CHAPTER 8

Toolbox
9. Adjoint Functors and the Yoneda Lemma

**Theorem 8.9.1.** (Yoneda Lemma) Let \(C\) be a category. Given a covariant functor \(\mathcal{F}: C \to \text{Set}\) and an object \(A \in C\). Then the map

\[
\pi: \text{Nat}(\text{Mor}_C(A, -), \mathcal{F}) \ni \phi \mapsto \phi(A)(1_A) \in \mathcal{F}(A)
\]

is bijective with the inverse map

\[
\pi^{-1}: \mathcal{F}(A) \ni a \mapsto h^a \in \text{Nat}(\text{Mor}_C(A, -), \mathcal{F}),
\]

where \(h^a(B)(f) = \mathcal{F}(f)(a)\).

**Proof.** For \(\phi \in \text{Nat}(\text{Mor}_C(A, -), \mathcal{F})\) we have a map \(\phi(A): \text{Mor}_C(A, A) \to \mathcal{F}(A)\), hence \(\pi\) with \(\pi(\phi) := \phi(A)(1_A)\) is a well defined map. For \(\pi^{-1}\) we have to check that \(h^a\) is a natural transformation. Given \(f: B \to C\) in \(C\). Then the diagram

\[
\begin{array}{ccc}
\text{Mor}_C(A, B) & \xrightarrow{\text{Mor}(A, f)} & \text{Mor}_C(A, C) \\
h^a(B) \downarrow & & \downarrow h^a(C) \\
\mathcal{F}(B) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(C)
\end{array}
\]

is commutative. In fact if \(g \in \text{Mor}_C(A, B)\) then \(h^a(C)\text{Mor}_C(A, f)(g) = h^a(C)(fg) = \mathcal{F}(fg)(a) = \mathcal{F}(f)\mathcal{F}(g)(a) = \mathcal{F}(f)h^a(B)(a)\). Thus \(\pi^{-1}\) is well defined.

Let \(\pi^{-1}(a) = h^a\). Then \(\pi\pi^{-1}(a) = h^a(A)(1_A) = \mathcal{F}(1_A)(a) = a\). Let \(\pi(\phi) = \phi(A)(1_A) = a\). Then \(\pi^{-1}(\phi) = h^a\) and \(h^a(B)(f) = \mathcal{F}(f)(a) = \mathcal{F}(f)(\phi(A)(1_A)) = \phi(B)\text{Mor}_C(A, f)(1_A) = \phi(B)(f)\), also \(h^a = \phi\).

**Corollary 8.9.2.** Given \(A, B \in C\). Then the following hold

1. \(\text{Mor}_C(A, B) \ni f \mapsto \text{Mor}_C(f, -) \in \text{Nat}(\text{Mor}_C(B, -), \text{Mor}_C(A, -))\) is a bijective map.

2. With the bijective map from 1. the isomorphisms from \(\text{Mor}_C(A, B)\) correspond to natural isomorphisms from \(\text{Nat}(\text{Mor}_C(B, -), \text{Mor}_C(A, -))\).

3. For contravariant functors \(\mathcal{F}: C \to \text{Set}\) we have \(\text{Nat}(\text{Mor}_C(-, A), \mathcal{F}) \cong \mathcal{F}(A)\).

4. \(\text{Mor}_C(A, B) \ni f \mapsto \text{Mor}_C(-, f) \in \text{Nat}(\text{Mor}_C(-, A), \text{Mor}_C(-, B))\) is a bijective map that defines a one-to-one correspondence between the isomorphisms from \(\text{Mor}_C(A, B)\) and the natural isomorphisms from \(\text{Nat}(\text{Mor}_C(-, A), \text{Mor}_C(-, B))\).

**Proof.** 1. follows from the Yoneda Lemma with \(\mathcal{F} = \text{Mor}_C(A, -)\).

2. Observe that \(h^J(C)(g) = \text{Mor}_C(A, g)(f) = gf = \text{Mor}_C(f, C)(g)\) hence \(h^f = \text{Mor}_C(f, -)\). Since we have \(\text{Mor}_C(f, -)\text{Mor}_C(g, -) = \text{Mor}_C(gf, -)\) and \(\text{Mor}_C(f, -) = \text{id}_{\text{Mor}_C(A, -)}\) if and only if \(f = 1_A\) we get the one-to-one correspondence between the isomorphisms from 1.

3. and 4. follow by dualizing.  \(\square\)
Remark 8.9.3. The map $\pi$ is a natural transformation in the arguments $A$ and $\mathcal{F}$. More precisely: if $f : A \to B$ and $\phi : \mathcal{F} \to \mathcal{G}$ are given then the following diagrams commute

$$
\begin{array}{ccc}
\text{Nat}(\text{Mor}_C(A,-), \mathcal{F}) & \xrightarrow{\pi} & \mathcal{F}(A) \\
\downarrow \text{Nat}(\text{Mor}(A,-), \phi) & & \downarrow \phi(A) \\
\text{Nat}(\text{Mor}_C(A,-), \mathcal{G}) & \xrightarrow{\pi} & \mathcal{G}(A) \\
\downarrow \text{Nat}(\text{Mor}(A,-), \mathcal{F}) & & \downarrow \mathcal{F}(f) \\
\text{Nat}(\text{Mor}_C(B,-), \mathcal{F}) & \xrightarrow{\pi} & \mathcal{F}(B).
\end{array}
$$

This can be easily checked. Furthermore we have for $\psi : \text{Mor}_C(A,-) \to \mathcal{F}$

$$
\pi \text{Nat}(\text{Mor}_C(A,-), \phi)(\psi) = \pi(\phi\psi) = (\phi\psi)(A)(1_A) = \phi(A)\psi(A)(1_A) = \phi(A)\pi(\psi)
$$

and

$$
\pi \text{Nat}(\text{Mor}_C(f,-), \mathcal{F})(\psi) = \pi(\psi\text{Mor}_C(f,-)) = (\psi\text{Mor}_C(f,-))(B)(1_B) = \psi(B)(f) = \psi(B)\text{Mor}_C(A,f)(1_A) = \mathcal{F}(f)\psi(A)(1_A) = \mathcal{F}(f)\pi(\psi).
$$

Remark 8.9.4. By the previous corollary the representing object $A$ is uniquely determined up to isomorphism by the isomorphism class of the functor $\text{Mor}_C(A,-)$.

Problem 8.9.1. 1. Determine explicitly all natural endomorphisms from $\mathbb{G}_a$ to $\mathbb{G}_a$ (as defined in Lemma 2.3.5).
2. Determine all additive natural endomorphisms of $\mathbb{G}_a$.
3. Determine all natural transformations from $\mathbb{G}_a$ to $\mathbb{G}_m$ (see Lemma 2.3.7).
4. Determine all natural automorphisms of $\mathbb{G}_m$.

Proposition 8.9.5. Let $\mathcal{G} : C \times \mathcal{D} \to \text{Set}$ be a covariant bifunctor such that the functor $\mathcal{G}(C, -) : \mathcal{D} \to \text{Set}$ is representable for all $C \in C$. Then there exists a contravariant functor $\mathcal{F} : C \to \mathcal{D}$ such that $\mathcal{G} \cong \text{Mor}_\mathcal{D}(\mathcal{F}(-), -)$ holds. Furthermore $\mathcal{F}$ is uniquely determined by $\mathcal{G}$ up to isomorphism.

Proof. For each $C \in C$ choose an object $\mathcal{F}(C) \in \mathcal{D}$ and an isomorphism $\xi_C : \mathcal{G}(C,-) \cong \text{Mor}_\mathcal{D}(\mathcal{F}(C),-)$. Given $f : C \to C'$ in $C$ then let $\mathcal{F}(f) : \mathcal{F}(C') \to \mathcal{F}(C)$ be the uniquely determined morphism (by the Yoneda Lemma) in $\mathcal{D}$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{G}(C,-) & \xrightarrow{\xi_C} & \text{Mor}_\mathcal{D}(\mathcal{F}(C),-) \\
\downarrow \mathcal{G}(f,-) & & \downarrow \text{Mor}(\mathcal{F}(f),-) \\
\mathcal{G}(C',-) & \xrightarrow{\xi_{C'}} & \text{Mor}_\mathcal{D}(\mathcal{F}(C'),-)
\end{array}
$$
commutes. Because of the uniqueness $F(f)$ and because of the functoriality of $G$ it is easy to see that $F(fg) = F(g)F(f)$ and $F(1_C) = 1_{F(C)}$ hold and that $F$ is a contravariant functor.

If $F' : C \to D$ is given with $G \cong \text{Mor}_D(F', -)$ then $\phi : \text{Mor}_D(F', -) \cong \text{Mor}_D(F', -)$. Hence by the Yoneda Lemma $\psi(C) : F(C) \cong F'(C)$ is an isomorphism for all $C \in C$. With these isomorphisms induced by $\phi$ the diagram

$$
\begin{array}{ccc}
\text{Mor}_D(F'(C), -) & \xrightarrow{\text{Mor}(\psi(C), -)} & \text{Mor}_D(F(C), -) \\
\downarrow \text{Mor}(F'(f), -) & & \downarrow \text{Mor}(F(f), -) \\
\text{Mor}_D(F'(C'), -, -) & \xrightarrow{\text{Mor}(\psi(C'), -, -)} & \text{Mor}_D(F(C), -)
\end{array}
$$

commutes. Hence the diagram

$$
\begin{array}{ccc}
F(C') & \xrightarrow{\psi(C')} & F'(C') \\
\downarrow F(f) & & \downarrow F(f) \\
F(C) & \xrightarrow{\psi(C)} & F'(C)
\end{array}
$$

commutes. Thus $\psi : F \to F'$ is a natural isomorphism. \qed

**Definition 8.9.6.** Let $C$ and $D$ be categories and $F : C \to D$ and $G : D \to C$ be covariant functors. $F$ is called leftadjoint to $G$ and $G$ rightadjoint to $F$ if there is a natural isomorphism of bifunctors $\phi : \text{Mor}_D(F, -) \to \text{Mor}_C(-, G)$ from $C^{op} \times D$ to $\text{Set}$.

**Lemma 8.9.7.** If $F : C \to D$ is leftadjoint to $G : D \to C$ then $F$ is uniquely determined by $G$ up to isomorphism. Similarly $G$ is uniquely determined by $F$ up to isomorphism.

**Proof.** Now we prove the first claim. Assume that also $F'$ is leftadjoint to $G$ with $\phi' : \text{Mor}_D(F', -) \to \text{Mor}_C(-, G')$. Then we have a natural isomorphism $\phi'^{-1} \phi : \text{Mor}_D(F, -) \to \text{Mor}_D(F', -)$. By Proposition 8.9.5 we get $F \cong F'$. \qed

**Lemma 8.9.8.** A functor $G : D \to C$ has a leftadjoint functor iff all functors $\text{Mor}_C(C, G)$ are representable.

**Proof.** follows from 8.9.5. \qed

**Lemma 8.9.9.** Let $F : C \to D$ and $G : D \to C$ be covariant functors. Then

$$
\text{Nat}(\text{Id}_C, GF) \ni \phi \mapsto G(\Phi) \in \text{Nat}(\text{Mor}_D(F, -), \text{Mor}_C(-, G))
$$

is a bijective map with inverse map

$$
\text{Nat}(\text{Mor}_D(F, -), \text{Mor}_C(-, G)) \ni \phi \mapsto \phi(-, F)(1_{\text{-}}) \in \text{Nat}(\text{Id}_C, GF).
$$
Furthermore

\[ \text{Nat}(\mathcal{F}, \text{Id}_\mathcal{C}) \ni \Psi \mapsto \Psi \cdot \mathcal{F} \cdot \in \text{Nat}(\text{Mor}_\mathcal{C}(-, \mathcal{G}), \text{Mor}_\mathcal{D}(\mathcal{F}, -)) \]

is a bijective map with inverse map

\[ \text{Nat}(\text{Mor}_\mathcal{C}(-, \mathcal{G}), \text{Mor}_\mathcal{D}(\mathcal{F}, -)) \ni \psi \mapsto \psi(\mathcal{G}, -)(1_{\mathcal{G}}) \in \text{Nat}(\mathcal{F}, \text{Id}_\mathcal{C}). \]

**Proof.** The natural transformation \( \mathcal{G} \cdot \Phi \cdot \) is defined as follows. Given \( C \in \mathcal{C}, D \in \mathcal{D} \) and \( f \in \text{Mor}_\mathcal{D}(\mathcal{F}(C), D) \) then let \((\mathcal{G} \cdot \Phi \cdot)(C, D)(f) := \mathcal{G}((f)\Phi(C)) : C \to \mathcal{G}\mathcal{F}(C) \to \mathcal{G}(D) \). It is easy to check the properties of a natural transformation.

Given \( \Phi \) then one obtains by composition of the two maps \( \mathcal{G}((1_{\mathcal{F}(C)})\Phi(C)) = \mathcal{G}\mathcal{F}(1_C)\Phi(C) \). Given \( \phi \) one obtains

\[
\begin{align*}
\mathcal{G}(f)(\phi(C, \mathcal{F}(C)))(1_{\mathcal{F}(C)}) &= \text{Mor}_\mathcal{C}(C, \mathcal{G}(f))\phi(C, \mathcal{F}(C))(1_{\mathcal{F}(C)}) \\
&= \phi(C, D)\text{Mor}_\mathcal{D}(\mathcal{F}(C), f)(1_{\mathcal{F}(C)}) = \phi(C, D)(f).
\end{align*}
\]

The second part of the lemma is proved similarly. \( \square \)

**Proposition 8.9.10.** Let

\[ \phi : \text{Mor}_\mathcal{D}(\mathcal{F}, -) \to \text{Mor}_\mathcal{C}(-, \mathcal{G}) \quad \text{and} \quad \psi : \text{Mor}_\mathcal{C}(-, \mathcal{G}) \to \text{Mor}_\mathcal{D}(\mathcal{F}, -) \]

be natural transformations with associated natural transformations (by Lemma 8.9.9)

\[ \Phi : \text{Id}_\mathcal{C} \to \mathcal{G}\mathcal{F} \text{ resp. } \Psi : \mathcal{F}\mathcal{G} \to \text{Id}_\mathcal{D}. \]

1) Then we have \( \phi \psi = \text{id}_{\text{Mor}(-, \mathcal{G})} \) if and only if \((\mathcal{G} \xrightarrow{\phi} \mathcal{G}\mathcal{F} \xrightarrow{\psi} \mathcal{G}) = \text{id}_\mathcal{G} \).

2) We also have \( \psi \phi = \text{id}_{\text{Mor}(\mathcal{F}, -)} \) if and only if \((\mathcal{F} \xrightarrow{\phi} \mathcal{F}\mathcal{G} \xrightarrow{\psi} \mathcal{F}) = \text{id}_\mathcal{F} \).

**Proof.** We get

\[
\begin{align*}
\mathcal{G}\Psi(D)\Phi\mathcal{G}(D) &= \mathcal{G}\Psi(D)\phi(\mathcal{G}(D), \mathcal{F}\mathcal{G}(D))(1_{\mathcal{F}\mathcal{G}(D)}) \\
&= \text{Mor}_\mathcal{C}(\mathcal{G}(D), \Psi(D))\phi(\mathcal{G}(D), \mathcal{F}\mathcal{G}(D))(1_{\mathcal{F}\mathcal{G}(D)}) \\
&= \phi(\mathcal{G}(D), D)\text{Mor}_\mathcal{D}(\mathcal{F}\mathcal{G}(D), \Psi(D))(1_{\mathcal{F}\mathcal{G}(D)}) \\
&= \phi(\mathcal{G}(D), D)(\Psi(D)) \\
&= \phi(\mathcal{G}(D), D)(\psi(\mathcal{G}(D), D))(1_{\mathcal{G}(D)}) \\
&= \phi(\mathcal{G}(D), D)(1_{\mathcal{G}(D)}).
\end{align*}
\]

Similarly we get

\[
\begin{align*}
\phi\psi(C, D)(f) &= \phi(C, D)\psi(C, D)(f) = \mathcal{G}(\Psi(D)\mathcal{F}(f))\Phi(C) \\
&= \mathcal{G}\Psi(D)\mathcal{G}\mathcal{F}(f)\Phi(C) = \mathcal{G}\Psi(D)\Phi\mathcal{G}(D)f. \quad \square
\end{align*}
\]

**Corollary 8.9.11.** Let \( \mathcal{F} : \mathcal{C} \to \mathcal{D} \) and \( \mathcal{G} : \mathcal{D} \to \mathcal{C} \) be functors. \( \mathcal{F} \) is leftadjoint to \( \mathcal{G} \) if and only if there are natural transformations \( \Phi : \text{Id}_\mathcal{C} \to \mathcal{G}\mathcal{F} \) and \( \Psi : \mathcal{F}\mathcal{G} \to \text{Id}_\mathcal{D} \)

such that \((\mathcal{G}\Psi)(\Phi\mathcal{G}) = \text{id}_\mathcal{C} \) and \((\Psi\mathcal{F})(\Phi) = \text{id}_\mathcal{D} \).

**Definition 8.9.12.** The natural transformations \( \Phi : \text{Id}_\mathcal{C} \to \mathcal{G}\mathcal{F} \) and \( \Psi : \mathcal{F}\mathcal{G} \to \text{Id}_\mathcal{D} \) given in 8.9.11 are called unit and counit resp. for the adjoint functors \( \mathcal{F} \) and \( \mathcal{G} \).
Problem 8.9.2. 1. Let \( R M_S \) be a bimodule. Show that the functor \( M \otimes_S - : sM \to R M \) is leftadjoint to \( \text{Hom}_R(M, -) : R M \to sM \). Determine the associated unit and counit.

b) Show that there is a natural isomorphism \( \text{Map}(A \times B, C) \cong \text{Map}(B, \text{Map}(A, C)) \). Determine the associated unit and counit.

c) Show that there is a natural isomorphism \( \mathbb{K} \text{-Alg}(\mathbb{K}G, A) \cong \text{Gr}(G, U(A)) \). Determine the associated unit and counit.

d) Show that there is a natural isomorphism \( \mathbb{K} \text{-Alg}(U(g), A) \cong \text{Lie-Alg}(g, A^L) \). Determine the corresponding leftadjoint functor and the associated unit and counit.

Definition 8.9.13. Let \( \mathcal{G} : \mathcal{D} \to \mathcal{C} \) be a covariant functor. \( \mathcal{G} \) generates a \((co-\)universal problem\) a follows:

Given \( C \in \mathcal{C} \). Find an object \( \mathcal{F}(C) \in \mathcal{D} \) and a morphism \( \iota : C \to \mathcal{G}(\mathcal{F}(C)) \) in \( \mathcal{C} \) such that there is a unique morphism \( \mathcal{g} : \mathcal{F}(C) \to D \) in \( \mathcal{D} \) for each object \( D \in \mathcal{D} \) and for each morphism \( f : C \to \mathcal{G}(D) \) in \( \mathcal{C} \) such that the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\mathcal{F}} & \mathcal{G}(\mathcal{F}(C)) \\
\downarrow{f} & & \downarrow{\mathcal{g}} \\
\mathcal{G}(D) & \xrightarrow{} & \mathcal{G}(D)
\end{array}
\]

commutes.

A pair \((\mathcal{F}(C), \iota)\) that satisfies the above conditions is called a \textit{universal solution} of the \((co-)universal problem\) defined by \( \mathcal{G} \) and \( C \).

Let \( \mathcal{F} : \mathcal{C} \to \mathcal{D} \) be a covariant functor. \( \mathcal{F} \) generates a \textit{universal problem} a follows:

Given \( D \in \mathcal{D} \). Find an object \( \mathcal{G}(D) \in \mathcal{C} \) and a morphism \( \nu : \mathcal{F}(\mathcal{G}(D)) \to D \) in \( \mathcal{D} \) such that there is a unique morphism \( \mathcal{g} : C \to \mathcal{G}(D) \) in \( \mathcal{C} \) for each object \( C \in \mathcal{C} \) and for each morphism \( f : \mathcal{F}(C) \to D \) in \( \mathcal{D} \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{F}(C) & \xrightarrow{} & \mathcal{G}(\mathcal{G}(D)) \\
\downarrow{\mathcal{F}(\iota)} & & \downarrow{\nu} \\
\mathcal{F}(\mathcal{G}(D)) & \xrightarrow{} & D
\end{array}
\]

commutes.

A pair \((\mathcal{G}(D), \nu)\) that satisfies the above conditions is called a \textit{universal solution} of the \((co-)universal problem\) defined by \( \mathcal{F} \) and \( D \).

Proposition 8.9.14. Let \( \mathcal{F} : \mathcal{C} \to \mathcal{D} \) be leftadjoint to \( \mathcal{G} : \mathcal{D} \to \mathcal{C} \). Then \( \mathcal{F}(C) \) and the unit \( \iota = \Phi(C) : C \to \mathcal{G}\mathcal{F}(C) \) form a \((co-)universal solution\) for the \((co-)universal problem\) defined by \( \mathcal{G} \) and \( C \).

Furthermore \( \mathcal{G}(D) \) and the counit \( \nu = \Psi(D) : \mathcal{F}\mathcal{G}(D) \to D \) form a universal solution for the universal problem defined by \( \mathcal{F} \) and \( D \).
9. ADJOURT FUNCTORS AND THE YONEDA LEMMA

PROOF. By Theorem 8.9.10 the morphisms \( \phi : \text{Mor}_D(\mathcal{F},-) \rightarrow \text{Mor}_C(-,-) \) and \( \psi : \text{Mor}_C(-,\mathcal{G}-) \rightarrow \text{Mor}_D(\mathcal{F},-) \) are inverses of each other. They are defined with unit and counit as \( \phi(C,D)(g) = \mathcal{G}(g)\Phi(C) \) resp. \( \psi(C,D)(f) = \Psi(D)\mathcal{F}(f) \). Hence for each \( f : C \rightarrow \mathcal{G}(D) \) there is a unique \( g : \mathcal{F}(C) \rightarrow D \) such that \( f = \phi(C,D)(g) = \mathcal{G}(g)\Phi(C) = \mathcal{G}(g) \).

The second statement follows analogously. \( \square \)

Remark 8.9.15. If \( \mathcal{G} : \mathcal{D} \rightarrow \mathcal{C} \) and \( C \in \mathcal{C} \) are given then the (co-)universal solution \( (\mathcal{F}(C),\iota : C \rightarrow \mathcal{G}(D)) \) can be considered as the best (co-)approximation of the object \( C \) in \( \mathcal{C} \) by an object \( D \) in \( \mathcal{D} \) with the help of a functor \( \mathcal{G} \). The object \( D \in \mathcal{D} \) turns out to be \( \mathcal{F}(C) \).

If \( \mathcal{F} : \mathcal{C} \rightarrow \mathcal{D} \) and \( D \in \mathcal{D} \) are given then the universal solution \( (\mathcal{G}(D),\nu : \mathcal{F}\mathcal{G}(D) \rightarrow D) \) can be considered as the best approximation of the object \( D \) in \( \mathcal{D} \) by an object \( C \) in \( \mathcal{C} \) with the help of a functor \( \mathcal{F} \). The object \( C \in \mathcal{C} \) turns out to be \( \mathcal{G}(D) \).

Proposition 8.9.16. Given \( \mathcal{G} : \mathcal{D} \rightarrow \mathcal{C} \). Assume that for each \( C \in \mathcal{C} \) the universal problem defined by \( \mathcal{G} \) and \( C \) is solvable. Then there is a leftadjoint functor \( \mathcal{F} : \mathcal{C} \rightarrow \mathcal{D} \) to \( \mathcal{G} \).

Given \( \mathcal{F} : \mathcal{C} \rightarrow \mathcal{D} \). Assume that for each \( D \in \mathcal{D} \) the universal problem defined by \( \mathcal{F} \) and \( D \) is solvable. Then there is a leftadjoint functor \( \mathcal{G} : \mathcal{D} \rightarrow \mathcal{C} \) to \( \mathcal{F} \).

PROOF. Assume that the (co-)universal problem defined by \( \mathcal{G} \) and \( C \) is solved by \( \iota : C \rightarrow \mathcal{F}(C) \). Then the map \( \text{Mor}_C(C,\mathcal{G}(D)) \ni f \mapsto g \in \text{Mor}_D(\mathcal{F}(C),D) \) with \( \mathcal{G}(g)\iota = f \) is bijective. The inverse map is given by \( g \mapsto \mathcal{G}(g)\iota \). This is a natural transformation since the diagram

\[
\begin{array}{ccc}
\text{Mor}_D(\mathcal{F}(C),D) & \rightarrow & \text{Mor}_C(C,\mathcal{G}(D)) \\
\downarrow \text{Mor}_D(\mathcal{F}(C),h) & & \downarrow \text{Mor}_C(C,\mathcal{G}(h)) \\
\text{Mor}_D(\mathcal{F}(C),D') & \rightarrow & \text{Mor}_C(C,\mathcal{G}(D')) \\
\downarrow \text{Mor}_D(\mathcal{F}(C),g) & & \downarrow \text{Mor}_C(C,\mathcal{G}(g)) \\
\end{array}
\]

commutes for each \( h \in \text{Mor}_D(D,D') \). In fact we have

\[
\text{Mor}_C(C,\mathcal{G}(h))(\mathcal{G}(g)\iota) = \mathcal{G}(h)\mathcal{G}(g)\iota = \mathcal{G}(h)\iota = \mathcal{G}(\text{Mor}_C(\mathcal{F}(C),h)(g))\iota.
\]

Hence for all \( C \in \mathcal{C} \) the functor \( \text{Mor}_C(-,\mathcal{G}(-)) : \mathcal{D} \rightarrow \text{Set} \) induced by the bifunctor \( \text{Mor}_C(-,\mathcal{G}(-)) : \mathcal{C}^{op} \times \mathcal{D} \rightarrow \text{Set} \) is representable. By Theorem 8.9.5 there is a functor \( \mathcal{F} : \mathcal{C} \rightarrow \mathcal{D} \) such that \( \text{Mor}_C(-,\mathcal{G}(-)) \cong \text{Mor}_D(\mathcal{F}(-),-) \).

The second statement follows analogously. \( \square \)

Remark 8.9.17. One can characterize the properties that \( \mathcal{G} : \mathcal{D} \rightarrow \mathcal{C} \) (resp. \( \mathcal{F} : \mathcal{C} \rightarrow \mathcal{D} \)) must have in order to possess a left-(right-)adjoint functor. One of the essential properties for this is that \( \mathcal{G} \) preserves limits (hence direct products and difference kernels).