CHAPTER 3

Hopf Algebras, Algebraic, Formal, and Quantum Groups
3. Dual Objects

At the end of the first section in Corollary 3.1.15 we saw that the dual of an $H$-module can be constructed. We did not show the corresponding result for comodules. In fact such a construction for comodules needs some finiteness conditions. With this restriction the notion of a dual object can be introduced in an arbitrary monoidal category.

**Definition 3.3.1.** Let $(C, \otimes)$ be a monoidal category $M \in C$ be an object. An object $M^* \in C$ together with a morphism $ev : M^* \otimes M \rightarrow I$ is called a *left dual* for $M$ if there exists a morphism $db : I \rightarrow M \otimes M^*$ in $C$ such that

\[
(M \xrightarrow{\text{db} \otimes 1} M \otimes M^* \otimes M \xrightarrow{1 \otimes ev} M) = 1_M
\]

\[
(M^* \xrightarrow{1 \otimes \text{db}} M^* \otimes M \otimes M^* \xrightarrow{ev \otimes 1} M^*) = 1_{M^*}.
\]

A monoidal category is called *left rigid* if each object $M \in C$ has a left dual.

Symmetrically we define: an object $^*M \in C$ together with a morphism $ev : M \otimes ^*M \rightarrow I$ is called a *right dual* for $M$ if there exists a morphism $db : I \rightarrow ^*M \otimes M$ in $C$ such that

\[
(M \xrightarrow{1 \otimes \text{db}} M \otimes ^*M \otimes M \xrightarrow{ev \otimes 1} M) = 1_M
\]

\[
(^*M \xrightarrow{\text{db} \otimes 1} ^*M \otimes M \otimes M \xrightarrow{1 \otimes ev} M^*) = 1_{M^*}.
\]

A monoidal category is called *right rigid* if each object $M \in C$ has a left dual.

The morphisms $ev$ and $db$ are called the *evaluation* and the *dual basis*.

**Remark 3.3.2.** If $(M^*, ev)$ is a left dual for $M$ then obviously $(M, ev)$ is a right dual for $M^*$ and conversely. One uses the same morphism $db : I \rightarrow M \otimes M^*$.

**Lemma 3.3.3.** Let $(M^*, ev)$ be a left dual for $M$. Then there is a natural isomorphism

\[
\text{Mor}_C(- \otimes M, -) \cong \text{Mor}_C(-, - \otimes M^*),
\]

i.e. the functor $- \otimes M : C \rightarrow C$ is left adjoint to the functor $- \otimes M^* : C \rightarrow C$.\]

**Proof.** We give the unit and the counit of the pair of adjoint functors. We define $\Phi(A) := 1_A \otimes \text{db} : A \rightarrow A \otimes M \otimes M^*$ and $\Psi(B) := 1_B \otimes ev : B \otimes M^* \otimes M \rightarrow B$. These are obviously natural transformations. We have commutative diagrams

\[
\begin{array}{ccc}
A \otimes M & \xrightarrow{\Phi(A)} & A \otimes M \otimes M^* \otimes M \\
\downarrow{1_A \otimes \text{db} \otimes 1_M} & & \downarrow{1_A \otimes 1_M \otimes ev} \\
A \otimes M \otimes M^* \otimes M & \xrightarrow{\Psi(A)} & A \otimes M \\
\end{array}
\]

and

\[
\begin{array}{ccc}
B \otimes M^* & \xrightarrow{\Phi(B)} & B \otimes M^* \otimes M \otimes M^* \\
\downarrow{1_B \otimes 1_M \otimes \text{db}} & & \downarrow{1_B \otimes ev \otimes 1_M} \\
B \otimes M^* \otimes M \otimes M^* & \xrightarrow{\Psi(B)} & B \otimes M^* \\
\end{array}
\]

thus the Lemma has been proved by Corollary A.9.11. \qed

The converse holds as well. If $- \otimes M$ is left adjoint to $- \otimes M^*$ then the unit $\Phi$ gives a morphism $\text{db} := \Phi(I) : I \rightarrow M \otimes M^*$ and the counit $\Psi$ gives a morphism $ev := \Psi(I) : M^* \otimes M \rightarrow I$ satisfying the required properties. Thus we have
Corollary 3.3.4. If $- \otimes M : C \to C$ is left adjoint to $- \otimes M^* : C \to C$ then $M^*$ is a left dual for $M$.

Corollary 3.3.5. $(M^*, ev)$ is a left dual for $M$ if and only if there is a natural isomorphism
\[
\text{Mor}_C(M^* \otimes -, -) \cong \text{Mor}_C(-, M \otimes -),
\]
i.e. the functor $M^* \otimes - : C \to C$ is left adjoint to the functor $M \otimes - : C \to C$. The natural isomorphism is given by
\[
(f : M^* \otimes N \to P) \mapsto ((1_M \otimes f)(\text{db} \otimes 1_N) : N \to M \otimes M^* \otimes N \to M \otimes P)
\]
and
\[
(g : N \to M \otimes P) \mapsto ((ev \otimes 1_P)(1_{M^*} \otimes g) : M^* \otimes N \to M^* \otimes M \otimes P \to P).
\]

Proof. We have a natural isomorphism
\[
\text{Mor}_C(M^* \otimes -, -) \cong \text{Mor}_C(-, M \otimes -),
\]
iff $(M, ev)$ is a right dual for $M^*$ (as a symmetric statement to Lemma 3.3.3) iff $(M^*, ev)$ is a left dual for $M$.

Corollary 3.3.6. If $M$ has a left dual then this is unique up to isomorphism.

Proof. Let $(M^*, ev)$ and $(M', ev')$ be left duals for $M$. Then the functors $- \otimes M^*$ and $- \otimes M'$ are isomorphic by Lemma A.9.7. In particular we have $M^* \cong I \otimes M^* \cong I \otimes M' \cong M'$. If we consider the construction of the isomorphism then we get in particular that $(ev' \otimes 1)(1 \otimes \text{db}) : M^* \to M^* \otimes M \otimes M^* \to M^*$ is the given isomorphism.

Problem 3.3.1. Let $(M^*, ev)$ be a left dual for $M$. Then there is a unique morphism $\text{db} : I \to M \otimes M^*$ satisfying the conditions of Definition 3.3.1.

Definition 3.3.7. Let $(M^*, ev_M)$ and $(N^*, ev_N)$ be left duals for $M$ resp. $N$. For each morphism $f : M \to N$ we define the transposed morphism
\[
(f^* : N^* \to M^* := (N^* \xrightarrow{1 \otimes \text{db}} N^* \otimes M \otimes M^* \xrightarrow{1 \otimes f \otimes 1} N^* \otimes N \otimes M^* \xrightarrow{ev_N \otimes 1} M^*).
\]

With this definition we get that the left dual is a contravariant functor, since we have

Lemma 3.3.8. Let $(M^*, ev_M)$, $(N^*, ev_N)$, and $(P^*, ev_P)$ be left duals for $M$, $N$ and $P$ respectively.
1. We have $(1_M)^* = 1_{M^*}$.
2. If $f : M \to N$ and $g : N \to P$ are given then $(gf)^* = f^*g^*$ holds.

Proof. 1. $(1_M)^* = (ev \otimes 1)(1 \otimes 1 \otimes 1)(1 \otimes \text{db}) = 1_{M^*}$. 

2. The following diagram commutes

\[
M \xrightarrow{\mathrm{db}_N \otimes 1} N \otimes N^* \otimes M \\
\downarrow f \quad \quad \quad \downarrow 1 \circ 1 \circ f \\
N \xrightarrow{\mathrm{db}_N \otimes 1} N \otimes N^* \otimes N \xrightarrow{1 \circ \mathrm{ev}_N} N \\
\downarrow g \circ 1 \circ 1 \\
P \otimes N^* \otimes N \xrightarrow{1 \circ \mathrm{ev}_N} P
\]

Hence we have \( gf = (1 \otimes \mathrm{ev}_N)(g \otimes 1 \otimes f)(\mathrm{db}_N \otimes 1) \). Thus the following diagram commutes

\[
P^* \xrightarrow{1 \circ \mathrm{db}} P^* \otimes N \otimes N^* \xrightarrow{1 \circ g \circ 1} P^* \otimes P \otimes N^* \\
P^* \otimes M \otimes M^* \xrightarrow{1 \circ \mathrm{db} \circ 1} P^* \otimes N \otimes N^* \otimes M \otimes M^* \\
P^* \otimes P \otimes M^* \xrightarrow{1 \circ \mathrm{ev} \circ 1} P^* \otimes P \otimes N^* \otimes N \otimes M^* \\
M^* \xrightarrow{1 \circ \mathrm{ev}} N^* \otimes N \otimes M^* \xrightarrow{1 \circ f \circ 1} N^* \otimes M \otimes M^* \\
\]

**Problem 3.3.2.**

1. In the category of \( \mathbb{N} \)-graded vector spaces determine all objects \( M \) that have a left dual.

2. In the category of chain complexes \( \mathbb{K} \text{-Comp} \) determine all objects \( M \) that have a left dual.

3. In the category of cochain complexes \( \mathbb{K} \text{-Cocomp} \) determine all objects \( M \) that have a left dual.

4. Let \((M^*, \mathrm{ev})\) be a left dual for \( M \). Show that \( \mathrm{db} : I \to M \otimes M^* \) is uniquely determined by \( M, M^*, \) and \( \mathrm{ev} \). (Uniqueness of the dual basis.)

5. Let \((M^*, \mathrm{ev})\) be a left dual for \( M \). Show that \( \mathrm{ev} : M^* \otimes M \to I \) is uniquely determined by \( M, M^*, \) and \( \mathrm{db} \).
**Corollary 3.3.9.** Let \( M, N \) have the left duals \((M^*, \text{ev}_M)\) and \((N^*, \text{ev}_N)\) and let \( f : M \to N \) be a morphism in \( C \). Then the following diagram commutes

\[
\begin{array}{c}
I & \xrightarrow{\text{db}_M} & M \otimes M^* \\
\downarrow \text{db}_N & & \downarrow f \otimes 1 \\
N \otimes N^* & \xrightarrow{1 \otimes f^*} & N \otimes M^*. \\
\end{array}
\]

**Proof.** The following diagram commutes

\[
\begin{array}{c}
M & \xrightarrow{\text{db}_1} & N \otimes N^* \otimes M \\
\downarrow f & & \downarrow 1 \otimes 1 \otimes f \\
N & \xrightarrow{\text{db}_1} & N \otimes N^* \otimes N \\
\downarrow 1 & & \downarrow 1 \otimes \text{ev} \\
& & N. \\
\end{array}
\]

This implies \((f \otimes 1_{M^*}) \text{db}_M = ((1_N \otimes \text{ev}_N)(1_N \otimes 1_{N^*} \otimes f)(\text{db}_N \otimes 1_M) \otimes 1_{M^*}) \text{db}_M = (1_N \otimes \text{ev}_N \otimes 1_{M^*})(1_N \otimes 1_N \otimes f \otimes 1_{M^*})(\text{db}_N \otimes 1_M \otimes 1_{M^*}) \text{db}_M = (1_N \otimes \text{ev}_N \otimes 1_{M^*})(1_N \otimes 1_N \otimes \text{db}_M) \text{db}_N = (1_N \otimes (\text{ev}_N \otimes 1_{M^*})(1_N \otimes f \otimes 1_{M^*}))(1_N \otimes \text{db}_M)) \text{db}_N = (1_N \otimes f^* \otimes \text{db}_N). \]

**Corollary 3.3.10.** Let \( M, N \) have the left duals \((M^*, \text{ev}_M)\) and \((N^*, \text{ev}_N)\) and let \( f : M \to N \) be a morphism in \( C \). Then the following diagram commutes

\[
\begin{array}{c}
N^* \otimes M & \xrightarrow{f \otimes 1} & M^* \otimes M \\
\downarrow 1 \otimes f & & \downarrow \text{ev}_M \\
N^* \otimes N & \xrightarrow{\text{ev}_N} & I. \\
\end{array}
\]

**Proof.** This statement follows immediately from the symmetry of the definition of a left dual.

**Example 3.3.11.** Let \( M \in {}_R \mathcal{M}_R \) be an \( R\)-\( R \)-bimodule. Then \( \text{Hom}_R(M, R) \) is an \( R\)-\( R \)-bimodule by \((rf)(x) = rf(sx)\). Furthermore we have the morphism \( \text{ev} : \text{Hom}_R(M, R) \otimes_R M \to R \) defined by \( \text{ev}(f \otimes_R m) = f(m) \).

(Dual Basis Lemma:) A module \( M \in \mathcal{M}_R \) is called *finitely generated and projective* if there are elements \( m_1, \ldots, m_n \in M \) and \( m^1, \ldots, m^n \in \text{Hom}_R(M, R) \) such that

\[
\forall m \in M : \sum_{i=1}^n m_i m^i(m) = m.
\]

Actually this is a consequence of the dual basis lemma. But this definition is equivalent to the usual definition.
Let $M \in R\mathcal{M}_R$. $M$ as a right $R$-module is finitely generated and projective iff $M$ has a left dual. The left dual is isomorphic to $\text{Hom}_R(M, R)$.

If $M_R$ is finitely generated projective then we use $\text{db} : R \to M \otimes_R \text{Hom}_R(M, R)$ with $\text{db}(1) = \sum_{i=1}^n m_i \otimes_R m^i$. In fact we have $(1 \otimes_R \text{ev})(\text{db} \otimes_R 1)(m) = (1 \otimes_R \text{ev})(\sum m_i \otimes_R m^i) = \sum m_i m^i(m) = m$. We have furthermore $(\text{ev} \otimes_R 1)(1 \otimes_R \text{db})(f)(m) = (\text{ev} \otimes_R 1)(\sum_{i=1}^n f \otimes_R m_i \otimes_R m^i)(m) = \sum f(m_i) m^i(m) = f(\sum m_i m^i(m)) = f(m)$ for all $m \in M$ hence $(\text{ev} \otimes_R 1)(1 \otimes_R \text{db})(f) = f$.

Conversely if $M$ has a left dual $M^*$ then $\text{ev} : M^* \otimes_R M \to R$ defines a homomorphism $\iota : M^* \to \text{Hom}_R(M, R)$ in $R\mathcal{M}_R$ by $\iota(m^*)(m) = \text{ev}(m^* \otimes_R m)$. We define $\sum_{i=1}^n m_i \otimes m^i := \text{db}(1) \in M \otimes M^*$, then $m = (1 \otimes \text{ev})(\text{db} \otimes 1)(m) = (1 \otimes \text{ev})(\sum m_i \otimes m^i \otimes m) = \sum m_i \iota(m^i)(m)$ so that $m_1, \ldots, m_n \in M$ and $\iota(m_1), \ldots, \iota(m_n) \in \text{Hom}_R(M, R)$ form a dual basis for $M$, i.e. $M$ is finitely generated and projective as an $R$-module. Thus $M^*$ and $\text{Hom}_R(M, R)$ are isomorphic by the map $\iota$.

Analogously $\text{Hom}_R(M, R)$ is a right dual for $M$ iff $M$ is finitely generated and projective as a left $R$-module.

**Problem 3.3.3.** Find an example of an object $M$ in a monoidal category $C$ that has a left dual but no right dual.

**Definition 3.3.12.** Given objects $M, N$ in $C$. An object $[M, N]$ is called a left inner $\text{Hom}$ of $M$ and $N$ if there is a natural isomorphism $\text{Mor}_C(- \otimes M, N) \cong \text{Mor}_C(-, [M, N])$, i.e. if it represents the functor $\text{Mor}_C(- \otimes M, N)$.

If there is an isomorphism $\text{Mor}_C(P \otimes M, N) \cong \text{Mor}_C(P, [M, N])$ natural in the three variable $M, N, P$ then the category $C$ is called monoidal and left closed.

If there is an isomorphism $\text{Mor}_C(M \otimes P, N) \cong \text{Mor}_C(P, [M, N])$ natural in the three variable $M, N, P$ then the category $C$ is called monoidal and right closed.

If $M$ has a left dual $M^*$ in $C$ then there are inner $\text{Homs}$ $[M, -]$ defined by $[M, N] := N \otimes M^*$. In particular left rigid monoidal categories are left closed.

**Example 3.3.13.**
1. The category of finite dimensional vector spaces is a monoidal category where each object has a (left and right) dual. Hence it is (left and right) closed and rigid.
2. Let $\text{Ban}$ be the category of (complex) Banach spaces where the morphisms satisfy $\| f(m) \| \leq \| m \|$ i.e. the maps are bounded by 1 or contracting. $\text{Ban}$ is a monoidal category by the complete tensor product $M \otimes N$. In $\text{Ban}$ exists an inner $\text{Hom}$ functor $[M, N]$ that consists of the set of bounded linear maps from $M$ to $N$ made into a Banach space by an appropriate topology. Thus $\text{Ban}$ is a monoidal closed category.
3. Let $H$ be a Hopf algebra. The category $H\text{-Mod}$ of left $H$-modules is a monoidal category (see Example 3.2.4 2). Then $\text{Hom}_K(M, N)$ is an object in $H\text{-Mod}$ by the multiplication $(hf)(m) := \sum h_{(1)} f(mS(h_{(2)}))$
as in Proposition 3.1.14.

\(\text{Hom}_\mathbb{K}(M, N)\) is an inner Hom functor in the monoidal category \(H\text{-Mod}\).

The isomorphism \(\phi : \text{Hom}_\mathbb{K}(P, \text{Hom}_\mathbb{K}(M, N)) \cong \text{Hom}_\mathbb{K}(P \otimes M, N)\) can be restricted to an isomorphism

\[
\text{Hom}_H(P, \text{Hom}_\mathbb{K}(M, N)) \cong \text{Hom}_H(P \otimes M, N),
\]

because \(\phi(f)(h(p \otimes m)) = \phi(f)(\sum h_{(1)}p \otimes h_{(2)}m) = \sum f(h_{(1)}p)(h_{(2)}m) = \sum h_{(1)}f(p)(S(h_{(2)})h_{(3)}m) = h(f(p)(m)) = h(\phi(f)(p \otimes m))\) and conversely \(h(f(p))(m) = \sum h_{(1)}(f(p)(S(h_{(2)})m)) = \sum h_{(1)}(\phi(f)(p \otimes S(h_{(2)})m)) = \sum \phi(f)(h_{(1)}p \otimes h_{(2)}S(h_{(3)})m) = \phi(f)(hp \otimes m) = f(hp)(m)\). Thus \(H\text{-Mod}\) is left closed.

If \(M \in H\text{-Mod}\) is a finite dimensional vector space then the dual vector space \(M^* := \text{Hom}_\mathbb{K}(M, \mathbb{K})\) again is an \(H\)-module by \((hf)(m) := f(S(h)m)\). Furthermore \(M^*\) is a left dual for \(M\) with the morphisms

\[
\text{db} : \mathbb{K} \ni 1 \mapsto \sum_i m_i \otimes m^i \in M \otimes M^*
\]

and

\[
\text{ev} : M^* \otimes M \ni f \otimes m \mapsto f(m) \in \mathbb{K}
\]

where \(m_i\) and \(m^i\) are a dual basis of the vector space \(M\). Clearly we have \((1 \otimes \text{ev})(\text{db} \otimes 1) = 1_M\) and \((\text{ev} \otimes 1)(1 \otimes \text{db}) = 1_{M*}\) since \(M\) is a finite dimensional vector space. We have to show that \(\text{db}\) and \(\text{ev}\) are \(H\)-module homomorphisms. We have

\[
(h \text{db}(1))(m) = (h(\sum_i m_i \otimes m^i))(m) = (\sum h_{(1)}m_i \otimes h_{(2)}m^i)(m) = \sum h_{(1)}m_i(h_{(2)}m^i)(m) = \sum h_{(1)}S(h_{(2)})m = \varepsilon(h)m = \varepsilon(h)(\sum m_i \otimes m^i)(m) = \varepsilon(h) \text{db}(1)(m) = \text{db}(\varepsilon(h))(m) = \text{db}(h1)(m),
\]

hence \(h \text{db}(1) = \text{db}(h1)\). Furthermore we have

\[
\text{h ev}(f \otimes m) = hf(m) = \sum h_{(1)}f(S(h_{(2)})h_{(3)}m) = \sum (h_{(1)}f)(h_{(2)}m) = \sum \text{ev}(h_{(1)}f \otimes h_{(2)}m) = \text{ev}(h(f \otimes m)).
\]

4. Let \(H\) be a Hopf algebra. Then the category of left \(H\)-comodules (see Example 3.2.4 3.) is a monoidal category. Let \(M \in H\text{-Comod}\) be a finite dimensional vector space. Let \(m_i\) be a basis for \(M\) and let the comultiplication of the comodule be \(\delta(m_i) = \sum h_{ij} \otimes m_j\). Then we have \(\Delta(h_{ik}) = \sum h_{ij} \otimes h_{jk}\). \(M^* := \text{Hom}_\mathbb{K}(M, \mathbb{K})\) becomes a left \(H\)-comodule \(\delta(m^i) := \sum S(h_{ij}) \otimes m^i\). One verifies that \(M^*\) is a left dual for \(M\).

**Lemma 3.3.14.** Let \(M \in \mathcal{C}\) be an object with left dual \((M^*, \text{ev})\). Then

1. \(M \otimes M^*\) is an algebra with multiplication

\[
\nabla := 1_M \otimes \text{ev} \otimes 1_{M^*} : M \otimes M^* \otimes M \otimes M^* \rightarrow M \otimes M^*
\]
The unit axiom is given by
\[ u := \text{db} : I \rightarrow M \otimes M^*; \]

2. \( M^* \otimes M \) is a coalgebra with comultiplication
\[ \Delta := 1_{M^*} \otimes \text{db} \otimes 1_{M^*} : M^* \otimes M \rightarrow M^* \otimes M \otimes M^* \otimes M \]
and counit
\[ \varepsilon := \text{ev} : M^* \otimes M \rightarrow I. \]

**Proof.** 1. The associativity is given by
\[
(\nabla \otimes 1)\nabla = (1_M \otimes \text{ev} \otimes 1_{M^*} \otimes 1_M \otimes 1_{M^*}) \cdot (1_M \otimes \text{ev} \otimes 1_{M^*} \otimes 1_M \otimes \text{ev} \otimes 1_{M^*}) = (1_M \otimes 1_{M^*} \otimes 1_M \otimes \text{ev} \otimes 1_{M^*} \otimes 1_M \otimes \text{ev} \otimes 1_{M^*}) \cdot (1_M \otimes \text{ev} \otimes 1_{M^*} \otimes 1_M \otimes \text{ev} \otimes 1_{M^*}).
\]
The axiom for the left unit is \( \nabla (u \otimes 1) = (1_M \otimes \text{ev} \otimes 1_{M^*}) \cdot (\text{db} \otimes 1_M \otimes 1_{M^*}) = 1_M \otimes 1_{M^*}. \)

2. is dual to the statement for algebras. \( \square \)

**Lemma 3.3.15.** 1. Let \( A \) be an algebra in \( C \) and left \( M \in C \) be a left rigid object with left dual \((M^*, \text{ev})\). There is a bijection between the set of morphisms \( f : A \otimes M \rightarrow M \) making \( M \) a left \( A \)-module and the set of algebra morphisms \( \tilde{f} : A \rightarrow M \otimes M^* \).

2. Let \( C \) be a coalgebra in \( C \) and left \( M \in C \) be a left rigid object with left dual \((M^*, \text{ev})\). There is a bijection between the set of morphisms \( f : M \rightarrow M \otimes C \) making \( M \) a right \( C \)-comodule and the set of coalgebra morphisms \( \tilde{f} : M^* \otimes M \rightarrow C \).

**Proof.** 1. By Lemma 3.3.14 the object \( M \otimes M^* \) is an algebra. Given \( f : A \otimes M \rightarrow M \) such that \( M \) becomes an \( A \)-module. By Lemma 3.3.3 we associate \( \tilde{f} := (f \otimes 1)(1 \otimes \text{db}) : A \rightarrow A \otimes M \otimes M^* \rightarrow M \otimes M^* \). The compatibility of \( \tilde{f} \) with the multiplication is given by the commutative diagram

![Diagram](https://via.placeholder.com/150)

The unit axiom is given by

\[
\begin{align*}
1 & \xrightarrow{\alpha} M \otimes M^* \\
A & \xrightarrow{1 \otimes \text{db}} A \otimes M \otimes M^* & \xrightarrow{f \otimes 1} M \otimes M^*
\end{align*}
\]
Conversely let $g : A \to M \otimes M^*$ be an algebra homomorphism and consider $\tilde{g} := (1 \otimes \text{ev})(g \otimes 1) : A \otimes M \to M \otimes M^* \otimes M \to M$. Then $M$ becomes a left $A$-module since

$$A \otimes A \otimes M \xrightarrow{\nabla \otimes 1} A \otimes M \xrightarrow{\tilde{g}} M \otimes M^* \otimes M \xrightarrow{1 \otimes \text{ev} \otimes 1} M \otimes M^* \otimes M \xrightarrow{1 \otimes \text{ev}} M$$

and

$$M \xrightarrow{u \otimes 1} A \otimes M \xrightarrow{g \otimes 1} M \otimes M^* \otimes M \xrightarrow{1 \otimes \text{ev}} M$$

commute.

2. $(M^*, \text{ev})$ is a left dual for $M$ in the category $\mathcal{C}$ if and only if $(M^*, \text{db})$ is the right dual for $M$ in the dual category $\mathcal{C}^{\text{op}}$. So if we dualize the result of part 1. we have to change sides, hence 2. \qed