CHAPTER 3

Hopf Algebras, Algebraic, Formal, and Quantum Groups
5. The coalgebra coend

**Proposition 3.5.1.** Let $C$ be a monoidal category and $\omega : D \to C$ be a diagram in $C$. Assume that there is a universal object $\text{coend}(\omega)$ and natural transformation $\delta : \omega \to \omega \otimes \text{coend}(\omega)$.

Then there is exactly one coalgebra structure on $\text{coend}(\omega)$ such that the diagrams

$$
\begin{array}{ccc}
\omega & \xrightarrow{\delta} & \omega \otimes \text{coend}(\omega) \\
\downarrow \delta & & \downarrow (1 \otimes \Delta) \\
\omega \otimes \text{coend}(\omega) & \xrightarrow{\Delta \otimes 1} & \omega \otimes \text{coend}(\omega) \otimes \text{coend}(\omega)
\end{array}
$$

and

$$
\begin{array}{ccc}
\omega & \xrightarrow{\delta} & \omega \otimes \text{coend}(\omega) \\
\downarrow \varepsilon & & \downarrow (1 \otimes \epsilon) \\
\omega \otimes I
\end{array}
$$

commute.

**Proof.** Because of the universal property of $\text{coend}(\omega)$ there are structure morphisms $\Delta : \text{coend}(\omega) \to \text{coend}(\omega) \otimes \text{coend}(\omega)$ and $\epsilon : \text{coend}(\omega) \to I$. This implies the coalgebra property similar to the proof of Corollary 3.3.8. \qed

Observe that by this construction all objects and all morphisms of the diagram $\omega : D \to C_0 \subseteq C$ are comodules or morphisms of comodules over the coalgebra $\text{coend}(\omega)$. In fact $C := \text{coend}(\omega)$ is the universal coalgebra over which the given diagram becomes a diagram of comodules.

**Corollary 3.5.2.** Let $(D, \omega)$ be a diagram $C$ with objects in $C_0$. Then all objects of the diagram are comodules over the coalgebra $C := \text{coend}(\omega)$ and all morphisms are morphisms of comodules. If $D$ is another coalgebra and all objects of the diagram are $D$-comodules by $\varphi(X) : \omega(X) \to \omega(X) \otimes D$ and all morphisms of the diagram are morphisms of $D$-comodules then there exists a unique morphism of coalgebras $\tilde{\varphi} : \text{coend}(\omega) \to D$ such that the diagram

$$
\begin{array}{ccc}
\omega & \xrightarrow{\varphi} & \omega \otimes \text{coend}(\omega) \\
\downarrow \tilde{\varphi} & & \downarrow (1 \otimes \tilde{\varphi}) \\
\omega \otimes D
\end{array}
$$

commutes.

**Proof.** The morphisms $\varphi(X) : \omega(X) \to \omega(X) \otimes D$ define a natural transformation since all morphisms of the diagram are morphisms of comodules. So the existence
and the uniqueness of a morphism $\bar{\varphi} : \text{coend}(\omega) \to D$ is clear. The only thing to show is that this is a morphism of coalgebras. This follows from the universal property of $C = \text{coend}(\omega)$ and the diagram

$$
\begin{array}{ccc}
\omega & \xrightarrow{\delta} & \omega \otimes C \\
\downarrow & & \downarrow \delta \\
\omega \otimes C & \xrightarrow{\delta \otimes 1} & \omega \otimes C \otimes C \\
\downarrow & & \downarrow 1 \otimes \Delta \\
\omega & \xrightarrow{\varphi} & \omega \otimes D \\
\downarrow & & \downarrow 1 \otimes \varphi \\
\omega \otimes D & \xrightarrow{\varphi \otimes 1} & \omega \otimes D \otimes D \\
\end{array}
$$

where the right side of the cube commutes by the universal property. Similarly we get that $\bar{\varphi}$ preserves the counit since the following diagram commutes

$$
\begin{array}{ccc}
\omega & \xrightarrow{\delta} & \omega \otimes C \\
\downarrow & & \downarrow 1 \\
\omega & \xrightarrow{\varphi} & \omega \otimes D \\
\downarrow & & \downarrow 1 \otimes \varphi \\
\omega \otimes D & \xrightarrow{\varphi \otimes 1} & \omega \otimes D \otimes D \\
\end{array}
$$

$\square$