CHAPTER 8

Toolbox

1. Categories

Definition 8.1.1. Let $\mathcal{C}$ consist of
1. a class $\text{Ob} \mathcal{C}$ whose elements $A, B, C, \ldots \in \text{Ob} \mathcal{C}$ are called \textit{objects},
2. a family $\{\text{Mor}_\mathcal{C}(A, B) \mid A, B \in \text{Ob} \mathcal{C}\}$ of mutually disjoint sets whose elements $f, g, \ldots \in \text{Mor}_\mathcal{C}(A, B)$ are called \textit{morphisms}, and
3. a family $\{\text{Mor}_\mathcal{C}(A, B) \times \text{Mor}_\mathcal{C}(B, C) \ni (f, g) \mapsto gf \in \text{Mor}_\mathcal{C}(A, C) \mid A, B, C \in \text{Ob} \mathcal{C}\}$ of maps called \textit{compositions}. 

$\mathcal{C}$ is called a \textit{category} if the following axioms hold for $\mathcal{C}$
1. Associative Law:
   \[ \forall A, B, C, D \in \text{Ob} \mathcal{C}, f \in \text{Mor}_\mathcal{C}(A, B), g \in \text{Mor}_\mathcal{C}(B, C), h \in \text{Mor}_\mathcal{C}(C, D) : \]
   \[ h(gf) = (hg)f; \]

2. Identity Law:
   \[ \forall A \in \text{Ob} \mathcal{C}, \exists 1_A \in \text{Mor}_\mathcal{C}(A, A) \forall B, C \in \text{Ob} \mathcal{C}, \forall f \in \text{Mor}_\mathcal{C}(A, B), \forall g \in \text{Mor}_\mathcal{C}(C, A) : \]
   \[ 1_A g = g \quad \text{and} \quad f 1_A = f. \]

Examples 8.1.2. 1. The category of sets $\text{Set}$. 
2. The categories of $R$-modules $\text{R-Mod}$, $k$-vector spaces $k$-$\text{Vec}$ or $k$-$\text{Mod}$, groups $\text{Gr}$, abelian groups $\text{Ab}$, monoids $\text{Mon}$, commutative monoids $\text{cMon}$, rings $\text{Ri}$, fields $\text{Fld}$, topological spaces $\text{Top}$. 

Since modules are highly important for all what follows, we recall the definition and some basic properties.

Definition and Remark 8.1.3. Let $R$ be a ring (always associative with unit). A \textit{left $R$-module} $RM$ is an (additively written) abelian group $M$ together with an operation $R \times M \ni (r, m) \mapsto rm \in M$ such that
1. $(rs)m = r(sm)$,
2. $(r + s)m = rm + sm$,
3. $r(m + m') = rm + rm'$,
4. $1m = m$
for all $r, s \in R$, $m, m' \in M$.

Each abelian group is a $\mathbb{Z}$-module in a unique way.
A homomorphism of left $R$-modules $f : _RM \rightarrow _RN$ is a group homomorphism such that $f(rm) = rf(m)$.

Analogously we define right $R$-modules $M_R$ and their homomorphisms.

We denote by $\text{Hom}_R(M,N)$ the set of homomorphisms of left $R$-modules $_RM$ and $_RN$. Similarly $\text{Hom}_R(M,N)$ denotes the set of homomorphisms of right $R$-modules $M_R$ and $N_R$. Both sets are abelian groups by $(f + g)(m) := f(m) + g(m)$.

For arbitrary categories we adopt many of the customary notations.

**Notation 8.1.4.** $f \in \text{Mor}_C(A,B)$ will be written as $f : A \rightarrow B$ or $A \xrightarrow{f} B$. $A$ is called the domain, $B$ the range of $f$.

The composition of two morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$ is written as $gf : A \rightarrow C$ or as $g \circ f : A \rightarrow C$.

**Definition and Remark 8.1.5.** A morphism $f : A \rightarrow B$ is called an isomorphism if there exists a morphism $g : B \rightarrow A$ in $C$ such that $fg = 1_B$ and $gf = 1_A$. The morphism $g$ is uniquely determined by $f$ since $g = gf = g$.

An object $A$ is said to be isomorphic to an object $B$ if there exists an isomorphism $f : A \rightarrow B$. If $f$ is an isomorphism then $f$ is $f^{-1} : B \rightarrow A$. If $f : A \rightarrow B$ and $g : B \rightarrow C$ are isomorphisms in $C$ then so is $gf : A \rightarrow C$. We have $(f^{-1})^{-1} = f$ and $(gf)^{-1} = f^{-1}g^{-1}$. The relation of being isomorphic between objects is an equivalence relation.

**Example 8.1.6.** In the categories $\text{Set}$, $R\text{-Mod}$, $k\text{-Vec}$, $\text{Gr}$, $\text{Ab}$, $\text{Mon}$, $\text{cMon}$, $\text{Ri}$, $\text{Fld}$ the isomorphisms are exactly those morphisms which are bijective as set maps.

In $\text{Top}$ the set $M = \{a, b\}$ with $\mathcal{X}_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and with $\mathcal{X}_2 = \{\emptyset, M\}$ defines two different topological spaces. The map $f = \text{id} : (M, \mathcal{X}_1) \rightarrow (M, \mathcal{X}_2)$ is bijective and continuous. The inverse map, however, is not continuous, hence $f$ is no isomorphism (homeomorphism).

Many well known concepts can be defined for arbitrary categories. We are going to apply some of them. Here are two examples.

**Definition 8.1.7.** 1. A morphism $f : A \rightarrow B$ is called a monomorphism if $\forall C \in \text{Ob} C$, $\forall g, h \in \text{Mor}_C(C, A)$:

$$fg = fh \Rightarrow g = h \quad (f \text{ is left cancellable}).$$

2. A morphism $f : A \rightarrow B$ is called an epimorphism if $\forall C \in \text{Ob} C$, $\forall g, h \in \text{Mor}_C(B, C)$:

$$gf = hf \Rightarrow g = h \quad (f \text{ is right cancellable}).$$

**Definition 8.1.8.** Given $A, B \in C$. An object $A \times B$ in $C$ together with morphisms $p_A : A \times B \rightarrow A$ and $p_B : A \times B \rightarrow B$ is called a (categorical) product of $A$ and $B$ if for every object $T \in C$ and every pair of morphisms $f : T \rightarrow A$ and
$g : T \to B$ there exists a unique morphism $(f, g) : T \to A \times B$ such that the diagram

\[
\begin{array}{ccc}
T & \xrightarrow{(f, g)} & A \times B \\
\downarrow f & & \downarrow p_B \\
A & \xrightarrow{p_A} & A
\end{array}
\]

commutes.

An object $E \in C$ is called a final object if for every object $T \in C$ there exists a unique morphism $e : T \to E$ (i.e. $\text{Mor}_C(T, E)$ consists of exactly one element).

A category $C$ which has a product for any two objects $A$ and $B$ and which has a final object is called a category with finite products.

**Remark 8.1.9.** If the product $(A \times B, p_A, p_B)$ of two objects $A$ and $B$ in $C$ exists then it is unique up to isomorphism.

If the final object $E$ in $C$ exists then it is unique up to isomorphism.

**Problem 8.1.1.** Let $C$ be a category with finite products. Give a definition of a product of a family $A_1, \ldots, A_n$ ($n \geq 0$). Show that products of such families exist in $C$.

**Definition and Remark 8.1.10.** Let $C$ be a category. Then $C^\text{op}$ with the following data $\text{Ob} C^\text{op} := \text{Ob} C$, $\text{Mor}_{C^\text{op}}(A, B) := \text{Mor}_C(B, A)$, and $f \circ_{C^\text{op}} g := g \circ f$ defines a new category, the dual category to $C$.

**Remark 8.1.11.** Any notion expressed in categorical terms (with objects, morphisms, and their composition) has a dual notion, i.e. the given notion in the dual category.

Monomorphisms $f$ in the dual category $C^\text{op}$ are epimorphisms in the original category $C$ and conversely. A final objects $I$ in the dual category $C^\text{op}$ is an initial object in the original category $C$.

**Definition 8.1.12.** The coproduct of two objects in the category $C$ is defined to be a product of the objects in the dual category $C^\text{op}$.

**Remark 8.1.13.** Equivalent to the preceding definition is the following definition.

Given $A, B \in C$. An object $A \coprod B$ in $C$ together with morphisms $j_A : A \to A \coprod B$ and $j_B : B \to A \coprod B \to B$ is a (categorical) coproduct of $A$ and $B$ if for every object $T \in C$ and every pair of morphisms $f : A \to T$ and $g : B \to T$ there exists a unique morphism $[f, g] : A \coprod B \to T$ such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{j_A} & A \coprod B & \xleftarrow{j_B} & B \\
\downarrow f & & \downarrow [j_A, j_B] & & \downarrow g \\
T & & & & T
\end{array}
\]
commutes.

The category $C$ is said to have finite coproducts if $C^{\text{op}}$ is a category with finite products. In particular coproducts are unique up to isomorphism.