Mathematisches Institut
der Universität München
Prof. Dr. B. Pareigis

SS 2002
Set 3

Problem set for Quantum Groups and Noncommutative Geometry

(9) Consider the following subset $\mathbb{H}$ of the set of complex 2×2-matrices:

$$\mathbb{H} := \left\{ \begin{pmatrix} x & -y \\ \bar{y} & \bar{x} \end{pmatrix} \in M_{\mathbb{C}}(2 \times 2) | x, y \in \mathbb{C} \right\}$$

We call $\mathbb{H}$ the set of *Hamiltonian quaternions*. For

$$h = \begin{pmatrix} x & -y \\ \bar{y} & \bar{x} \end{pmatrix}$$

we define:

$$\bar{h} := \begin{pmatrix} \bar{x} & y \\ -\bar{y} & x \end{pmatrix}$$

Show:
(a) $h\bar{h} = (|x|^2 + |y|^2)E$ (E the unit matrix),
(b) $\mathbb{H}$ is a real subalgebra of the complex algebra of 2×2-matrices.
(c) $\mathbb{H}$ is a division algebra, i. e. each element different from zero has an inverse under the multiplication.
(d) Let

$$I := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad K := \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

Then E, I, J, K is an $\mathbb{R}$-basis of $\mathbb{H}$ an we have the following multiplication table:

$$I^2 = J^2 = K^2 = -1$$

$$IJ = -JI = K \quad JK = -KJ = I \quad KI = -IK = J.$$

(10) Compute the $\mathbb{H}$-points $A^{2|0}(\mathbb{H})$ of the quantum plane.

(11) **Definition:** Let $\mathcal{X}$ be an geometric space with affine algebra A. Let D be an algebra. A natural transformation $\rho : D \times \mathcal{X} \longrightarrow \mathbb{A}$ is called an *algebra action* if $\rho(B)(\text{-}, p) : D \longrightarrow \mathbb{A}(B) = B$ is an algebra homomorphism for all B and all $p \in \mathcal{X}(B)$.
Give proofs for:
Lemma: The natural transformation $\psi : A \times \mathcal{X} \longrightarrow \mathbb{A}$ is an algebra action.

Theorem: Let D be an algebra and $\rho : D \times \mathcal{X}(\text{-}) \longrightarrow \mathbb{A}(\text{-})$ be an algebra action. Then there exists a unique algebra homomorphism $f : D \longrightarrow A$ such that the diagram

$$\begin{tikzcd} D \times \mathcal{X}(B) \arrow[d, "f \times 1"'] \arrow[dr, "\rho(B)"] & \\ A \times \mathcal{X}(B) \arrow[r, "\psi(B)"'] & B \end{tikzcd}$$

commutes.

(12) Determine explicitly the dual coalgebra A^* of the algebra $A := \mathbb{K}\langle x \rangle / (x^2)$. (Hint: Find a basis for A.)

Due date: Tuesday, 07.05.2002, 16:15 in Lecture Hall E41