CHAPTER 2

Hopf Algebras, Algebraic, Formal, and Quantum Groups
6. Quantum Automorphism Groups

**Lemma 2.6.1.** The category $\mathbb{K}$-$\text{Alg}$ of $\mathbb{K}$-algebras has arbitrary coproducts.

**Proof.** This is a well-known fact from universal algebra. In fact all equationally defined algebraic categories are complete and cocomplete. We indicate the construction of the coproduct of a family $(A_i|i \in I)$ of $\mathbb{K}$-algebras.

Define $\prod_{i \in I} A_i := T(\bigoplus_{i \in I} A_i)/L$ where $T$ denotes the tensor algebra and where $L$ is the two-sided ideal in $T(\bigoplus_{i \in I} A_i)$ generated by the set

$$J := \{ \epsilon j_k(x_k y_k) - \delta(j_k(x_k))\delta(j_k(y_k)), 1_{T(\bigoplus A_i)} - \epsilon j_k(1_{A_k}) | x_k, y_k \in A_k, k \in I \}.$$

Then one checks easily for a family of algebra homomorphisms $(f_k : A_k \to B | k \in I)$ that the following diagram gives the required universal property

$$\begin{array}{ccc}
A_k & \xrightarrow{j_k} & \bigoplus A_i \\
\downarrow & & \downarrow \\
T(\bigoplus A_i) & \xrightarrow{\pi} & T(\bigoplus A_i)/L \\
\downarrow & & \downarrow \\
B & & \\
\end{array}$$

Corollary 2.6.2. The category of bialgebras has finite coproducts.

**Proof.** The coproduct $\prod B_i$ of bialgebras $(B_i|i \in I)$ in $\mathbb{K}$-$\text{Alg}$ is an algebra. For the diagonal and the counit we obtain the following commutative diagrams

$$\begin{array}{ccc}
B_k & \xrightarrow{j_k} & \prod B_i \\
\downarrow & & \downarrow \\
B_k \otimes B_k & \xrightarrow{j_k \otimes j_k} & \prod B_i \otimes \prod B_i \\
\downarrow & & \downarrow \\
B_k & \xrightarrow{j_k} & \prod B_i \\
\downarrow & & \downarrow \\
\mathbb{K} & & \\
\end{array}$$

since in both cases $\prod B_i$ is a coproduct in $\mathbb{K}$-$\text{Alg}$. Then it is easy to show that these homomorphisms define a bialgebra structure on $\prod B_i$ and that $\prod B_i$ satisfies the universal property for bialgebras.

**Theorem 2.6.3.** Let $B$ be a bialgebra. Then there exists a Hopf algebra $H(B)$ and a homomorphism of bialgebras $\iota : B \to H(B)$ such that for every Hopf algebra $H$ and for every homomorphism of bialgebras $f : B \to H$ there is a unique
homomorphism of Hopf algebras $g : H(B) \rightarrow H$ such that the diagram

$$
\begin{array}{ccc}
B & \xrightarrow{t} & H(B) \\
\downarrow{f} & & \downarrow{g} \\
H & & H
\end{array}
$$

commutes.

**Proof.** Define a sequence of bialgebras $(B_i | i \in \mathbb{N})$ by

$$
B_0 := B, \\
B_{i+1} := B_i^{op,cop}, i \in \mathbb{N}.
$$

Let $B'$ be the coproduct of the family $(B_i | i \in \mathbb{N})$ with injections $\iota_i : B_i \rightarrow B'$. Because $B'$ is a coproduct of bialgebras there is a unique homomorphism of bialgebras $S' : B' \rightarrow B'^{op,cop}$ such that the diagrams

$$
\begin{array}{ccc}
B_i & \xrightarrow{i_i} & B' \\
\downarrow{id} & & \downarrow{S'} \\
B_i^{op,cop} & \xrightarrow{i_{i+1}} & B'^{op,cop}
\end{array}
$$

commute.

Now let $I$ be the two sided ideal in $B'$ generated by

$$
\{(S' \ast 1 - u \varepsilon)(x_i),(1 \ast S' - u \varepsilon)(x_i)| x_i \in \iota_i(B_i), i \in \mathbb{N}\}.
$$

$I$ is a coideal, i.e. $\varepsilon_B(I) = 0$ and $\Delta_B(I) \subseteq I \otimes B' + B' \otimes I$.

Since $\varepsilon_B$ and $\Delta_B$ are homomorphisms of algebras it suffices to check this for the generating elements of $I$. Let $x \in B_i$ be given. Then we have

$$
\varepsilon((1 \ast S')\iota_i(x)) = \varepsilon(\nabla(1 \otimes S')\Delta_i(x)) = \nabla_2(\varepsilon \otimes \varepsilon S')(\iota_i \otimes \iota_i)\Delta_i(x) = (\varepsilon \otimes \varepsilon)\Delta_i(x) = \varepsilon_i(x) = \varepsilon(u \varepsilon \iota_i(x)).
$$

Symmetrically we have $\varepsilon((S' \ast 1)\iota_i(x)) = \varepsilon(u \varepsilon \iota_i(x))$. Furthermore we have

$$
\begin{align*}
\Delta((1 \ast S')\iota_i(x)) & = \Delta \nabla(1 \otimes S')\Delta_i(x) \\
& = (\nabla \otimes \nabla)(1 \otimes \tau \otimes 1) (\Delta \otimes \Delta) (1 \otimes S')(\iota_i \otimes \iota_i)\Delta_i(x) \\
& = (\nabla \otimes \nabla)(1 \otimes \tau \otimes 1) (\Delta \otimes \tau (S' \otimes S') \Delta) (\iota_i \otimes \iota_i)\Delta_i(x) \\
& = \sum (\nabla \otimes \nabla)(1 \otimes \tau \otimes 1) (\iota_i(x_{(1)}) \otimes \iota_i(x_{(2)}) \otimes S' t_i(x_{(3)}) \otimes S' t_i(x_{(3)})) \\
& = \sum \iota_i(x_{(1)}) S' t_i(x_{(4)}) \otimes \iota_i(x_{(2)}) S' t_i(x_{(3)}) \\
& = \sum \iota_i(x_{(1)}) S' t_i(x_{(3)}) \otimes (1 \ast S')\iota_i(x_{(2)}).
\end{align*}
$$
Hence we have
\[
\Delta((1 \ast S' - u \varepsilon) \iota_i(x)) \\
= \sum \iota_i(x(1))S' \iota_i(x(3)) \otimes (1 \ast S' - u \varepsilon) \iota_i(x(2)) \\
+ \sum \iota_i(x(1))S' \iota_i(x(3)) \otimes (1 \ast S' - u \varepsilon) \iota_i(x(2)) \\
= \sum \iota_i(x(1))S' \iota_i(x(3)) \otimes (1 \ast S' - u \varepsilon) \iota_i(x(2)) \\
+ \sum \iota_i(x(1))S' \iota_i(x(3)) \otimes (1 \ast S' - u \varepsilon) \iota_i(x(2)) \\
= \sum \iota_i(x(1))S' \iota_i(x(3)) \otimes (1 \ast S' - u \varepsilon) \iota_i(x(2)) \\
+ (1 \ast S' - u \varepsilon) \iota_i(x) \otimes 1_{B'} \\
\in B' \otimes I + I \otimes B'.
\]

Thus \(I\) is a coideal and a biideal of \(B'\).

Now let \(H(B) := B'/I\) and let \(\nu : B' \to H(B)\) be the residue class homomorphism. We show that \(H(B)\) is a bialgebra and \(\nu\) is a homomorphism of bialgebras. \(H(B)\) is an algebra and \(\nu\) is a homomorphism of algebras since \(I\) is a two sided ideal.

Since \(I \subseteq \text{Ker}(\varepsilon)\) there is a unique factorization

\[
\begin{array}{ccc}
B' & \xrightarrow{\nu} & B'/I \\
\varepsilon' & \downarrow & \varepsilon \\
\mathbb{K} & & \\
\end{array}
\]

where \(\varepsilon : B'/I \to \mathbb{K}\) is a homomorphism of algebras. Since \(\Delta(I) \subseteq B' \otimes I + I \otimes B' \subseteq \text{Ker}(\nu \otimes \nu : B' \otimes B' \to B'/I \otimes B'/I)\) and thus \(I \subseteq \text{Ker}(\Delta(\nu \otimes \nu))\) we have a unique factorization

\[
\begin{array}{ccc}
B' & \xrightarrow{\nu} & B'/I \\
\Delta_B & \downarrow & \Delta \\
B' \otimes B' & \xrightarrow{\nu \otimes \nu} & B'/I \otimes B'/I \\
\end{array}
\]

by an algebra homomorphism \(\Delta : B'/I \to B'/I \otimes B'/I\). Now it is easy to verify that \(B'/I\) becomes a bialgebra and \(\nu\) a bialgebra homomorphism.

We show that the map \(\nu S' : B' \to B'/I\) can be factorized through \(B'/I\) in the commutative diagram

\[
\begin{array}{ccc}
B' & \xrightarrow{\nu} & B'/I \\
\nu S' & \downarrow & S \\
B'/I & & \\
\end{array}
\]
This holds if \( I \subseteq \text{Ker}(\nu S') \). Since \( \text{Ker}(\nu) = I \) it suffices to show \( S'(I) \subseteq I \). We have
\[
S'((S' \ast 1)\iota_i(x)) = \\
= \nabla \tau (S' \ast_1 S' \iota_i \Delta_i(x)) \\
= \nabla \tau (S' \otimes 1)(\iota_i \otimes \iota_i) \Delta_i(x) \\
= \nabla (1 \otimes S')(\iota_i \otimes \iota_i) \tau \Delta_i(x) \\
= (1 \otimes S')(\iota_i \otimes \iota_i) \Delta_i(x) \\
= (1 \ast S')\iota_i(x)
\]
and
\[
S'(u\varepsilon\iota_i(x)) = S'(1)\iota_i(x) = S'(1)\iota_{i+1}(x) = S'(u\varepsilon\iota_{i+1}(x))
\]
hence we get
\[
S'((S' \ast 1 - u\varepsilon)\iota_i(x)) = (1 \ast S' - u\varepsilon)\iota_{i+1}(x) \in I.
\]
This shows \( S'(I) \subseteq I \). So there is a unique homomorphism of bialgebras \( S : H(B) \to H(B)_{\text{opcop}} \) such that the diagram
\[
\begin{array}{ccc}
B' & \xrightarrow{\nu} & H(B) \\
\downarrow{S'} & & \downarrow{s} \\
B_{\text{opcop}} & \xrightarrow{\nu} & H(B)_{\text{opcop}}
\end{array}
\]
commutes.

Now we show that \( H(B) \) is a Hopf algebra with antipode \( S \). By Proposition 2.1.3 it suffices to test on generators of \( H(B) \) hence on images \( \nu \iota_i(x) \) of elements \( x \in B_i \).

We have
\[
(1 \ast S)\nu \iota_i(x) = \nabla (\nu \otimes \nu S) \Delta_i(x) = \nabla (\nu \otimes \nu)(1 \otimes S') \Delta_i(x) = \\
= \nu(1 \ast S')\iota_i(x) = \nu \varepsilon \iota_i(x) = u \varepsilon \nu \iota_i(x).
\]
By Proposition 2.1.3 \( S \) is an antipode for \( H(B) \).

We prove now that \( H(B) \) together with \( \iota := \nu_0 : B \to H(B) \) is a free Hopf algebra over \( B \). Let \( H \) be a Hopf algebra and let \( f : B \to H \) be a homomorphism of bialgebras. We will show that there is a unique homomorphism \( \tilde{f} : H(B) \to H \) such that
\[
\begin{array}{ccc}
B & \xrightarrow{\iota} & H(B) \\
\downarrow{f} & & \downarrow{\tilde{f}} \\
H & & 
\end{array}
\]
commutes.

We define a family of homomorphisms of bialgebras \( f_i : B_i \to H \) by
\[
\begin{align*}
\tilde{f}_0 & := f, \\
\tilde{f}_{i+1} & := S_H \tilde{f}_i, \ i \in \mathbb{N}.
\end{align*}
\]
We have in particular $f_i = S_H^i f$ for all $i \in \mathbb{N}$. Thus there is a unique homomorphism of bialgebras $f' : B' = \coprod B_i \to H$ such that $f'_i = f_i$ for all $i \in \mathbb{N}$.

We show that $f'(I) = 0$. Let $x \in B_i$. Then

$$f'((1*S')\iota_i(x)) = f'((\nabla(1 \otimes S')(\iota_i \otimes \iota_i)\Delta_i(x)))$$
$$= \sum f'_i(x_{(1)}) f'_i S' \iota_i(x_{(2)})$$
$$= \sum f'_i(x_{(1)}) f'_i(x_{(2)} + (1_S) f_i(x_{(2)}))$$
$$= \sum f_i(x_{(1)}) f_i(x_{(2)})$$
$$= (1*S)f_i(x) = u \varepsilon_i(x) = u \varepsilon(x)$$
$$= f'(u \varepsilon_i(x)).$$

This together with the symmetric statement gives $f'(I) = 0$. Hence there is a unique factorization through a homomorphism of algebras $\tilde{f} : H(B) \to H$ such that $f' = \tilde{f} \nu$.

The homomorphism $\tilde{f} : H(B) \to H$ is a homomorphism of bialgebras since the diagram

$$\begin{array}{ccc}
B' & \xrightarrow{f'} & B'/I \\
\Delta & \downarrow & \Delta' \\
B' \otimes B' & \xrightarrow{\epsilon \otimes \epsilon} & B'/I \otimes B'/I \\
\cdot f' \otimes f' & \xrightarrow{f \otimes f} & H \otimes H
\end{array}$$

commutes with the possible exception of the right hand square $\Delta \tilde{f}$ and $(\tilde{f} \otimes \tilde{f}) \Delta'$. But $\nu$ is surjective so also the last square commutes. Similarly we get $\varepsilon_H \tilde{f} = \varepsilon_{H(B)}$. Thus $\tilde{f}$ is a homomorphism of bialgebras and hence a homomorphism of Hopf algebras. \(\square\)

**Remark 2.6.4.** In chapter 1 we have constructed universal bialgebras $M(A)$ with coaction $\delta : A \to M(A) \otimes A$ for certain algebras $A$ (see 1.3.12). This induces a homomorphism of algebras

$$\delta' : A \to H(M(A)) \otimes A$$

such that $A$ is a comodule-algebra over the Hopf algebra $H(M(A))$. If $H$ is a Hopf algebra and $A$ is an $H$-comodule algebra by $\delta : A \to H \otimes A$ then there is a unique homomorphism of bialgebras $f : M(A) \to H$ such that

$$\begin{array}{ccc}
A & \xrightarrow{\delta} & M(A) \otimes A \\
\cdot f \otimes 1 & \xrightarrow{} & H \otimes A
\end{array}$$
commutes. Since the \( f : M(A) \to H \) factorizes uniquely through \( \tilde{f} : H(M(A)) \to H \) we get a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{g} & H(M(A)) \otimes A \\
\downarrow{a} & & \downarrow{f \otimes 1} \\
H \otimes A & & 
\end{array}
\]

with a unique homomorphism of Hopf algebras \( \tilde{f} : H(M(A)) \to H \).

This proof depends only on the existence of a universal algebra \( M(A) \) for the algebra \( A \). Hence we have

**Corollary 2.6.5.** Let \( \mathcal{X} \) be a quantum space with universal quantum space (and quantum monoid) \( \mathcal{M}(\mathcal{X}) \). Then there is a unique (up to isomorphism) quantum group \( \mathcal{H}(\mathcal{M}(\mathcal{X})) \) acting universally on \( \mathcal{X} \).

This quantum group \( \mathcal{H}(\mathcal{M}(\mathcal{X}')) \) can be considered as the “quantum subgroup of invertible elements” of \( \mathcal{M}(\mathcal{X}) \) or the quantum group of “quantum automorphisms” of \( \mathcal{X} \).