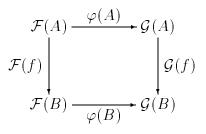
## CHAPTER 8

## Toolbox

## 3. Natural Transformations

**Definition 8.3.1.** Let  $\mathcal{F} : \mathcal{C} \to \mathcal{D}$  and  $\mathcal{G} : \mathcal{C} \to \mathcal{D}$  be two functors. A *natural transformation* or a *functorial morphism*  $\varphi : \mathcal{F} \to \mathcal{G}$  is a family of morphisms  $\{\varphi(A) : \mathcal{F}(A) \to \mathcal{G}(A) | A \in \mathcal{C}\}$  such that the diagram



commutes for all  $f : A \to B$  in  $\mathcal{C}$ , i.e.  $\mathcal{G}(f)\varphi(A) = \varphi(B)\mathcal{F}(f)$ .

**Lemma 8.3.2.** Given covariant functors  $\mathcal{F} = \mathrm{Id}_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$  and  $\mathcal{G} = \mathrm{Mor}_{\mathbf{Set}}(\mathrm{Mor}_{\mathbf{Set}}(-, A), A) : \mathbf{Set} \to \mathbf{Set}$  for a set A. Then  $\varphi : \mathcal{F} \to \mathcal{G}$  with

$$\varphi(B): B \ni b \mapsto (\operatorname{Mor}_{\operatorname{\mathbf{Set}}}(B, A) \ni f \mapsto f(b) \in A) \in \mathcal{G}(B)$$

is a natural transformation.

**PROOF.** Given  $g: B \to C$ . Then the following diagram commutes

since

$$\begin{aligned} \varphi(C)\mathcal{F}(g)(b)(f) &= \varphi(C)g(b)(f) = fg(b) = \varphi(B)(b)(fg) \\ &= [\varphi(B)(b)\mathrm{Mor}_{\mathbf{Set}}(g,A)](f) = [\mathrm{Mor}_{\mathbf{Set}}(\mathrm{Mor}_{\mathbf{Set}}(g,A),A)\varphi(A)(b)](f). \end{aligned}$$

**Lemma 8.3.3.** Let  $f : A \to B$  be a morphism in  $\mathcal{C}$ . Then  $\operatorname{Mor}_{\mathcal{C}}(f, -) : \operatorname{Mor}_{\mathcal{C}}(B, -) \to \operatorname{Mor}_{\mathcal{C}}(A, -)$  given by  $\operatorname{Mor}_{\mathcal{C}}(f, C) : \operatorname{Mor}_{\mathcal{C}}(B, C) \ni g \mapsto gf \in \operatorname{Mor}_{\mathcal{C}}(A, C)$  is a natural transformation of covariant functors.

Let  $f : A \to B$  be a morphism in C. Then  $Mor_{\mathcal{C}}(-, f) : Mor_{\mathcal{C}}(-, A) \to Mor_{\mathcal{C}}(-, B)$ given by  $Mor_{\mathcal{C}}(C, f) : Mor_{\mathcal{C}}(C, A) \ni g \mapsto fg \in Mor_{\mathcal{C}}(C, B)$  is a natural transformation of contravariant functors.

## 8. TOOLBOX

**PROOF.** Let  $h: C \to C'$  be a morphism in  $\mathcal{C}$ . Then the diagrams

 $\begin{array}{c|c} \operatorname{Mor}_{\mathcal{C}}(B,C) & \xrightarrow{\operatorname{Mor}_{\mathcal{C}}(f,C)} \operatorname{Mor}_{\mathcal{C}}(A,C) \\ & & & & & \\ \operatorname{Mor}_{\mathcal{C}}(B,h) & & & & & \\ \operatorname{Mor}_{\mathcal{C}}(B,C') & & & & \\ \operatorname{Mor}_{\mathcal{C}}(f,C') & & & \operatorname{Mor}_{\mathcal{C}}(A,C') \end{array}$ 

and

commute.

**Remark 8.3.4.** The composition of two natural transformations is again a natural transformation. The identity  $id_{\mathcal{F}}(A) := 1_{\mathcal{F}(A)}$  is also a natural transformation.

**Definition 8.3.5.** A natural transformation  $\varphi : \mathcal{F} \to \mathcal{G}$  is called a *natural iso*morphism if there exists a natural transformation  $\psi : \mathcal{G} \to \mathcal{F}$  such that  $\varphi \circ \psi = \mathrm{id}_{\mathcal{G}}$ and  $\psi \circ \varphi = \mathrm{id}_{\mathcal{F}}$ . The natural transformation  $\psi$  is uniquely determined by  $\varphi$ . We write  $\varphi^{-1} := \psi$ .

A functor  $\mathcal{F}$  is said to be *isomorphic* to a functor  $\mathcal{G}$  if there exists a natural isomorphism  $\varphi : \mathcal{F} \to \mathcal{G}$ .

**Problem 8.3.1.** 1. Let  $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$  be functors. Show that a natural transformation  $\varphi : \mathcal{F} \to \mathcal{G}$  is a natural isomorphism if and only if  $\varphi(A)$  is an isomorphism for all objects  $A \in \mathcal{C}$ .

2. Let  $(A \times B, p_A, p_B)$  be the product of A and B in C. Then there is a natural isomorphism

$$Mor(-, A \times B) \cong Mor_{\mathcal{C}}(-, A) \times Mor_{\mathcal{C}}(-, B).$$

3. Let  $\mathcal{C}$  be a category with finite products. For each object A in  $\mathcal{C}$  show that there exists a morphism  $\Delta_A : A \to A \times A$  satisfying  $p_1 \Delta_A = 1_A = p_2 \Delta_A$ . Show that this defines a natural transformation. What are the functors?

4. Let  $\mathcal{C}$  be a category with finite products. Show that there is a bifunctor  $-\times -: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$  such that  $(-\times -)(A, B)$  is the object of a product of A and B. We denote elements in the image of this functor by  $A \times B := (-\times -)(A, B)$  and similarly  $f \times g$ .

5. With the notation of the preceding problem show that there is a natural transformation  $\alpha(A, B, C) : (A \times B) \times C \cong A \times (B \times C)$ . Show that the diagram

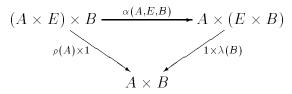
8

(coherence or constraints)

$$\begin{array}{cccc} ((A \times B) \times C) \times D & \xrightarrow{\alpha(A,B,C) \times 1} & (A \times (B \times C)) \times D & \xrightarrow{\alpha(A,B \times C,D)} & A \times ((B \times C) \times D) \\ & & & & & \\ & & & & \\ & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ &$$

commutes.

6. With the notation of the preceding problem show that there are a natural transformations  $\lambda(A) : E \times A \to A$  and  $\rho(A) : A \times E \to A$  such that the diagram (coherence or constraints)



**Definition 8.3.6.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. A covariant functor  $\mathcal{F} : \mathcal{C} \to \mathcal{D}$  is called an *equivalence of categories* if there exists a covariant functor  $\mathcal{G} : \mathcal{D} \to \mathcal{C}$  and natural isomorphisms  $\varphi : \mathcal{GF} \cong \mathrm{Id}_{\mathcal{C}}$  and  $\psi : \mathcal{FG} \cong \mathrm{Id}_{\mathcal{D}}$ .

A contravariant functor  $\mathcal{F} : \mathcal{C} \to \mathcal{D}$  is called a *duality of categories* if there exists a contravariant functor  $\mathcal{G} : \mathcal{D} \to \mathcal{C}$  and natural isomorphisms  $\varphi : \mathcal{GF} \cong \mathrm{Id}_{\mathcal{C}}$  and  $\psi : \mathcal{FG} \cong \mathrm{Id}_{\mathcal{D}}$ .

A category  $\mathcal{C}$  is said to be *equivalent* to a category  $\mathcal{D}$  if there exists an equivalence  $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ . A category  $\mathcal{C}$  is said to be *dual* to a category  $\mathcal{D}$  if there exists a duality  $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ .

**Problem 8.3.2.** 1. Show that the dual category  $\mathcal{C}^{op}$  is dual to the category  $\mathcal{C}$ .

2. Let  $\mathcal{D}$  be a category dual to the category  $\mathcal{C}$ . Show that  $\mathcal{D}$  is equivalent to the dual category  $\mathcal{C}^{op}$ .

3. Let  $\mathcal{F} : \mathcal{C} \to \mathcal{D}$  be an equivalence with respect to  $\mathcal{G} : \mathcal{D} \to \mathcal{C}, \varphi : \mathcal{GF} \cong \mathrm{Id}_{\mathcal{C}},$ and  $\psi : \mathcal{FG} \cong \mathrm{Id}_{\mathcal{D}}$ . Show that  $\mathcal{G} : \mathcal{D} \to \mathcal{C}$  is an equivalence. Show that  $\mathcal{G}$  is uniquely determined by  $\mathcal{F}$  up to a natural isomorphism.