"'] & \mathcal{F}(D') \arrow[d, "\mathcal{F}(f)"] \\ & \mathcal{F}(D'') \end{tikzcd}$$

is commutative because of $\mathcal{F}(f)\rho(D') = \mathcal{F}(f)\pi_{\mathcal{F}(D')}\psi = q_f\psi = \pi_f q\psi = \pi_f p\psi = p_f\psi = \pi_{\mathcal{F}(D'')}\psi = \rho(D'')$. Thus we have obtained a natural transformation $\rho : \mathcal{K}_{\mathrm{Ker}(p,q)} \longrightarrow \mathcal{F}$.

Now let an object C' and a natural transformation $\varphi : \mathcal{K}_{C'} \longrightarrow \mathcal{F}$ be given. Then this defines a unique morphism $g : C' \longrightarrow \prod_{D \in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D)$ with $\pi_{\mathcal{F}(D)} g = \varphi(D)$ for all $D \in \mathcal{D}$. Since φ is a natural transformation we have $\varphi(D'') = \mathcal{F}(f)\varphi(D')$ for each morphism $f : D' \longrightarrow D''$. Thus we obtain $\pi_f pg = p_f g = \pi_{\mathcal{F}(D'')} g = \varphi(D'') = \mathcal{F}(f)\varphi(D') = \mathcal{F}(f)\pi_{\mathcal{F}(D')} g = q_f g = \pi_f qg$ for all morphisms $f : D' \longrightarrow D''$ hence $pg = qg$. Thus g can be uniquely factorized through the equalizer $\psi : \mathrm{Ker}(p,q) \longrightarrow \prod_{D \in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D)$ in the form $g = \psi h$ with $h : C' \longrightarrow \mathrm{Ker}(p,q)$. Then we have $\rho(D)h = \pi_{\mathcal{F}(D)}\psi h = \pi_{\mathcal{F}(D)} g = \varphi(D)$ for all $D \in \mathcal{D}$ hence $\rho\mathcal{K}_h = \varphi$.

Finally let another morphism $h' : C' \longrightarrow \mathrm{Ker}(p,q)$ with $\rho\mathcal{K}_{h'} = \varphi$ be given. Then we have $\pi_{\mathcal{F}(D)}\psi h' = \rho(D)h' = \varphi(D) = \rho(D)h = \pi_{\mathcal{F}(D)}\psi h$ hence $\psi h' = \psi h = g$. Because of the uniqueness of the factorization of g through ψ we get $h = h'$. Thus $(\mathrm{Ker}(p,q), \rho)$ is the limit of $\mathcal{F}$. □

Remark 8.10.11. The proof of the preceding Theorem gives an explicit construction of the limit of $\mathcal{F}$ as an equalizer

$$\mathrm{Ker}(p,q) \xrightarrow{\ \psi\ } \prod_{D \in \mathrm{Ob}\,\mathcal{D}} \mathcal{F}(D) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{f \in \mathrm{Mor}\mathcal{D}} \mathcal{F}(\mathrm{Codom}(f))$$

Hence the limit can be represented as a subobject of a suitable product. Dually the colimit can be represented as a quotient object of a suitable coproduct. This construction will be used in chapter 3.

Another fact is very important for us, the fact that certain functors preserve limits resp. colimits. We say that a functor $\mathcal{G} : \mathcal{C} \longrightarrow \mathcal{C}'$ preserves limits over the diagramscheme $\mathcal{D}$ if $\varprojlim(\mathcal{G}\mathcal{F}) \cong \mathcal{G}(\varprojlim(\mathcal{F}))$ for each diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$.

Proposition 8.10.12. Covariant representable functors preserve limits. Contravariant representable functors map colimits into limits.

Proof. We only prove the first assertion. The second assertion is dual to the first one. For a diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Set}$ the set

$$\{(x_D | D \in \mathrm{Ob}\,\mathcal{D}, x_D \in \mathcal{F}(D)) | \forall (f : D \longrightarrow D') \in \mathcal{D} : \mathcal{F}(f)(x_D) = x_{D'}\}$$

is a limit of $\mathcal{F}$ by Problem 8.1. Now let a diagram $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ be given and let $\varprojlim(\mathcal{F})$ be the limit. Furthermore let $\mathrm{Mor}_{\mathcal{C}}(C', \text{-}) : \mathcal{C} \longrightarrow \mathrm{Set}$ be a representable functor. By the definition of the limit of $\mathcal{F}$ there is a unique morphism $f : C' \longrightarrow \varprojlim(\mathcal{F})$ with $\pi\mathcal{K}_f = \varphi$ for each natural transformation $\varphi : \mathcal{K}_{C'} \longrightarrow \mathcal{F}$. This defines an isomorphism $\mathrm{Nat}(\mathcal{K}_{C'}, \mathcal{F}) \cong \mathrm{Mor}_{\mathcal{C}}(C', \varprojlim(\mathcal{F}))$. Hence we have

$$\begin{aligned}
&\varprojlim(\mathrm{Mor}_{\mathcal{C}}(C, \mathcal{F})) \cong \\
&\{(\varphi(D) : C' \longrightarrow \mathcal{F}(D) | D \in \mathcal{D}) | \forall (f : D \longrightarrow D') \in \mathcal{D} : \mathcal{F}(f)\varphi(D) = \varphi(D')\} \\
&= \mathrm{Nat}(\mathcal{K}_{C'}, \mathcal{F}) \cong \mathrm{Mor}_{\mathcal{C}}(C', \varprojlim(\mathcal{F})). \quad \Box
\end{aligned}$$

Corollary 8.10.13. Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{C}'$ be left adjoint to $\mathcal{G} : \mathcal{C}' \longrightarrow \mathcal{C}$. Then $\mathcal{F}$ preserves colimits and $\mathcal{G}$ preserves limits.

Proof. For a diagram $\mathcal{H} : \mathcal{D} \longrightarrow \mathcal{C}$ we have

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{C}}(\text{-}, \varprojlim(\mathcal{G}\mathcal{H})) &\cong \varprojlim \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}\mathcal{H}) \cong \varprojlim \mathrm{Mor}_{\mathcal{C}'}(\mathcal{F}\text{-}, \mathcal{H}) \cong \\
&\mathrm{Mor}_{\mathcal{C}'}(\mathcal{F}\text{-}, \varprojlim(\mathcal{H})) \cong \mathrm{Mor}_{\mathcal{C}}(\text{-}, \mathcal{G}(\varprojlim(\mathcal{H}))),
\end{aligned}$$

hence $\varprojlim(\mathcal{G}\mathcal{H}) \cong \mathcal{G}(\varprojlim(\mathcal{H}))$ as representing objects. The proof for the left adjoint functor is analogous. $\Box$