Mathematisches Institut
der Universität München
Prof. Dr. B. Pareigis

WS 2001/02
Set 4

Problem set for Advanced Algebra

(13) Let X be a set and $V := \mathbb{K}X$ be the free $\mathbb{K}$-module over X. Show that $X \to V \to T(V)$ defines a *free algebra* over X, i.e. for every $\mathbb{K}$-algebra A and every map $f : X \to A$ there is a unique homomorphism of $\mathbb{K}$-algebras $g : T(V) \to A$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & T(V) \arrow[d, "g"] \\ & A \end{tikzcd}$$

commutes.

We write $\mathbb{K}\langle X\rangle := T(\mathbb{K}X)$ and call it the *polynomial ring over* $\mathbb{K}$ *in the non-commuting variables* X.

(14) Let X be a set and $V := \mathbb{K}X$ be the free $\mathbb{K}$-module over X. Show that $X \to V \to S(V)$ defines a *free commutative algebra* over X, i.e. for every commutative $\mathbb{K}$-algebra A and every map $f : X \to A$ there is a unique homomorphism of $\mathbb{K}$-algebras $g : S(V) \to A$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & S(V) \arrow[d, "g"] \\ & A \end{tikzcd}$$

commutes.

(15) (a) Let $S(V)$ and $\iota : V \to S(V)$ be a symmetric algebra. Show that there is a unique homomorphism of algebras $\Delta : S(V) \to S(V) \otimes S(V)$ with $\Delta(v) = v \otimes 1 + 1 \otimes v$ for all $v \in V$.
(b) Show that $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta : S(V) \to S(V) \otimes S(V) \otimes S(V)$.

(16) Let V be a $\mathbb{K}$-module and A be a $\mathbb{K}$-algebra.
(a) Let $f : V \to A$ be a homomorphism of $\mathbb{K}$-modules satisfying $f(v)^2 = 0$ for all $v \in V$. Then $f(v)f(v') = -f(v')f(v)$ for all $v, v' \in V$.
(b) Let 2 be invertible in $\mathbb{K}$ (e.g. $\mathbb{K}$ a field of characteristic $\neq 2$). Let $f : V \to A$ be a homomorphism of $\mathbb{K}$-modules satisfying $f(v)f(v') = -f(v')f(v)$ for all $v, v' \in V$. Then $f(v)^2 = 0$ for all $v \in V$.

Due date: Tuesday, 13.11.2001, 16:15 in Lecture Hall 138