Antipodes in the Theory of Noncommutative Torsors

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Introduction

The essence of Galois' achievements in modern algebra was to associate to a given polynomial equation $f(x) = 0$ over a field K a finite group, called Galois group, and to show that the equation can be solved by radicals if and only if its Galois group is solvable. In the same spirit, the fundamental theorem of Galois theory states that the subfields of a finite separable and normal field extension $K \subset L$ are in one-to-one correspondence with the subgroups of the automorphism group $\mathrm{Aut}(L/K)$. This concept of gaining information about an algebraic system by considering a group that acts on it, is widely applied in today's mathematics.

Let X be a set that is endowed with an action of a group G. The action is called simply transitive if for each pair of elements in X there exists a unique element in the group G taking the first element to the second. In this case, X is called a principal homogeneous space under the action of G. Equivalently, one could also require that the canonical map

$$\alpha : X \times G \longrightarrow X \times X \ , \ (x, g) \mapsto (x, x \cdot g)$$

be an isomorphism. This map appears also in the definition of a principal homogeneous space in algebraic geometry. An action of an affine algebraic group scheme G on an affine scheme X is called free, if the map α is a closed embedding [29]. The image of α is the fiber product $X \times_Y X$, where $Y = X/G$ is the affine quotient of X by the action of G. A principal homogeneous space is an affine scheme X with $X \longrightarrow Y$ faithfully flat and $X \times G \longrightarrow X \times_Y X$ an isomorphism [29, 48]. It is also called a torsor in a terminology that goes back to work of Grothendieck [17] and Demazure-Gabriel [12].

Torsors in algebraic geometry possess naturally a counterpart in the theory of Hopf algebras. Essentially, commutative Hopf algebras are the function algebras of affine algebraic groups. Like groups their representations allow tensor products and duals. This last property holds even for noncommutative Hopf algebras, the function algebras of quantum groups introduced in 1986 by Drinfeld [14]. Indeed, the main feature of a Hopf algebra is a so-called antipode map. It can be interpreted as the noncommutative analogue of a map which assigns an inverse to each element of a group. This property of the antipode is part of the well-known connection between Hopf algebras and algebraic varieties: Each affine scheme X over a field k is given as $X = \mathrm{Spec}(A)$ for a commutative affine k-algebra A, and each affine group scheme G is given as $G = \mathrm{Spec}(H)$ for a commutative k-Hopf algebra H. Thus, an action $X \times G \longrightarrow X$ is uniquely determined by a coaction $\rho : A \longrightarrow A \otimes_k H$, and it is free if the map

$$\beta : A \otimes_k A \longrightarrow A \otimes_k H \ , \ a \otimes b \mapsto (a \otimes 1)\rho(b)$$

is surjective.

This map β appeared in 1969 in lecture notes by Chase and Sweedler [10]. It was studied there in connection with Galois objects, which are essentially torsors in the category of commutative algebras. An inspiration for this was the Chase-Harrison-Rosenberg approach to Galois theory for groups acting on commutative rings [9]. There, the finite group of automorphisms in the classical theory is replaced by the coaction of a Hopf algebra. In 1981, Kreimer and Takeuchi [25] gave a definition of Hopf-Galois extension that is based on bijectivity of the map $A \otimes_B A \longrightarrow A \otimes_k H$, where $B \subset A$ is the set of invariants under the coaction of H.

Ever since, Hopf-Galois extensions have been objects of fundamental interest. Apart from being an important tool in the investigation of Hopf algebras themselves, they are also studied in connection with affineness theorems for algebraic groups [46], inseparable field extensions [16], representation theory [47] and duality theorems [4], just to name a few.
The notion of Hopf-Galois extension comprises classical Galois field extensions when H is chosen as the dual of a group algebra. Also, as expected, the coordinate rings of affine torsors correspond precisely to the faithfully flat Hopf-Galois extensions with respect to Hopf algebras that are coordinate rings of affine group schemes. By analogy, noncommutative and noncocommutative Hopf algebras are coordinate rings of quantum groups, and lead to Hopf-Galois extensions that can be interpreted as noncommutative torsors.

In 1999 Kontsevich [24] suggested to use another definition of a torsor, a non-empty set X that is endowed with a map $\lambda : X \times X \times X \longrightarrow X$ satisfying the same identities as the ternary operation $(a, b, c) \mapsto ab^{-1}c$ in groups. This is equivalent to the definition of a principal homogeneous space given above. The corresponding axioms are

$$\lambda(a, a, b) = b \quad , \quad \lambda(a, b, b) = a$$
$$\lambda(\lambda(a, b, c), d, e) = \lambda(a, b, \lambda(c, d, e)) = \lambda(a, \lambda(d, c, b), e) \ .$$

This approach is based on an old intrinsic reformulation of affine structures that was originally suggested by Reinhold Baer [1] in 1929 under the name "Schar". As a consequence of these axioms, a torsor has always two isomorphic groups that act simply transitively on it (one from the left and the other one from the right). They can both be recovered using a method of Weinstein [58] which is based on the idea that to any $(a, b, c) \in X^3$ there exits a unique fourth vertex $ab^{-1}c \in X$ such that $(a, b, c, ab^{-1}c)$ forms a parallelogram. Weinstein's interest in what he calls affinoid structures lies in the study of compatible geometric structures on Poisson manifolds.

In an attempt to unify the various notions of torsor that appear in algebraic geometry and Poisson geometry, Grunspan recently introduced the concept of quantum torsor in [18]. His definition follows the usual procedure of using noncommutative algebras and reversing arrows in order to obtain quantized objects. Consequently, a quantum torsor T is an algebra equipped with an algebra homomorphism $\mu : T \longrightarrow T \otimes T^{op} \otimes T$ satisfying axioms dual to those of the map λ above. In dualizing the last equality, however, a difficulty appears. This lies in the fact that although we have $(g^{-1})^{-1} = g$ for each element of a group, this is no longer true in the quantum group case. So a quantum torsor has as an additional data an algebra map $\theta : T \longrightarrow T$ that is supposed to play the role of a squared inverse.
One of Grunspan's main results is the noncommutative analogue to what holds for Kontsevich's torsors. It says that one can associate to each faithfully flat quantum torsor T two (non-isomorphic) Hopf algebras, which both coact on T. Moreover, the coactions are such that, under each of them, T becomes a Hopf-Galois extension.
This situation can be described with the notion of a Hopf bi-Galois extension introduced by Schauenburg in [38]. There, it is shown that the structure maps of each faithfully flat H-Galois extension A determine a coaction of another Hopf algebra L, such that A becomes also an L-Galois extension. So classical torsors and Hopf-Galois extensions really show the same behaviour: Although one starts from just one action of a group resp. one coaction of a Hopf algebra, there will always be another group resp. another Hopf algebra acting as a counterpart.
In the noncommutative setting, this is also encoded in the concept of a Hopf-Galois system due to Bichon [3]. Its data contains all the structure maps that make A into an L-H-bicomodule algebra. In addition, there occurs one particular map that satisfies properties similar to those of an antipode.

Each of these approaches to the concept of noncommutative torsor is based on its own particular features. It is interesting to see, though, that in two cases there occur maps that generalize either an antipode or its square.

This thesis is devoted to the study of these generalized antipode maps in the theory of noncommutative torsors. We are particularly interested in the θ-map for quantum torsors and the generalized antipode map in a Hopf-Galois system. The latter reveals a striking connection between the structure of torsors and Hopf algebras. We use two different reconstruction methods due to Tannaka-Krein [11] and Takeuchi [51], respectively, to show how torsor structures appear in "nature": Quantum torsors can be built up from a pair of comodule algebras by adjoining generalized antipodes. Moreover, we answer a question of Bichon [3]. Generalized antipodes are uniquely determined and they are algebra anti-morphisms.
We also study an extended version of quantum torsor that has been suggested by Schauenburg [42]. Such a B-torsor is naturally endowed with the coaction of a Hopf algebra. But apart from this, we uncover Hopf algebroid structures. They come together with antipode maps in the sense of Lu [26], and those again are connected to the above θ-map. We derive Grunspan's axioms for the θ-map from the Hopf algebroid axioms. Thus, the θ-map can be interpreted in terms of Hopf algebroid structures that are encoded in the torsor structure map.
But what are the advantages of the intrinsic description of principal homogeneous spaces? We give an application of noncommutative torsors in the extension theory of algebras: Extensions of depth two, arising from subfactor theory [22], can be embedded into the concept of torsor. Here, the intrinsic definition proves to be very effective in detecting Hopf-Galois extensions without even knowing about possible coactions of Hopf algebras. Our approach allows us to simplify proofs and extend results of [21, 22] about Jones towers of Frobenius extensions. As a consequence, we arrive at a new general definition of noncommutative torsor that comes along with coactions of $\times_A$-Hopf algebras rather than ordinary Hopf algebras.

What follows is a detailed description of the contents in this thesis:

There are three concepts of noncommutative torsor known in the literature: Hopf-Galois extensions, Hopf-Galois systems and quantum torsors. Each of them takes a different approach in describing the properties of a noncommutative principal homogeneous space.
We present their definitions and main results in the first chapter. Hopf-Galois extensions are the oldest of the three notions, defined in terms of a Hopf algebra and a comodule algebra by requiring that a certain map be bijective. This notion goes back to work of Chase and Sweedler in 1969. The notion of quantum torsor as introduced by Grunspan [18] does not use a Hopf algebra at all in the definition. The coactions are "hidden" in the

torsor structure map. The concept of Hopf-Galois system was introduced by Bichon in [3]. It is the most explicit one, since its data describes both the torsor structure and the coactions of two Hopf algebras. We discuss several examples of each of these three structures, and indicate how they are related to each other and to the notion of Hopf bi-Galois extension [38].

By a classical result of Saavedra Rivano [36], the torsors with respect to a commutative k-Hopf algebra H correspond to the k-valued fiber functors on the category of finite dimensional H-comodules. A noncommutative version of this theorem for Hopf-Galois extensions was proved by Ulbrich [54, 55].
In the second chapter, we indicate how reconstruction techniques due to Tannaka-Krein can be applied to describe Hopf-Galois extensions as cohomomorphism objects of two particular fiber functors. We give an example which shows that even two arbitrary fiber functors lead to Hopf-Galois extensions.
Quite a few authors have suggested notations that allow to carry out calculations with tensor products. The well-known Sweedler notation [50] is one of them. Another one, which goes back to Penrose [34], is the graphical notation. It has the advantage of not using "elements", and is thus valid in any monoidal category.
We use this graphical notation to prove that quantum torsors arise naturally as cohomomorphism objects. We consider two arbitrary functors $\omega, \nu : \mathcal{C} \longrightarrow \mathcal{M}$ from a monoidal category $\mathcal{C}$ to a small rigid monoidal category $\mathcal{M}$. Then the cohomomorphism object $\mathrm{cohom}(\nu, \omega)$ from [32] turns out to be a torsor. The construction of a θ-map for $\mathrm{cohom}(\nu, \omega)$ requires that $\mathcal{C}$ be rigid. It also shows how the θ-map is connected to the antipode of a Hopf algebra constructed in [56]. In fact, the whole construction method reveals that the axioms of a Hopf algebra can be naturally interpreted in terms of quantum torsor structures: If we start with functors $\omega = \nu$, then $\mathrm{cohom}(\omega, \omega)$ is a Hopf algebra.

In the third chapter we consider total Hopf-Galois systems. They were defined by Grunspan [18] as an extended version of Bichon's Hopf-Galois systems. Their additional data is a second bicomodule algebra with structure maps, such that there are two Hopf algebras and two bicomodule algebras carrying all the data needed to recover two Hopf bi-Galois extensions.
Our approach to the concept of total Hopf-Galois system is different. We show that the axioms of a Hopf-Galois system are modelled analogously to those of a Hopf algebra. The latter is a bialgebra which has an antipode map. So we start from a bicomodule algebra system, and add generalized antipodes to its set of axioms. We use the crucial observation that generalized antipodes are units for suitably defined multiplications on certain

homomorphism sets. Thus, we develop a technique to prove properties for generalized antipodes, that correspond to those of an ordinary antipode. In particular, we give a direct proof of the fact that generalized antipodes are uniquely determined algebra anti-morphisms. This answers questions of Bichon in [3].
We give an explicit example of a total Hopf-Galois system by computing all the structure maps that arise out of a faithfully flat Hopf-Galois extension. Our previous results on generalized antipodes allow us to generalize Takeuchi's result on free Hopf algebras over bialgebras [51] to the case of total Hopf-Galois systems. Each bicomodule algebra system possesses a universal Hopf-Galois system. Hence, it is possible, in a certain sense, to "adjoin" generalized antipodes to a bicomodule algebra system. The resulting total Hopf-Galois system is universal with respect to this property.

Above, we have listed axioms for the ternary operation $(a, b, c) \mapsto ab^{-1}c$ in groups. In fact, we can show that the last axiom

$$\lambda(a, b, \lambda(c, d, e)) = \lambda(a, \lambda(d, c, b), e) \ ,$$

which is responsible for the map θ in the quantized case, can be deduced from the first three ones. It is interesting to compare this to Schauenburg's result in [41]. It says that the structure of a faithfully flat quantum torsor is already determined by those axioms that do not make use of the θ-map. This allows to extend the notion of quantum torsor to the case of an arbitrary algebra extension $B \subset T$, leading to the notion of B-torsor with a torsor structure map $T \longrightarrow T \otimes (T \otimes_B T)^B$.
We start the fourth chapter with a review of $\times_A$-bialgebras introduced by Takeuchi [52], and their equivalent notion of bialgebroids due to Lu [26]. These objects can be seen as a generalization of ordinary bialgebras and have recently aroused much interest in the literature [5, 30, 44]. We compare the two different concepts of $\times_A$-Hopf algebra [43] and Hopf algebroid [26], which both make a different approach in generalizing Hopf algebras. For further applications we introduce the analogue of a right bialgebroid in the terminology of $\times_A$-bialgebras, and call it a $\times^A$-bialgebra. Furthermore, we prove that each $\times_A$-bialgebra that admits a Hopf-Galois extension is a $\times_A$-Hopf algebra.
Then we turn to the study of B-torsors and their associated algebraic structures. Each B-torsor T is endowed with the coaction of a Hopf algebra H, that can be recovered by faithfully flat descent [42]. This makes T into an H-Galois extension of B. We discover that, apart from this, the torsor map induces Hopf algebroid structures. Each B-torsor T gives rise to a right T^B-bialgebroid structure on the centralizer $(T \otimes_B T)^B$. The Galois map

then induces a bialgebroid structure on the smash product algebra $T^B \# H$. A non-finite version of Lu's theorem on the construction of Hopf algebroid structures for smash product algebras [26] implies that $T^B \# H$ is in fact a Hopf algebroid. It turns out that the antipode of this Hopf algebroid is closely connected to an endomorphism of the centralizer T^B. This endomorphism is given by the same formula as the θ-map of a quantum torsor, and it is also involved in the antipode map $\Theta : (T \otimes_B T)^B \longrightarrow (T \otimes_B T)^B$ for the Hopf algebroid $(T \otimes_B T)^B$.
Now the Hopf algebroid axioms imply certain properties for Θ that generalize Grunspan's axioms. We obtain a connection between Θ and the torsor structure map. Another property joins Θ, the torsor map and the θ-map. This yields a reasonable interpretation of Grunspan's axioms. The occurrence of generalized antipodes resp. their squares can be understood in terms of "hidden" Hopf algebroid structures.

What is the advantage of describing the properties of principal homogeneous spaces in terms of torsors structures? At first, it seems like a loss of information to hide Hopf algebras and their coactions in torsor structure maps. But then we can look at it from the opposite direction: Assume that we are given an algebra extension $N \subset M$. Then it might neither be obvious whether there exists a Hopf algebra that coacts on M, nor that it makes M into a Hopf-Galois extension of N. However, if we can find an N-torsor structure map for M, then our previous results provide a lot of information. They allow us to construct a Hopf algebra with a coaction on M such that the latter becomes a Hopf-Galois extension of N.
We show in the fifth chapter how this approach indeed provides new results for a particular type of algebra extensions. The intrinsic definition of a noncommutative torsor is connected to a notion of depth two that has its source in the classification of subfactors [35]. Finite depth is a property of the standard invariant of the Jones tower for a subfactor $N \subset M$ [19]. Such a Jones tower is obtained by iterating the fundamental construction described in [15]. It can be used to investigate inclusions of one finite semisimple algebra in another. This procedure was applied on Frobenius extensions in [21] to study Hopf algebra actions on strongly separable extensions.
Our starting-point is a notion of depth two for ring extensions $N \subset M$ that was introduced by Kadison and Szlachányi in [22]. Their definition of a depth two extension of algebras possesses an equivalent formulation in terms of a quasibasis. It involves an equality that resembles one of the first two torsor axioms. We prove that irreducible depth two extensions $N \subset M$ with trivial centralizer $R := M^N \cong k$ can be embedded into the concept of N-torsor. Then previous results about torsors imply that any depth two ex-

tension $N \subset M$ is Hopf-Galois with respect to a coaction of the Hopf algebra $(M \otimes_N M)^N$. For this we only have to require that M be a faithfully flat N-module.
Such an extension $N \subset M$ gives rise to a generalized Jones tower for depth two Frobenius extensions. The same reasoning as above can be successively applied to each of its components. Even the assumption on faithful flatness can be dropped. We obtain that the tower consists entirely of Hopf-Galois extensions. Our approach extends results and simplifies proofs of [21] and [22], where just the first three components of the tower were considered under more restrictive assumptions.
We introduce the concept of A-B-torsor in order to study depth two extensions that are not necessarily irreducible. It is based on an $A \otimes B$-ring with a torsor structure map $T \longrightarrow T \otimes_A T \otimes_B T$. One example of such a structure is given by $\times_A$-Hopf algebras, and another by depth two extensions (which then become R-N-torsors). We prove that an A-B-torsor T gives rise to a left and a right bialgebroid. Via faithfully flat descent we can moreover recover the coactions of both a $\times_B$-Hopf algebra and a $\times^A$-Hopf algebra on T. These are different from the two bialgebroids we constructed at first. They coact on T such that it becomes a left and right Hopf-Galois extension in the sense of a definition given in [39]. For a $\times_A$-Hopf algebra L, this construction gives back L itself as well as the $\times^A$-Hopf algebra L^{opcop}.
We eventually apply our general concept of noncommutative torsor to the Jones tower for a depth two Frobenius extension $N \subset M$. Again, we can extend the results in [22], where just the first three components were proved to carry bialgebroid coactions. It turns out that in this particular case the previously constructed right bialgebroid is equal to the $\times^R$-Hopf algebra that coacts on T. This yields, as our final result, a Frobenius tower consisting entirely of right $\times^R$-Hopf-Galois extensions.
Hence, we see that the intrinsic reformulation of a Hopf-Galois extension as a noncommutative torsor can be indeed of practical use. With the method of faithfully flat descent at hand, it allows to recover principal homogeneous spaces and coactions of quantum groupoids in cases where such structures might not be obvious at first.

The appendix contains an exposition on some tools and notions that are used in the main part of the text, such as faithfully flat descent and reconstruction theorems for cohomomorphism objects. We give a short introduction to the theory of braided monoidal categories and inner hom-functors. We also explain the main features of the graphical notation that is used in the second and third chapter.

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Chapter 1

Three Concepts of Noncommutative Torsor

1.1 Preliminaries

In this section we fix some notations concerning Hopf algebras and their modules and comodules. A detailed discussion of the theory of Hopf algebras can be found in the literature, see for example [28], [23], [49] and others.

Throughout, k will denote a commutative ring. In some specific cases we will work over a field, which we then denote by $\mathbb{K}$. Unless otherwise mentioned, will write $\mathcal{M} = \mathcal{M}_k$ for the category of k-modules, and $\otimes = \otimes_k$ for the tensor product over k. If no confusion can arise, we denote for two k-modules $M, N \in \mathcal{M}$ by $\mathrm{Hom}(M, N) = \mathrm{Hom}_k(M, N)$ the k-module of k-linear maps $M \longrightarrow N$.

Some of our results are also valid in monoidal categories resp. braided monoidal categories. The basic definitions concerning these are given in the appendix, where also an introduction to graphical calculus in monoidal categories can be found.

Let R be a ring. Recall, for instance from [6], that a right R-module M is called *flat over R* if, whenever

$$N' \xrightarrow{f} N \xrightarrow{g} N'' \tag{1.1}$$

is an exact sequence of left R-modules, then the sequence

$$M \otimes_R N' \xrightarrow{M \otimes_R f} M \otimes_R N \xrightarrow{M \otimes_R g} M \otimes_R N'' \tag{1.2}$$

is also exact. A right R-module M for which (1.2) is exact if and only if (1.1)

is exact, is called *faithfully flat over* R. Another equivalent characterization of faithful flatness is that M is flat and $M \otimes_R N = 0$ implies $N = 0$ for each left R-module N.
Flatness and faithful flatness for left R-modules is defined analogously. If M is an R-bimodule, then we use the expressions *right faithfully flat* resp. *left faithfully flat* over R to distinguish faithful flatness of M as a right R module from faithful flatness of M as a left R-module.
It is well-known that projective modules are flat and that free modules are faithfully flat. More examples of flat and faithfully flat modules can be found in [6]. Faithfully flat modules have a property that is known as faithfully flat descent. This mechanism is explained in the appendix.

We will always denote the multiplication and unit map of a k-algebra A by $\nabla : A \otimes A \longrightarrow A$ and $\eta : k \longrightarrow A$, and the comultiplication and counit of a k-coalgebra C by $\Delta : C \longrightarrow C \otimes C$ and $\varepsilon : C \longrightarrow k$.

By definition, a k-bialgebra H is both an algebra and a coalgebra over k such that Δ and ε are algebra morphisms. Equivalently, one could also require ∇ and η to be morphisms of coalgebras. We note that a k-bialgebra H which is flat as a k-module, is already faithfully flat over k. This follows from the observation that k is a direct summand of H because of $\varepsilon\eta = \mathrm{id}_k$, which holds since ε is an algebra morphism.

Let A be a k-algebra, and $M \in {}_A\mathcal{M}$ a left A-module. We will usually denote the left action of A on M by am or $a \cdot m$, but if necessary, we also use symbols like $a \rhd m$ or $a \blacktriangleright m$ for $a \in A$ and $m \in M$.

For a k-coalgebra C and a right C-comodule $N \in \mathcal{M}^C$, we use Sweedler notation (see [49]) to write the comultiplication of C as $\Delta(c) = c_{(1)} \otimes c_{(2)}$ for $c \in C$, and the coaction $\delta : N \longrightarrow N \otimes C$ as $\delta(n) = n_{(0)} \otimes n_{(1)}$ for $n \in N$. Then the comodule axioms $(\delta \otimes \mathrm{id}_C)\delta = (\mathrm{id}_N \otimes \Delta)\delta$ and $(\mathrm{id}_N \otimes \varepsilon)\delta = \mathrm{id}_N$ can be expressed as $\delta(n_{(0)}) \otimes n_{(1)} = n_{(0)} \otimes n_{(1)} \otimes n_{(2)}$ and $n_{(0)}\varepsilon(n_{(1)}) = n$ for all $n \in N$. For a left C-comodule N', the notation is $\delta(n') = n'_{(-1)} \otimes n'_{(0)}$ for $n' \in N'$.

Let H be a bialgebra over k. A right H-comodule algebra A is by definition an algebra in the monoidal category $\mathcal{M}^H$ of right H-comodules. This means that A is both an algebra and an H-comodule such that the multiplication $\nabla : A \otimes A \longrightarrow A$ and the unit $\eta : k \longrightarrow A$ are H-colinear maps, where $A \otimes A$ becomes an H-comodule via the codiagonal action $\delta(a \otimes b) := a_{(0)} \otimes b_{(0)} \otimes a_{(1)}b_{(1)}$ for $a, b \in A$, and k has the trivial comodule structure induced by the unit of H. In other words, we then have $\delta(ab) = (ab)_{(0)} \otimes (ab)_{(1)} = a_{(0)}b_{(0)} \otimes a_{(1)}b_{(1)}$ for $a, b \in A$, and $\delta(1) = 1_{(0)} \otimes 1_{(1)} = 1 \otimes 1$. Equivalently, one could also require that the comodule structure map $\delta : A \longrightarrow A \otimes H$ be an algebra morphism.

For any right H-comodule M we let

$$M^{coH} := \{m \in M \mid m_{(0)} \otimes m_{(1)} = m \otimes 1\}$$

be the set of H*-coinvariant elements* of M. It is easy to see that for an H-comodule algebra A, the set A^{coH} is a subalgebra of A. The set of H-coinvariant elements for a left H-comodule N is defined by

$$^{coH}N := \{n \in N \mid n_{(-1)} \otimes n_{(0)} = 1 \otimes n\} \; .$$

Now let H be a Hopf algebra. This means by definition that the identity map $\mathrm{id}_H : H \longrightarrow H$ is convolution invertible in $\mathrm{Hom}(H, H)$, i.e. there exists a k-linear map $S : H \longrightarrow H$, the so-called *antipode* of H, such that $S * \mathrm{id}_H = \eta\varepsilon = \mathrm{id}_H * S$. The convolution is an associative and unitary (with unit $\eta\varepsilon$) multiplication on $\mathrm{Hom}(H, H)$, defined by $(f * g)(h) := f(h_{(1)})g(h_{(2)})$ for $f, g \in \mathrm{Hom}(H, H)$ and $h \in H$. Consequently, the antipode S satisfies the equation

$$S(h_{(1)})h_{(2)} = \varepsilon(h)1 = h_{(1)}S(h_{(2)})$$

for all $h \in H$. It is a basic result in Hopf algebra theory that the antipode, if it exists, is an algebra morphism $S : H \longrightarrow H^{opcop}$, and uniquely determined by the other structure morphisms of H.
If a bialgebra H possesses an antipode S, then it is also an antipode for the opposite coopposite bialgebra H^{opcop}. The opposite bialgebra H^{op} and the coopposite bialgebra H^{cop} become Hopf algebras if and only if the antipode of H is invertible. Then S^{-1} is an antipode for both H^{op} and H^{cop}.

1.2 Hopf-Galois Extensions

Hopf-Galois extensions were introduced by Chase and Sweedler [10] in the commutative case and by Kreimer and Takeuchi [25] for the case of finite dimensional Hopf algebras. Their properties dualize the axioms of a G-torsor when H is chosen as the coordinate ring of an affine group scheme $G = \mathrm{Spec}(H)$. The same definition is also valid in the noncommutative case and leads to the first concept of noncommutative torsor:

Definition 1.2.1 Let H be a k-bialgebra.
A right H-comodule algebra A is called a *right* H*-Galois extension* of $B := A^{coH}$, if the Galois map

$$\beta_r : A \otimes_B A \longrightarrow A \otimes H \; , \; x \otimes y \mapsto xy_{(0)} \otimes y_{(1)}$$

is a bijection.
A left H-comodule algebra A is called a *left H-Galois extension* of $B := {}^{coH}A$, if the Galois map

$$\beta_l : A \otimes_B A \longrightarrow H \otimes A \ , \ x \otimes y \mapsto x_{(-1)} \otimes x_{(0)}y$$

is a bijection.

By [42] and references therein, a k-flat bialgebra that admits a faithfully flat Hopf-Galois extension is a Hopf algebra. This can be proved by taking advantage of a connection between the Galois map β_r and the map β_H in the following example. It shows that every k-Hopf algebra H can be characterized as bialgebra H that becomes an H-Galois extension of k in a natural way.

Example 1.2.2 Let H be a k-Hopf algebra with antipode S. Then H is naturally a right H-comodule algebra.
It is clear that $k \subset H^{coH} = \{x \in H \mid x_{(1)} \otimes x_{(2)} = x \otimes 1\}$. For $x \in H^{coH}$ we have $x = (\varepsilon \otimes \mathrm{id})\Delta(x) = \varepsilon(x) \cdot 1 \in k$, and thus the coinvariants are $H^{coH} = k$. The Galois map is given by

$$\beta_H : H \otimes H \longrightarrow H \otimes H \ , \ h \otimes g \mapsto hg_{(1)} \otimes g_{(2)} \ .$$

It follows from the antipode axioms that this map is bijective with its inverse given by ${\beta_H}^{-1}(h \otimes g) = hS(g_{(1)}) \otimes g_{(2)}$. This implies that H is an H-Galois extension of k. □

Let H be a k-bialgebra and A a right H-comodule algebra. Whenever A is faithfully flat over B, then bijectivity of $\beta_r : A \otimes_B A \longrightarrow A \otimes H$ implies that $A^{coH} \cong B$. This is shown for the special case $B = k$ in [55]. Moreover, we see that then also H is faithfully flat over k, since $A \otimes H \cong A \otimes A$ is faithfully flat over A.

Definition 1.2.3 Let H be a k-flat bialgebra, and A a right H-Galois extension of k. If A is moreover a faithfully flat k-module, we will call it an *H-Galois object*.

Example 1.2.4 The notion of Hopf-Galois extension generalizes the classical notion of a Galois extension of fields, see also [28]:
Let $\mathbb{K} \subset F$ be a field extension and let G be a finite group that acts on F via a group homomorphism $\rho : G \longrightarrow \mathrm{Aut}(F)$ with $\rho(g)(x) := g \cdot x$ for all $g \in G$ and $x \in F$. Let $H := \mathbb{K}[G]^*$ be the dual of the group ring $\mathbb{K}[G]$ with the canonical $\mathbb{K}$-basis $(e_g)_{g \in G}$. The the action of G determines an H-comodule structure on F by $\delta(x) = \sum_{g \in G} g \cdot x \otimes e_g$ for all $x \in F$. Let $R := \{x \in F \mid g \cdot x = x \ \forall \ g \in G\}$.

Assume that F is an H-Galois extension of R, that is, the Galois map $\beta : F \otimes_R F \longrightarrow F \otimes H\ ,\ x \otimes y \mapsto \sum_{g\in G} x(g \cdot y) \otimes e_g$ is an R-linear isomorphism. We have $\mathrm{Im}(\rho) = \mathrm{Aut}(F/R)$ by definition of R, and the induced map $\tilde{\rho} : G \longrightarrow \mathrm{Aut}(F/R)$ is surjective. But we really have $|G| = |\mathrm{Aut}(F/R)|$, since the assumption $|\mathrm{Aut}(F/R)| = [F : R] =: m \neq n := |G|$ implies $\dim_R(A \otimes_R A) = m^2 \neq nm = \dim_R(A \otimes H)$ in contradiction to β being bijective. Hence, it follows that $R \subset F$ is a Galois field extension with Galois group G.

Conversely, assume that $R \subset F$ is a Galois field extension. Then we have $[F : R] = |G|$ and the Galois map is given by $\beta : F \otimes_R F \longrightarrow F \otimes H\ ,\ x \otimes y \mapsto \sum_{g\in G} x(g \cdot y) \otimes e_g$. If $\sum_i x_i \otimes y_i \in Ke(\beta)$, then $\sum_i x_i(g \cdot y_i) = 0$ for all $g \in G$. Now Dedekind's lemma on independence of characters implies that the matrix $(g \cdot y_i)_{i,g}$ is invertible. This means that $x_i = 0$ for all i and so β is injective. Since $F \otimes_R F$ and $F \otimes H$ have the same dimension over R, it follows that β is bijective. Therefore, F is an H-Galois extension of R. □

Let A be a right H-Galois extension of B. For calculations with the inverse of β_r, we use the notation

$$\beta_r{}^{-1}(1 \otimes h) := h^{[1]} \otimes h^{[2]} \in A \otimes_B A\ .$$

Then we have by definition

$$h^{[1]} h^{[2]}{}_{(0)} \otimes h^{[2]}{}_{(1)} = 1 \otimes h \tag{1.3}$$

for all $h \in H$, and the following formulas hold for $g, h \in H$, $a \in A$ and $b \in B$ by [47]:

$$h^{[1]} \otimes_B h^{[2]}{}_{(0)} \otimes h^{[2]}{}_{(1)} = h_{(1)}{}^{[1]} \otimes_B h_{(1)}{}^{[2]} \otimes h_{(2)} \tag{1.4}$$

$$h^{[1]}{}_{(0)} \otimes_B h^{[2]} \otimes h^{[1]}{}_{(1)} = h_{(2)}{}^{[1]} \otimes_B h_{(2)}{}^{[2]} \otimes S(h_{(1)}) \tag{1.5}$$

$$h^{[1]} h^{[2]} = \varepsilon(h) 1_A \tag{1.6}$$

$$(gh)^{[1]} \otimes_B (gh)^{[2]} = h^{[1]} g^{[1]} \otimes_B g^{[2]} h^{[2]} \tag{1.7}$$

$$b h^{[1]} \otimes_B h^{[2]} = h^{[1]} \otimes_B h^{[2]} b \tag{1.8}$$

$$h^{[1]} \otimes_B 1 \otimes_B h^{[2]} = h_{(1)}{}^{[1]} \otimes_B h_{(1)}{}^{[2]} h_{(2)}{}^{[1]} \otimes_B h_{(2)}{}^{[2]} \tag{1.9}$$

$$a_{(0)} a_{(1)}{}^{[1]} \otimes_B a_{(1)}{}^{[2]} = 1 \otimes_B a \tag{1.10}$$

We note for further applications that in case the antipode S of H is invertible, also the equality

$$S^{-1}(a_{(1)})^{[1]} \otimes S^{-1}(a_{(1)})^{[2]} a_{(0)} = a \otimes 1 \tag{1.11}$$

holds in $A \otimes_B A$, as can be seen by applying the isomorphism β_r to both sides, and using (1.4):

$$\begin{aligned}
&\beta_r(S^{-1}(a_{(1)})^{[1]} \otimes S^{-1}(a_{(1)})^{[2]} a_{(0)}) = \\
&\qquad = S^{-1}(a_{(2)})^{[1]} S^{-1}(a_{(2)})^{[2]}{}_{(0)} a_{(0)} \otimes S^{-1}(a_{(2)})^{[2]}{}_{(1)} a_{(1)} = \\
&\qquad = S^{-1}(a_{(2)})_{(1)}{}^{[1]} S^{-1}(a_{(2)})_{(1)}{}^{[2]} a_{(0)} \otimes S^{-1}(a_{(2)})_{(2)} a_{(1)} = \\
&\qquad = S^{-1}(a_{(3)})^{[1]} S^{-1}(a_{(3)})^{[2]} a_{(0)} \otimes S^{-1}(a_{(2)}) a_{(1)} = \\
&\qquad\qquad = a \otimes 1 = \beta_r(a \otimes 1) \ .
\end{aligned}$$

The following definition is taken from [38]:

Definition 1.2.5 Let L and H be Hopf algebras. An algebra A is called an *L-H-bi-Galois extension of k*, if A is both a left L-Galois extension of k and a right H-Galois extension of k such that the two comodule structures make it an L-H-bicomodule.

The following result of [38] says that each right H-Galois object A is in fact a Hopf-bi-Galois extension:

Theorem 1.2.6 *([38]) Let H be a Hopf algebra and let A be a faithfully flat H-Galois extension of k. Then $L := L(A,H) := (A \otimes A)^{coH}$, where the H-comodule structure on $A \otimes A$ is the codiagonal one, is a subalgebra of $A \otimes A^{op}$ and has the structure of a Hopf algebra with the comultiplication, counit and antipode given by*

$$\begin{aligned}
\Delta(\sum x_i \otimes y_i) &= \sum x_{i(0)} \otimes x_{i(1)}{}^{[1]} \otimes x_{i(1)}{}^{[2]} \otimes y_i \\
\varepsilon(\sum x_i \otimes y_i) &= \sum x_i y_i \in A^{coH} = k \\
S(\sum x_i \otimes y_i) &= \sum y_{i(0)} \otimes y_{i(1)}{}^{[1]} x_i y_{i(1)}{}^{[2]} \ .
\end{aligned}$$

A becomes a left L-Galois extension of k under the left L-coaction

$$\delta_{L,A} : A \longrightarrow L \otimes A \ , \ a \mapsto a_{(0)} \otimes a_{(1)}{}^{[1]} \otimes a_{(1)}{}^{[2]} \ ,$$

such that A is an L-H-bi-Galois extension of k.
Moreover, $L(A,H)$ satisfies the following universal property: For any bialgebra B and left B-comodule structure $\delta_{B,A}$ on A that makes A a B-H-bi-Galois extension, there is a unique isomorphism of bialgebras $\varphi : L \longrightarrow B$ such that $\delta_{B,A} = (\varphi \otimes A)\delta_{L,A}$.

If H is cocommutative and A is a faithfully flat right H-Galois extension of k, then $L(A,H) \cong H$ by [38]. Starting from a left L-Galois object A, it is possible to construct a Hopf algebra $R := R(A,L)$ such that A becomes an L-R-bi-Galois extension of k. Applying the construction $L(A,-)$ again yields $L(A,R(A,L)) \cong L$.

Hopf-Galois extensions have an interesting connection to the braided monoidal category of Yetter-Drinfeld modules introduced in [60].

We recall from [60] or [28] that a right-right *Yetter-Drinfeld module* $M \in \mathcal{YD}_H^H$ over a Hopf algebra H is a right H-module and right H-comodule satisfying the compatibility condition

$$m_{(0)} \triangleleft h_{(1)} \otimes m_{(1)}h_{(2)} = (m \triangleleft h_{(2)})_{(0)} \otimes h_{(1)}(m \triangleleft h_{(2)})_{(1)} \tag{1.12}$$

for all $m \in M$, where $\triangleleft$ denotes the right H-module structure of M. Morphisms in $\mathcal{YD}_H^H$ are the H-linear and H-colinear maps.
The category $\mathcal{YD}_H^H$ is monoidal with the tensor product of the underlying category $\mathcal{M}$, and prebraided by

$$\sigma : M \otimes N \longrightarrow N \otimes M \ , \ m \otimes n \mapsto n_{(0)} \otimes m \triangleleft n_{(1)} \ . \tag{1.13}$$

This morphism is invertible if the antipode of H is bijective, and in this case the inverse is given by

$$\sigma^{-1}(n \otimes m) = m \triangleleft S_H^{-1}(n_{(1)}) \otimes n_{(0)}$$

for $m \in M$, $n \in N$, and so the category $\mathcal{YD}_H^H$ is braided.

One can also define the category ${}_H\mathcal{YD}^H$ of left-right Yetter-Drinfeld modules over H. Its objects are left H-modules and right H-comodules M satisfying the condition

$$h_{(1)} \triangleright m_{(0)} \otimes h_{(2)}m_{(1)} = (h_{(2)} \triangleright m)_{(0)} \otimes (h_{(2)} \triangleright m)_{(1)}h_{(1)} \tag{1.14}$$

for all $m \in M$ and $h \in H$, which can easily be shown to be equivalent to

$$h_{(2)} \triangleright m_{(0)} \otimes h_{(3)}m_{(1)}S^{-1}(h_{(1)}) = (h \triangleright m)_{(0)} \otimes (h \triangleright m)_{(1)} \ . \tag{1.15}$$

Left-right Yetter-Drinfeld modules can be interpreted as right-right Yetter-Drinfeld modules over the opposite Hopf algebra H^{op}.

Let H be a Hopf algebra with bijective antipode and A a right H-Galois extension of B. We denote by

$$A^B = \{a \in A \mid ab = ba \ \forall \ b \in B\}$$

the centralizer of B in A, which is clearly a subalgebra of A. The *Miyashita-Ulbrich action* of H on A^B is defined by

$$a \triangleleft h := h^{[1]} a h^{[2]} \tag{1.16}$$

for $a \in A^B$ and $h \in H$, and makes A^B into a right H-module.

It was shown in [13] that the right Miyashita-Ulbrich action and the restricted right coaction of A make A^B into a Yetter-Drinfeld module. Moreover, A^B is a commutative algebra in the category $\mathcal{YD}_H^H$ with respect to the braiding (1.13).

1.3 Hopf-Galois Systems

We have seen in the previous section that a k-faithfully flat right H-Galois extension A of k always comes together with another Hopf algebra L that makes it into a left L-Galois extension of k. This behaviour is naturally described in the following notion of a Hopf-Galois system introduced by Bichon in [3]:

Definition 1.3.1 A *Hopf-Galois system* consists of four non-zero algebras (L, H, T, Z) such that

- L and H are bialgebras
- T is an L-H-bicomodule algebra
- There are algebra morphisms $\rho_L : L \longrightarrow T \otimes Z$ and $\rho_H : H \longrightarrow Z \otimes T$ such that the following diagrams commute:

$$\begin{CD}
T @>{\delta_{L,T}}>> L \otimes T\\
@V{\delta_{T,H}}VV @VV{\rho_L \otimes \mathrm{id}}V\\
T \otimes H @>>{\mathrm{id} \otimes \rho_H}> T \otimes Z \otimes T
\end{CD}$$

$$\begin{CD}
L @>{\Delta_L}>> L \otimes L\\
@V{\rho_L}VV @VV{\mathrm{id} \otimes \rho_L}V\\
T \otimes Z @>>{\delta_{L,T} \otimes \mathrm{id}}> L \otimes T \otimes Z
\end{CD}
\qquad
\begin{CD}
H @>{\Delta_H}>> H \otimes H\\
@V{\rho_H}VV @VV{\rho_H \otimes \mathrm{id}}V\\
Z \otimes T @>>{\mathrm{id} \otimes \delta_{T,H}}> Z \otimes T \otimes H
\end{CD}$$

- There is a linear map $S : Z \longrightarrow T$ such that the following diagrams commute:

$$\begin{array}{ccccc} L & \xrightarrow{\varepsilon_L} & I & \xrightarrow{\eta_T} & T \\ {\scriptstyle\rho_L}\downarrow & & & & \uparrow{\scriptstyle\nabla_T} \\ T \otimes Z & & \xrightarrow[\mathrm{id} \otimes S_Z]{} & & T \otimes T \end{array} \qquad \begin{array}{ccccc} H & \xrightarrow{\varepsilon_H} & I & \xrightarrow{\eta_T} & T \\ {\scriptstyle\rho_H}\downarrow & & & & \uparrow{\scriptstyle\nabla_T} \\ Z \otimes T & & \xrightarrow[S_Z \otimes \mathrm{id}]{} & & T \otimes T \end{array}$$

It turns out that a Hopf-Galois system contains exactly the data needed in order to obtain a Bi-Galois extension:

Theorem 1.3.2 *([3]) Let (L, H, T, Z) be a Hopf-Galois system. Then T is an L-H-bi-Galois extension. The inverse for the Galois map $\beta : T \otimes T \longrightarrow T \otimes H$ is given by*

$$\beta^{-1} : T \otimes H \longrightarrow T \otimes T \ , \ \beta^{-1} := (\nabla_T \otimes \mathrm{id}_T)(\mathrm{id}_T \otimes S \otimes \mathrm{id}_T)(\mathrm{id}_T \otimes \rho_H) \ .$$

We note that the above definition of a Hopf-Galois system can be stated in any braided monoidal category. In particular no assumptions on faithful flatness have to be made in order to prove Theorem 1.3.2.

Bichon [3] showed that one can also go in the opposite direction by using reconstruction methods due to Tannaka-Krein: Given an H-Galois extension A over a field $\mathbb{K}$, one can recover a Hopf-Galois system from it.

The existence and properties of the algebra Z in a Hopf-Galois system can be understood as follows:
It is shown in [38] that all faithfully flat Hopf algebras form a category $\mathcal{H}$ with morphisms the isomorphism classes of faithfully flat bi-Galois extensions of k. More precisely, the morphism set $\mathrm{Mor}_{\mathcal{H}}(L, R)$ of two Hopf algebras L and R consists of the isomorphism classes of all L-R-bi-Galois extensions. Let L, H and R be faithfully flat Hopf algebras and A a faithfully flat L-H-bi-Galois extension of k and B a faithfully flat H-R-bi-Galois extension of k. Then the composition of morphisms in $\mathcal{H}$ is given by the cotensor product $A \otimes_H B$, that is the equalizer

$$A \square_H B \longrightarrow A \otimes B \underset{\delta_{A,H} \otimes B}{\overset{A \otimes \delta_{H,B}}{\rightrightarrows}} A \otimes H \otimes B \ ,$$

and $A \otimes_H B$ is shown to be a faithfully flat L-R-bi-Galois extension of k. The inverse for A with respect to this composition is given by $A^{-1} := (H \otimes A)^{coH}$, which is a left H-subcomodule-algebra of $H \otimes A^{op}$ and a faithfully flat H-L-Bi-Galois extension of k. If the antipode of H is bijective, it turns out that $A^{-1} \cong A^{op}$. The algebra Z in a Hopf-Galois system (L, H, T, Z) plays the role of T^{-1}.

1.4 Quantum Torsors

We now arrive at the definition of our main subject of investigation, namely quantum torsors.
Let X be a principal homogeneous space under the action of a group G. By Kontsevich's definition in [24] it is possible to encode the action of G on X in a map $X \times X \times X \longrightarrow X$ which satisfies certain "parallelogram" axioms. The group G can then be reconstructed from this structure map. In the noncommutative setting, we have seen above that both definitions of Hopf-Galois extension and Hopf-Galois system made use of the coaction of one resp. two bialgebras. The concept of quantum torsor presents a way, dual to Kontsevich's approach, to formulate these properties in just one structure map. This map has the advantage of "hiding" the coactions of both Hopf algebras. Consequently, it allows to write down the properties of a Hopf-Galois extension without having to know about the coaction of a Hopf algebra. This was worked out by Grunspan in [18]:

Definition 1.4.1 A *quantum torsor* over k is a k-algebra (T, ∇, η) together with algebra morphisms

$$\mu : T \longrightarrow T \otimes T^{op} \otimes T \quad \text{and} \quad \theta : T \longrightarrow T$$

that satisfy the axioms

1) $(\nabla \otimes \mathrm{id})\mu = \eta \otimes \mathrm{id}$

2) $(\mathrm{id} \otimes \nabla)\mu = \mathrm{id} \otimes \eta$

3) $(\mathrm{id} \otimes \mathrm{id} \otimes \mu)\mu = (\mu \otimes \mathrm{id} \otimes \mathrm{id})\mu$

4) $(\mathrm{id} \otimes \mathrm{id} \otimes \theta \otimes \mathrm{id} \otimes \mathrm{id})(\mu \otimes \mathrm{id} \otimes \mathrm{id})\mu = (\mathrm{id} \otimes \mu^{op} \otimes \mathrm{id})\mu$

5) $(\theta \otimes \theta \otimes \theta)\mu = \mu \circ \theta$,

where $\mu^{op} := \tau_{13} \circ \mu$ is defined as the composition of μ with the twist $\sigma_{13} : T \otimes T \otimes T \longrightarrow T \otimes T \otimes T$, $x \otimes y \otimes z \mapsto z \otimes y \otimes x$ that interchanges the first and the third tensorand.

The torsor T is called *commutative*, if its underlying algebra structure is commutative. If $\mu = \mu^{op}$, then the torsor is said to be *endowed with a commutative law.*

Clearly, the notion of quantum torsor could also be defined in any braided monoidal category. In $\mathcal{M}_k$, we use generalized Sweedler notation to write

$$\mu(t) := t^{(1)} \otimes t^{(2)} \otimes t^{(3)} \in T \otimes T^{op} \otimes T$$

for each $t \in T$. Since μ is required to be an algebra morphism, we have for $t, s \in T$

$$\mu(ts) = (ts)^{(1)} \otimes (ts)^{(2)} \otimes (ts)^{(3)} = t^{(1)}s^{(1)} \otimes s^{(2)}t^{(2)} \otimes t^{(3)}s^{(3)} \ .$$

Furthermore, we have $\mu^{op}(t) = t^{(3)} \otimes t^{(2)} \otimes t^{(1)}$ for all $t \in T$, and so the axioms of a quantum torsor read as

$$\begin{aligned} t^{(1)}t^{(2)} \otimes t^{(3)} &= 1 \otimes t \\ t^{(1)} \otimes t^{(2)}t^{(3)} &= t \otimes 1 \\ t^{(1)} \otimes t^{(2)} \otimes t^{(3)(1)} \otimes t^{(3)(2)} \otimes t^{(3)(3)} &= t^{(1)(1)} \otimes t^{(1)(2)} \otimes t^{(1)(3)} \otimes t^{(2)} \otimes t^{(3)} \\ t^{(1)(1)} \otimes t^{(1)(2)} \otimes \theta(t^{(1)(3)}) \otimes t^{(2)} \otimes t^{(3)} &= t^{(1)} \otimes t^{(2)(3)} \otimes t^{(2)(2)} \otimes t^{(2)(1)} \otimes t^{(3)} \\ \theta(t^{(1)}) \otimes \theta(t^{(2)}) \otimes \theta(t^{(3)}) &= \theta(t)^{(1)} \otimes \theta(t)^{(2)} \otimes \theta(t)^{(3)} \ . \end{aligned}$$

Axiom 3) is a "coassociativity" condition on μ, and hence it is well-defined to introduce the notation

$$\begin{aligned} t^{(1)} \otimes t^{(2)} \otimes t^{(3)} \otimes t^{(4)} \otimes t^{(5)} &:= t^{(1)(1)} \otimes t^{(1)(2)} \otimes t^{(1)(3)} \otimes t^{(2)} \otimes t^{(3)} \\ &= t^{(1)} \otimes t^{(2)} \otimes t^{(3)(1)} \otimes t^{(3)(2)} \otimes t^{(3)(3)} \ . \end{aligned}$$

It is shown in [18] that the map θ is fully determined by μ and the algebra structure of T, and given by the formula

$$\theta(t) = t^{(1)}t^{(2)(3)}t^{(2)(2)}t^{(2)(1)}t^{(3)} = t^{(1)}t^{(4)}t^{(3)}t^{(2)}t^{(5)} \tag{1.17}$$

for all $t \in T$.

Example 1.4.2 ([18]) Let H be a Hopf algebra with antipode S. Then the algebra morphisms

$$\mu_H : H \longrightarrow H \otimes H^{op} \otimes H \ , \ h \mapsto h_{(1)} \otimes S(h_{(2)}) \otimes h_{(3)}$$

and

$$\theta_H : H \longrightarrow H \ , \ h \mapsto S^2(h)$$

make H into a quantum torsor.

By the antipode axioms we have $h_{(1)}S(h_{(2)}) \otimes h_{(3)} = 1 \otimes h$ and $h_{(1)} \otimes$

$S(h_{(2)})h_{(3)} = h \otimes 1$, which yields axioms 1) and 2). It is obvious that μ_H satisfies the coassociativity axiom 3). For the properties of θ_H, we get

$$\begin{aligned} h^{(1)} \otimes h^{(2)} \otimes S^2(h^{(3)}) \otimes h^{(4)} \otimes h^{(5)} \\ &= h_{(1)} \otimes S(h_{(2)}) \otimes S^2(h_{(3)}) \otimes S(h_{(4)}) \otimes h_{(5)} \\ &= h_{(1)} \otimes S(h_{(2)})_{(3)} \otimes S(S(h_{(2)})_{(2)}) \otimes S(h_{(2)})_{(1)} \otimes h_{(3)} \\ &= h^{(1)} \otimes h^{(2)(3)} \otimes h^{(2)(2)} \otimes h^{(2)(1)} \otimes h^{(3)} \end{aligned}$$

and

$$S^2(h_{(1)}) \otimes S^2(S(h_{(2)})) \otimes S^2(h_{(3)}) = S^2(h)_{(1)} \otimes S(S^2(h)_{(2)}) \otimes S^2(h)_{(3)} \ ,$$

showing 4) and 5). Grunspan calls this example the trivial torsor of a Hopf algebra. □

Remark 1.4.3 Given a quantum torsor T with structure maps μ and θ, the opposite algebra T^{op} becomes again a quantum torsor with the structure maps

$$\mu_{T^{op}} := \mu^{op} : T^{op} \otimes T \otimes T^{op} \ , \ t \mapsto t^{(3)} \otimes t^{(2)} \otimes t^{(1)}$$

and

$$\theta_{T^{op}} : T^{op} \longrightarrow T^{op} \ , \ t \mapsto \theta(t) \ .$$

From the axioms of a quantum torsor it is quite obvious what morphisms of quantum torsors should be:

Definition 1.4.4 ([18]) Let T and R be two quantum torsors. An algebra morphism $f : T \longrightarrow R$ is called a *morphism of quantum torsors* if it satisfies the equations

$$\mu_R \circ f = (f \otimes f \otimes f)\mu_T \tag{1.18}$$

$$\theta_R \circ f = f \circ \theta_T \ . \tag{1.19}$$

According to this definition, the map θ from Definition 1.4.1 can be considered as a morphism of quantum torsors $\theta : T \longrightarrow T$.

The main result of [18] can be summarized as follows:

Theorem 1.4.5 *([18]) Let T be a quantum torsor with the structure maps μ and θ. Assume that T is a faithfully flat k-module. Then*

$$\begin{aligned} H_r(T) := \{\textstyle\sum x_i \otimes y_i \in T^{op} \otimes T \mid \\ \textstyle\sum x_i \otimes \theta({y_i}^{(1)}) \otimes {y_i}^{(2)} \otimes {y_i}^{(3)} = \sum {x_i}^{(3)} \otimes {x_i}^{(2)} \otimes {x_i}^{(1)} \otimes y_i\} \end{aligned}$$

is a subalgebra of $T^{op} \otimes T$ *and becomes a Hopf algebra with the coalgebra structure maps*

$$\Delta : H_r(T) \longrightarrow H_r(T) \otimes H_r(T) \ , \ \sum x_i \otimes y_i \mapsto \sum x_i \otimes {y_i}^{(1)} \otimes {y_i}^{(2)} \otimes {y_i}^{(3)}$$

$$\varepsilon : H_r(T) \longrightarrow k \ , \ \sum x_i \otimes y_i \mapsto \sum x_i y_i$$

and the antipode

$$S : H_r(T) \longrightarrow H_r(T) \ , \ \sum x_i \otimes y_i \mapsto \sum \theta(y_i) \otimes x_i \ .$$

Symmetrically, also

$$H_l(T) := \{\sum x_i \otimes y_i \in T \otimes T^{op} \mid \sum {x_i}^{(1)} \otimes {x_i}^{(2)} \otimes \theta({x_i}^{(3)}) \otimes y_i = \sum x_i \otimes {y_i}^{(3)} \otimes {y_i}^{(2)} \otimes {y_i}^{(1)}\}$$

becomes a Hopf algebra, and both these Hopf algebras coact on T *such that* T *is an* $(H_l(T), H_r(T))$*-bicomodule algebra.*
Moreover, T *is a left* $H_l(T)$*-Galois extension of* k *and a right* $H_r(T)$*-Galois extension of* k*, which makes* T *an* $(H_l(T), H_r(T))$*-bi-Galois extension.*

Hence, each faithfully flat quantum torsor is in fact a Hopf-Galois extension, and the Hopf algebras coacting on the left resp. right are uniquely determined by the torsor structure map. The converse is also true, as Schauenburg has shown in [40]:

Theorem 1.4.6 *Let* H *be a* k*-faithfully flat Hopf algebra with antipode* S*, and let* T *be an* H*-Galois object. Then* T *is a quantum torsor with the structure maps given by*

$$\mu(x) = x_{(0)} \otimes {x_{(1)}}^{[1]} \otimes {x_{(1)}}^{[2]} \tag{1.20}$$

$$\theta(x) = S(x_{(1)})^{[1]} x_{(0)} S(x_{(1)})^{[2]} \ . \tag{1.21}$$

If the antipode of H *is invertible, then also* θ *is invertible with the inverse given by*

$$\theta^{-1}(x) = S^{-2}(x_{(1)})^{[1]} x_{(0)} S^{-2}(x_{(1)})^{[2]} \ . \tag{1.22}$$

This theorem allows us now to construct torsor structure maps for Hopf-Galois extensions. We show explicitly in the following example how one object can carry two different quantum torsor structures. They arise from coactions of two different Hopf algebras.

Example 1.4.7 We consider the field extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt[4]{2})$, that was also studied in [16]: Although it is not classically Galois (this is because $|\operatorname{Aut}(\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q})| = 2$, but $[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 4$), it is a Hopf-Galois extension of $\mathbb{Q}$ for two different Hopf algebras. Let $\omega := \sqrt[4]{2}$.

We consider $H := \mathbb{Q}[c,s]/(c^2 + s^2 - 1, cs)$, the so-called circle Hopf algebra, with the coalgebra structure given by $\Delta(c) = c \otimes c - s \otimes s$, $\Delta(s) = c \otimes s + s \otimes c$, $\varepsilon(c) = 1$, $\varepsilon(s) = 0$, and antipode $S(c) = c$, $S(s) = -s$. This Hopf algebra acts on $\mathbb{Q}(\omega)$ by the following table that can be found in [28]:

$\cdot$	1	ω	ω^2	ω^3
c	1	0	$-\omega^2$	0
s	0	$-\omega$	0	ω^3

The action of H is such that $\mathbb{Q}(\omega)$ becomes an H^*-Galois extension of $\mathbb{Q}$. As was shown in [16], there exists also an action of the Hopf algebra $K := \mathbb{Q}[a,b]/(b^2 - 2a^2 + 2, ab)$ on $\mathbb{Q}(\omega)$ such that $\mathbb{Q}(\omega)$ becomes an K^*-Galois extension of $\mathbb{Q}$. The structure maps of K are $\Delta(a) = a \otimes a - \frac{1}{2} b \otimes b$, $\Delta(b) = a \otimes b + b \otimes a$, $\varepsilon(a) = 1$, $\varepsilon(b) = 0$, $S(a) = a$ and $S(b) = b$, and its action on $\mathbb{Q}(\omega)$ is given by

$\cdot$	1	ω	ω^2	ω^3
a	1	0	$-\omega^2$	0
b	0	ω^3	0	-2ω

The Hopf algebras H and K are not isomorphic and lead to two different Hopf-Galois extensions $\mathbb{Q}(\omega)$ over $\mathbb{Q}$. So the above theorem says that we should get two different quantum torsor structures on $\mathbb{Q}(\omega)$. In fact, we obtain by a tedious calculation that the H^*-Galois extension induces the torsor structure

$$\mu_H(\omega) = \frac{1}{2} \omega \otimes \omega^3 \otimes \omega \ ,$$

on $\mathbb{Q}(\omega)$, while the action of K leads to the torsor structure map

$$\mu_K(\omega) = \frac{1}{4}(\omega \otimes \omega \otimes \omega^3 + \omega \otimes \omega^3 \otimes \omega + \omega^3 \otimes \omega \otimes \omega) - \frac{1}{8} \omega^3 \otimes \omega^3 \otimes \omega^3 \ .$$

We note that it is sufficient to define the respective maps $\mu : \mathbb{Q}(\omega) \longrightarrow \mathbb{Q}(\omega) \otimes \mathbb{Q}(\omega) \otimes \mathbb{Q}(\omega)$ on the algebra generator ω, since μ is supposed to be an algebra morphism. □

In [41] Schauenburg made the key observation that each algebra T, for which there exists an algebra morphism $\mu : T \longrightarrow T \otimes T^{op} \otimes T$ satisfying the properties 1), 2) and 3) from Definition 1.4.1, gives rise to a descent data on $T \otimes T$. Then he proved the following result using faithfully flat descent.

For the sake of simplicity, we abuse notation below, and denote elements in the subset of a tensor product as if they were decomposable tensors.

Theorem 1.4.8 *([41]) Let T be a k-faithfully flat algebra with an algebra morphism $\mu : T \longrightarrow T \otimes T^{op} \otimes T$, $\mu(t) := t^{(1)} \otimes t^{(2)} \otimes t^{(3)}$, such that 1), 2) and 3) in Definition 1.4.1 hold. Then*

$$H := \{x \otimes y \in T \otimes T \mid xy^{(1)} \otimes y^{(2)} \otimes y^{(3)} = 1 \otimes x \otimes y\}$$

is a Hopf algebra. The algebra structure is that of a subalgebra of $T^{op} \otimes T$, and the comultiplication and counit are given by

$$\begin{aligned} \Delta(x \otimes y) &= x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)} \\ \varepsilon(x \otimes y) &= xy \ . \end{aligned}$$

The algebra T is a right H-Galois object under the coaction

$$\delta(x) = x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \subset T \otimes H \ .$$

This yields together with Theorem 1.4.6, that the existence of μ as in Definition 1.4.1 already implies that the map θ given by (1.17) satisfies the axioms 4) and 5). Hence, for faithfully flat torsors, the existence of the map θ can be dropped from the set of torsor axioms.

As proposed in [41], we are going to call the algebra endomorphism $\theta : T \longrightarrow T$ from Definition 1.4.1 the *Grunspan map* of the quantum torsor T.

At this point we can observe a similarity between quantum torsors and commutative torsors. We recall that by a definition of Kontsevich's in [24], a torsor is a non-empty set X that is endowed with a map $\lambda : X \times X \times X \longrightarrow X$ satisfying the identities

1) $\lambda(a, a, b) = b$
2) $\lambda(a, b, b) = a$
3) $\lambda(\lambda(a, b, c), d, e) = \lambda(a, b, \lambda(c, d, e))$
4) $\lambda(a, b, \lambda(c, d, e)) = \lambda(a, \lambda(d, c, b), e)$

for all $a, b, c, d, e \in X$. These are the properties of the ternary operation $(a, b, c) \mapsto ab^{-1}c$ in groups [1, 8]. The axioms of a quantum torsor are such that they dualize these identities. The Grunspan map θ is actually needed for stating a noncommutative analogue to equality 4). But we can prove that this equality can already be derived from the first three axioms as follows:

$$\begin{aligned}\lambda(a,\lambda(d,c,b),e) &= \lambda(a,\lambda(d,c,b),\lambda(d,d,e))\\ &= \lambda(a,\lambda(d,c,b),\lambda(d,c,\lambda(c,d,e)))\\ &= \lambda(a,\lambda(d,c,b),\lambda(d,c,\lambda(b,b,\lambda(c,d,e))))\\ &= \lambda(a,\lambda(d,c,b),\lambda(\lambda(d,c,b),b,\lambda(c,d,e)))\\ &= \lambda(\lambda(a,\lambda(d,c,b),\lambda(d,c,b)),b,\lambda(c,d,e))\\ &= \lambda(a,b,\lambda(c,d,e))\ .\end{aligned}$$

So in this context, it is not such a great surprise that a definition of quantum torsor can really do without the existence of a Grunspan map. Nevertheless, such a map is always there.

Chapter 2

Torsor Structures in “Nature”

2.1 Fiber Functors and Cohomomorphism Objects

Let H be a k-flat Hopf algebra. As was shown by Ulbrich in [54] and [55], the H-Galois extensions of k are in one-to-one correspondence with so-called fiber functors from the category of left H-comodules to the category of k-modules. This is a noncommutative version of a result for classical torsors shown by Saavedra Rivano and Deligne-Milne.

We recall the definition of fiber functor from [55]. It is the noncommutative analogue to the notion of fiber functor for commutative Hopf algebras as introduced in [36].

Definition 2.1.1 Let H be a k-flat Hopf algebra. A k-linear functor $\omega : {}^H\mathcal{M} \longrightarrow \mathcal{M}_k$ from the category of left H-comodules to the category of k-modules is called a *fiber functor*, if it is monoidal, faithful, exact and preserves colimits.

Let H be a Hopf algebra over a field $\mathbb{K}$. Then a $\mathbb{K}$-linear functor $\omega : {}^H\mathcal{M}_f \longrightarrow \mathbb{K}$-Vec from the category ${}^H\mathcal{M}_f$ of finite dimensional H-comodules to the category of $\mathbb{K}$-vector spaces is called a *fiber functor* if it is monoidal and exact.

It is shown in [55], that for a field $\mathbb{K}$ each fiber functor on ${}^H\mathcal{M}_f$ is faithful and takes values in the category of finite dimensional $\mathbb{K}$-vector spaces. Moreover, due to the finiteness theorem for comodules, the restriction functor induces a category equivalence between the category of fiber functors on ${}^H\mathcal{M}$ and fiber functors on ${}^H\mathcal{M}_f$.

Now let H be again a k-flat Hopf algebra over a commutative ring k. Together with morphisms the monoidal natural transformations, the fiber functors $\omega : {}^H\mathcal{M} \longrightarrow \mathcal{M}_k$ form a category. The main results from [55] can be summarized as follows, see also [38]:

Theorem 2.1.2 *([55]) Let H be a k-flat Hopf algebra. There is a bijection between isomorphism classes of fiber functors $\omega : {}^H\mathcal{M} \longrightarrow \mathcal{M}_k$ and isomorphism classes of faithfully flat right H-Galois extensions A of k.*
The functor $\omega_A : {}^H\mathcal{M} \longrightarrow \mathcal{M}_k$ corresponding to a right H-Galois extension A is given by $\omega_A(M) := A\Box_H M$ for $M \in {}^H\mathcal{M}$.
The H-Galois extension corresponding to a fiber functor $\omega : {}^H\mathcal{M} \longrightarrow \mathcal{M}_k$ is given by $A := \omega(H)$.

In [55] Ulbrich points out another way of determining an H-Galois extension A from the fiber functor $\omega : {}^H\mathcal{M} \longrightarrow \mathcal{M}_k\ ,\ M \mapsto A\Box_H M$:
Let $\nu : {}^H\mathcal{M} \longrightarrow \mathcal{M}_k$ denote the forgetful functor. Then the k-module A turns out to be the representing object of the functor $M \mapsto \mathrm{Nat}(\omega, M \otimes \nu)$, where the latter denotes the set of all natural transformations $\omega \longrightarrow M \otimes \nu$ with $(M \otimes \nu)(N) := M \otimes \nu(N)$ for all $N \in {}^H\mathcal{M}$. The corresponding isomorphism $\rho : \mathrm{Hom}(A, M) \longrightarrow \mathrm{Nat}(\omega, M \otimes \nu)$ is given by $\rho(f)(\sum a_i \otimes m_i) = \sum f(a_i) \otimes m_i$ for all $\sum a_i \otimes m_i \in A\Box_H M = \omega(M)$ and $f \in \mathrm{Hom}(A, M)$.

At this point we have to recall the definition and properties of cohomomorphism objects from [32] or the appendix (where their main features are summarized). Then we see immediately that A is the cohomomorphism object of the functors ω and ν, that is $A \cong \mathrm{cohom}(\nu, \omega)$. This implies that there is a structure of $\mathrm{coend}(\omega)$-$\mathrm{coend}(\nu)$-bicomodule on A.
Since $\omega, \nu : {}^H\mathcal{M} \longrightarrow \mathcal{M}_k$ are monoidal functors, it moreover follows that A is a right $\mathrm{coend}(\nu)$-comodule algebra. By Tannaka duality we have $\mathrm{coend}(\nu) \cong H$, since $\nu : {}^H\mathcal{M} \longrightarrow \mathcal{M}_k$ is the forgetful functor.

The following theorem from [38] shows the role of the bialgebra $\mathrm{coend}(\omega)$ in this context:

Theorem 2.1.3 *([38]) Let A be an L-H-bi-Galois extension of k. Then the fiber functor $\omega_A : {}^H\mathcal{M} \longrightarrow \mathcal{M}_k\ ,\ M \mapsto A\Box_H M$ induces an equivalence of monoidal categories*

$$\mathcal{F}_A : {}^H\mathcal{M} \ni M \mapsto A\Box_H M \in {}^L\mathcal{M}$$

and we have

$$\mathrm{coend}(\omega_A) \cong L\ .$$

In particular, every fiber functor $\omega : {}^H\mathcal{M} \longrightarrow \mathcal{M}_k$ factors as $\omega \cong \nu\mathcal{F}$, where $\nu : {}^L\mathcal{M} \longrightarrow \mathcal{M}_k$ is the underlying functor with L a Hopf algebra, and $\mathcal{F}$ is an equivalence.

Schauenburg [38] calls two k-bialgebras H and F *monoidally co-Morita equivalent*, if the monoidal categories ${}^H\mathcal{M}$ and ${}^F\mathcal{M}$ are equivalent as monoidal k-linear categories. We state a consequence of the previous theorem, also from [38]:

Corollary 2.1.4 *([38]) Let H and F be k-flat Hopf algebras. The following are equivalent:*

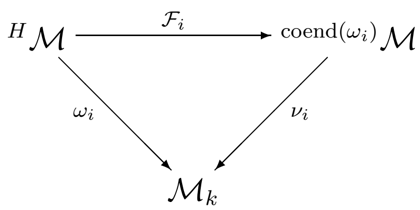
1) *H and F are monoidally co-Morita equivalent.*
2) *There exists a faithfully flat F-H-bi-Galois extension of k.*

As a motivation for the next section, we investigate the properties of the H-Galois extension A arising from a fiber functor:

Let H be a k-flat Hopf algebra, and let P, Q be two faithfully flat right H-Galois extensions of k. By Theorem 2.1.2, P and Q give rise to two fiber functors

$$\omega_1 : {}^H\mathcal{M} \longrightarrow \mathcal{M}_k \ , \ M \mapsto P\square_H M$$
$$\omega_2 : {}^H\mathcal{M} \longrightarrow \mathcal{M}_k \ , \ M \mapsto Q\square_H M \ .$$

We apply Theorem 2.1.3 to see that there exist equivalences $\mathcal{F}_1, \mathcal{F}_2$ such that the diagrams



commute for $i = 1, 2$, where $\nu_i : {}^{\mathrm{coend}(\omega_i)}\mathcal{M} \longrightarrow \mathcal{M}_k$ denotes the respective forgetful functor. Since $\mathcal{F}_1$ is an equivalence, there exists a functor $\mathcal{G}_1 : {}^{\mathrm{coend}(\omega_1)}\mathcal{M} \longrightarrow {}^H\mathcal{M}$ such that $\mathcal{F}_1\mathcal{G}_1 \cong \mathrm{Id}$ and $\mathcal{G}_1\mathcal{F}_1 \cong \mathrm{Id}$, and we can immediately conclude that $\mathrm{coend}(\omega_1) \cong \mathrm{coend}(\omega_2)$ through the equivalence $\mathcal{F}_2\mathcal{G}_1$.

On the other hand, we know that each such equivalence of comodule categories is given as

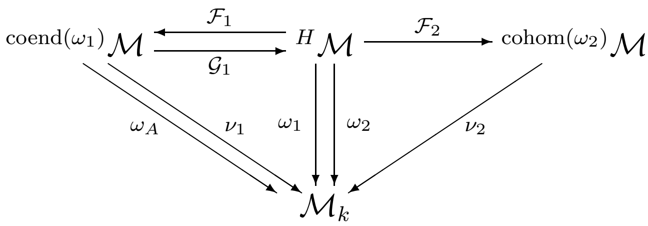
$$\phi_A : {}^{\mathrm{coend}(\omega_1)}\mathcal{M} \longrightarrow {}^{\mathrm{coend}(\omega_2)}\mathcal{M} \ , \ M \mapsto A \Box_{\mathrm{coend}(\omega_1)} M$$

for some faithfully flat $\mathrm{coend}(\omega_2)$-$\mathrm{coend}(\omega_1)$-bi-Galois extension A of k. Let $\nu : {}^H\mathcal{M} \longrightarrow \mathcal{M}_k$ denote the underlying functor. As shown above, we have $A \cong \mathrm{cohom}(\nu_1, \phi_A)$, $P \cong \mathrm{cohom}(\nu, \omega_1)$ and $Q \cong \mathrm{cohom}(\nu, \omega_2)$.

We now claim that $A \cong \mathrm{cohom}(\omega_1, \omega_2)$. Let $\omega_A : {}^{\mathrm{coend}(\omega_1)}\mathcal{M} \longrightarrow \mathcal{M}_k$ be the functor that arises through the equivalence ϕ_A by defining $\omega_A := \nu_2 \phi_A$. Then we have

$$\omega_A \mathcal{F}_1 = \nu_2 \phi_A \mathcal{F}_1 \cong \nu_2 \mathcal{F}_2 \mathcal{G}_1 \mathcal{F}_1 \cong \nu_2 \mathcal{F}_2 = \omega_2 \ .$$

The following diagram describes the relations between the functors that arise in this situation:



Since A is a right $\mathrm{coend}(\omega_1)$-Galois extension, we have $A \cong \mathrm{cohom}(\nu_1, \omega_A)$. Using that the functor $\mathcal{F}_1$ is an equivalence, we obtain by Corollary D.7 and the formulas above that

$$A \cong \mathrm{cohom}(\nu_1, \omega_A) \cong \mathrm{cohom}(\nu_1 \mathcal{F}_1, \omega_A \mathcal{F}_1) \cong \mathrm{cohom}(\omega_1, \omega_2) \ .$$

Corollary 2.1.5 *The cohomomorphism object* $\mathrm{cohom}(\omega_1, \omega_2)$ *describes the category equivalence* $\phi_A : {}^{\mathrm{coend}(\omega_1)}\mathcal{M} \longrightarrow {}^{\mathrm{coend}(\omega_2)}\mathcal{M}$ *that is induced by the functors* ω_1 *and* ω_2*. Consequently, the quasiinverse of* ϕ_A *is given by* $\phi_{A^{-1}}$ *with* $A^{-1} \cong \mathrm{cohom}(\omega_2, \omega_1)$*.*

What we have just seen is that not only cohomomorphism objects $\mathrm{cohom}(\nu, \omega)$ with ν the forgetful functor and ω a fiber functor carry the structure of a bi-Galois extension. On the contrary, it is also possible for an arbitrary choice of functors ω_1 and ω_2 to create a cohomomorphism object $\mathrm{cohom}(\omega_1, \omega_2)$ that possesses a structure of bi-Galois extension. So we will keep this generality in the next section, where we construct quantum torsor structures on arbitrary cohomomorphism objects.

2.2 Quantum Torsor Structures on Cohomomorphism Objects

The main result from Ulbrich in [56] is a reconstruction theorem for Hopf algebras. Given any monoidal functor $\omega : \mathcal{C} \longrightarrow {\mathcal{M}_k}^f$ from a small monoidal category $\mathcal{C}$ into the category of finitely generated projective k-modules, one can always reconstruct a bialgebra structure on $\mathrm{coend}(\omega)$. Ulbrich's theorem says that in case the category $\mathcal{C}$ is rigid, there exists an antipode map for $\mathrm{coend}(\omega)$ such that it becomes a Hopf algebra.
Bichon showed in [3] that the same assumption on rigidity of $\mathcal{C}$ allows to reconstruct a Hopf-Galois system from two functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \longrightarrow \mathbb{K}\text{-Vec}$.

In this section, we recover quantum torsor structures on cohomomorphism objects. We use essentially the same reconstruction methods as Ulbrich, but apply graphical calculus for the proofs. Apart from being very intuitive, graphical notation has the advantage of providing results that are valid in any monoidal category. This is because the calculations are carried out on the objects of the category and do not use their "elements". So we consider to following (most) general situation (see also the appendix) in which reconstruction is possible:

Let $(\mathcal{C}, \otimes, I_{\mathcal{C}})$ be a small (right) rigid monoidal category, and let $(\mathcal{M}, \otimes, I_{\mathcal{M}})$ be a small symmetric monoidal abelian category, which is cocomplete such that colimits commute with tensor products. We consider two monoidal functors

$$\omega : \mathcal{C} \longrightarrow \mathcal{M} \quad \text{and} \quad \nu : \mathcal{C} \longrightarrow \mathcal{M}$$

that both factor through the full subcategory $\mathcal{M}_0$ of objects in $\mathcal{M}$ that have right duals. As explained in the appendix, this implies that both coendomorphism objects $\mathrm{coend}(\omega)$ and $\mathrm{coend}(\nu)$ as well as both cohomomorphism objects $\mathrm{cohom}(\nu, \omega)$ and $\mathrm{cohom}(\omega, \nu)$ exist.

Recall that by Proposition D.3 there exist comultiplication maps on the cohomomorphism objects, compatible with the algebra structures, such that the diagram

$$\begin{array}{ccc}
\mathrm{cohom}(\nu,\omega) & \xrightarrow{\quad\Delta_\omega\quad} & \mathrm{coend}(\omega) \otimes \mathrm{cohom}(\nu,\omega) \\
\Big\downarrow{\scriptstyle \Delta_\nu} & & \Big\downarrow{\scriptstyle \Delta\otimes\mathrm{id}} \\
\mathrm{cohom}(\nu,\omega) \otimes \mathrm{coend}(\nu) & \xrightarrow{\ \mathrm{id}\otimes\Delta\ } & \mathrm{cohom}(\nu,\omega) \otimes \mathrm{cohom}(\omega,\nu) \otimes \mathrm{cohom}(\nu,\omega)
\end{array}$$

commutes. Let $\mu_0 := (\mathrm{id} \otimes \Delta)\Delta_\nu = (\Delta \otimes \mathrm{id})\Delta_\omega$ be the resulting map

$$\mu_0 : \mathrm{cohom}(\nu, \omega) \longrightarrow \mathrm{cohom}(\nu, \omega) \otimes \mathrm{cohom}(\omega, \nu) \otimes \mathrm{cohom}(\nu, \omega) \ .$$

It is clear that μ_0 is an algebra morphism and coassociative in the sense that $(\mu_0 \otimes \mathrm{id} \otimes \mathrm{id})\mu_0 = (\mathrm{id} \otimes \mathrm{id} \otimes \mu_0)\mu_0$.

Since we assume that the category $\mathcal{C}$ is rigid, there exists by definition a contravariant functor $\mathcal{C} \longrightarrow \mathcal{C}$, $X \mapsto X^*$ assigning to each object $X \in \mathcal{C}$ its right dual $X^* \in \mathcal{C}$.
The dual basis $\mathrm{db} : I_\mathcal{C} \longrightarrow X^* \otimes X$ in $\mathcal{C}$ gives rise to a map

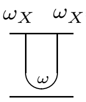
$$\mathrm{db}_\omega(X) : I_\mathcal{M} \cong \omega(I_\mathcal{C}) \xrightarrow{\omega(\mathrm{db})} \omega(X^* \otimes X) \xrightarrow{\xi} \omega(X^*) \otimes \omega(X) \ ,$$

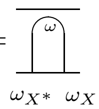
where ξ is the natural isomorphism belonging to the monoidal functor ω.

The image $\omega(X^*)$ of a right dual $X^* \in \mathcal{C}$ under the functor ω is naturally a right dual for $\omega(X)$ in $\mathcal{M}$ by the natural isomorphism

$$\kappa(X) : \omega(X)^* \xrightarrow{\mathrm{db}_\omega \otimes \omega(X)^*} \omega(X^*) \otimes \omega(X) \otimes \omega(X)^* \xrightarrow{\omega(X^*) \otimes \mathrm{ev}} \omega(X^*) \ \cdot$$

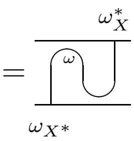
We are going to use the graphical notation

$\mathrm{ev}_\omega :=$ 

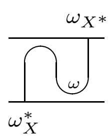
, $\mathrm{db}_\omega :=$ 

for the induced duality morphisms $\mathrm{ev}_\omega : \omega(X) \otimes \omega(X^*) \cong \omega(X \otimes X^*) \longrightarrow \omega(I_\mathcal{C}) \cong I_\mathcal{M}$ and $\mathrm{db}_\omega : I_\mathcal{M} \longrightarrow \omega(X^*) \otimes \omega(X)$ in $\mathcal{M}$. The natural isomorphism ξ is omitted in the notation.
To avoid overlapping labels, we use symbols like ω_X that stands for $\omega(X)$. In the same manner, we use the labels ω_{X^*} for $\omega(X^*)$ and ω_X^* for $\omega(X)^*$. Cohomomorphism objects $\mathrm{cohom}(\nu, \omega)$ will be denoted simply by (ν, ω).

So the above isomorphism $\kappa(X)$ is given by

$\kappa_X =$ 

with inverse

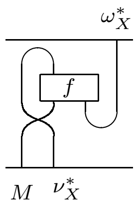
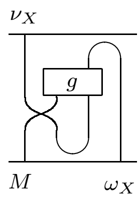
$\kappa_X^{-1} =$ 

.

It is clear that we have a similar map for the functor ν, which shall we also call κ.

For each $M \in \mathcal{M}$ and $X \in \mathcal{C}$ we have the canonical isomorphism

$$\Psi : \mathrm{Hom}(\nu(X), M \otimes \omega(X)) \xrightarrow{\cong} \mathrm{Hom}(\omega(X)^*, M \otimes \nu(X)^*) \ ,$$

$\Psi(f) :=$  with inverse $\Psi^{-1}(g) :=$  .

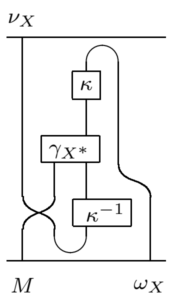
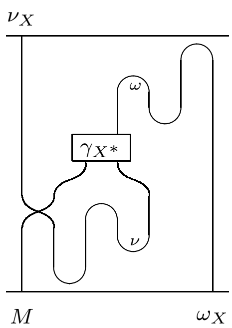
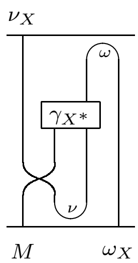
Hence, we can define a map

$$\mathrm{Nat}(\omega, M \otimes \nu) \longrightarrow \mathrm{Nat}(\nu, M \otimes \omega) \ , \ \gamma \mapsto \tilde{\gamma} \tag{2.1}$$

as follows: For $\gamma \in \mathrm{Nat}(\omega, M \otimes \nu)$ we let $\tilde{\gamma}(X) \in \mathrm{Hom}(\nu(X), M \otimes \omega(X))$ be the preimage of the map

$$\gamma^*(X) : \omega(X)^* \xrightarrow{\kappa_X} \omega(X^*) \xrightarrow{\gamma_{X^*}} M \otimes \nu(X^*) \xrightarrow{M \otimes \kappa_X^{-1}} M \otimes \nu(X)^*$$

under the isomorphism Ψ. This means that

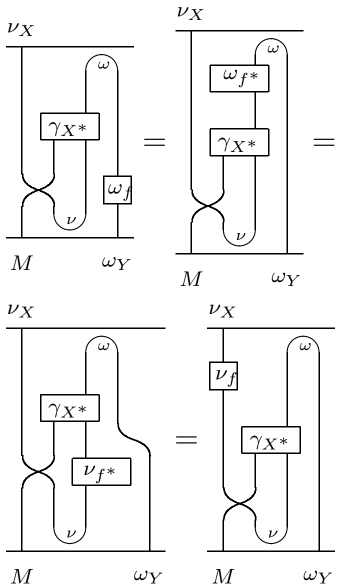
$\tilde{\gamma}_X =$  $=$  $=$  .

The square

$$\begin{array}{ccc} \nu(X) & \xrightarrow{\tilde{\gamma}(X)} & M \otimes \omega(X) \\ \Big\downarrow{\scriptstyle \nu(f)} & & \Big\downarrow{\scriptstyle M \otimes \omega(f)} \\ \nu(Y) & \xrightarrow{\tilde{\gamma}(Y)} & M \otimes \omega(Y) \end{array}$$

commutes for all $X, Y \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, since

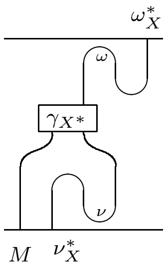
$$(M \otimes \omega(f)) \circ \tilde{\gamma}(X) = \quad = \quad = \quad = \quad = \tilde{\gamma}_Y \circ \nu_f$$



by naturality of γ and the symmetry in $\mathcal{M}$, and the definition of the transposed morphism $f^* : Y^* \longrightarrow X^*$. Thus, $\tilde{\gamma}$ is a natural transformation in $\mathrm{Nat}(\nu, M \otimes \omega)$.

We observe that $\gamma^*(X) : \omega(X)^* \longrightarrow M \otimes \nu(X)^*$, which is the image of $\tilde{\gamma}$ under Ψ, is given by

$$\gamma^*(X) = \qquad , \tag{2.2}$$



for each $X \in \mathcal{C}$. It is also a natural transformation.

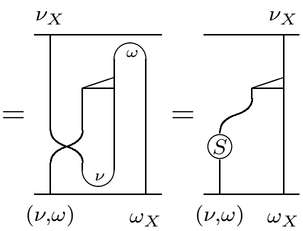
By the Yoneda Lemma and the universal property of the cohomomorphism objects $\mathrm{cohom}(\nu, \omega)$ and $\mathrm{cohom}(\omega, \nu)$, we obtain that the mapping $\gamma \mapsto \tilde{\gamma}$ from (2.1) is uniquely determined by a morphism

$$S : \mathrm{cohom}(\omega, \nu) \longrightarrow \mathrm{cohom}(\nu, \omega) \tag{2.3}$$

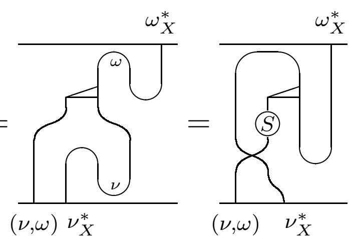
such that the diagram

$$\begin{array}{ccc} \mathrm{Nat}(\omega, M \otimes \nu) & \xrightarrow{\gamma \mapsto \tilde{\gamma}} & \mathrm{Nat}(\nu, M \otimes \omega) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}(\mathrm{cohom}(\nu, \omega), M) & \xrightarrow{\mathrm{Hom}(S, \mathrm{id})} & \mathrm{Hom}(\mathrm{cohom}(\omega, \nu), M) \end{array}$$

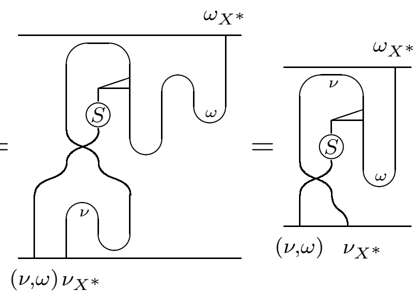
commutes. We can now insert $\mathrm{cohom}(\nu, \omega)$ instead of M into this diagram and use the fact that the natural transformation $\delta_{\omega,\nu} : \omega \longrightarrow \mathrm{cohom}(\nu, \omega) \otimes \nu$ is induced by the identity $\mathrm{id} : \mathrm{cohom}(\nu, \omega) \longrightarrow \mathrm{cohom}(\nu, \omega)$. This implies $\tilde{\delta}_{\omega,\nu} = (S \otimes \mathrm{id}) \circ \delta_{\nu,\omega}$, which is

$\tilde{\delta}_{\omega,\nu}(X) =$  . (2.4)

Applying the isomorphism Ψ to $\tilde{\delta}$ yields the following formula for δ^* from (2.2). It can be considered as an induced comultiplication on $\omega(X)^*$.

$\delta^*_{\omega,\nu}(X) =$  . (2.5)

Now we can express $\delta_{\omega,\nu}(X^*)$ as

$\delta_{\omega,\nu}(X^*) =$  . (2.6)

Proposition 2.2.1 *The morphism* $S : \mathrm{cohom}(\omega, \nu) \longrightarrow \mathrm{cohom}(\nu, \omega)$ *in (2.3) is an algebra anti-morphism.*

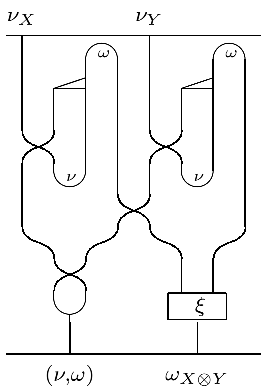
Proof. We know from Corollary D.15 that there is an isomorphism

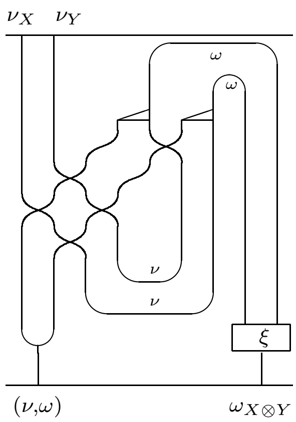
$$\mathrm{Nat}^{\otimes}(\nu, \mathrm{cohom}(\nu, \omega)^{op} \otimes \omega) \cong \mathrm{Alg}(\mathrm{cohom}(\omega, \nu), \mathrm{cohom}(\nu, \omega)^{op})$$

and that the image of $\tilde{\delta}$ under this isomorphism is S. So we have to show that $\tilde{\delta}_{\omega,\nu} \in \mathrm{Nat}(\nu, \mathrm{cohom}(\nu, \omega) \otimes \omega)$ is a monoidal natural transformation when $\mathrm{cohom}(\nu, \omega)$ is endowed with its opposite algebra structure. In other words, we have to check that the diagram

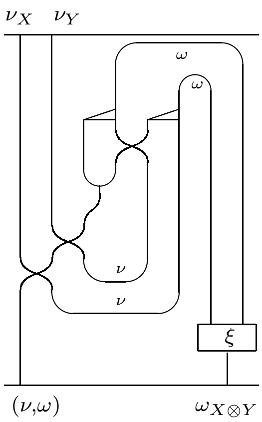
$$\begin{array}{ccc}
\nu(X) \otimes \nu(Y) & \xrightarrow{\tilde{\delta}(X)\otimes\tilde{\delta}(Y)} & \mathrm{cohom}(\nu, \omega) \otimes \mathrm{cohom}(\nu, \omega) \otimes \omega(X) \otimes \omega(Y) \\
\Big\downarrow \xi & & \Big\downarrow \nabla^{op}\otimes\xi \\
\nu(X \otimes Y) & \xrightarrow{\tilde{\delta}(X\otimes Y)} & \mathrm{cohom}(\nu, \omega) \otimes \omega(X \otimes Y)
\end{array}$$

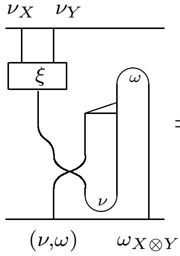
commutes, where ξ denotes in either case the isomorphism that belongs to the monoidal functor ω resp. ν. With formula (2.4) we get

$(\tilde{\delta}(X) \otimes \tilde{\delta}(Y))(\nabla^{op} \otimes \xi) =$ 

$=$ 

$=$

$=$ 

$=$ 

$= \tilde{\delta}(X \otimes Y)\xi$

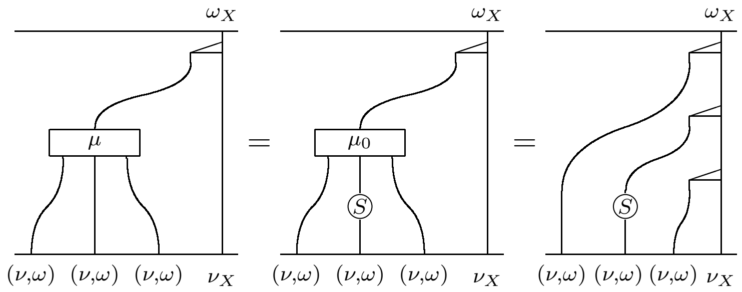
using naturality of the duality morphisms and the fact that δ is a monoidal natural transformation. Hence, the diagram commutes and we obtain that S is an algebra anti-morphism. □

Now that we have constructed an algebra morphism between the cohomomorphism objects $\mathrm{cohom}(\omega,\nu)$ and $\mathrm{cohom}(\nu,\omega)^{op}$, we can define an algebra morphism

$$\mu : \mathrm{cohom}(\nu,\omega) \longrightarrow \mathrm{cohom}(\nu,\omega) \otimes \mathrm{cohom}(\nu,\omega)^{op} \otimes \mathrm{cohom}(\nu,\omega)$$

by $\mu := (\mathrm{id} \otimes S \otimes \mathrm{id})\mu_0$.
It is obvious that μ inherits the coassociativity property from μ_0 and hence satisfies the coassociativity axiom 3) in Definition 1.4.1.

For further calculations we note that μ is induced via the universal property of $\mathrm{cohom}(\nu,\omega)$ by



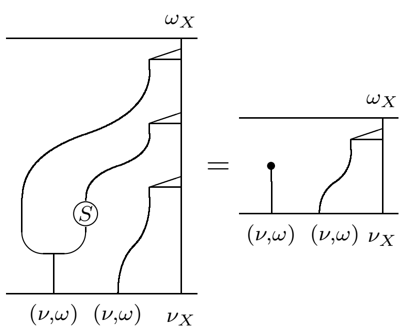
, (2.7)

as follows from Proposition D.3.

Proposition 2.2.2 *The coendomorphism object* $\mathrm{cohom}(\nu,\omega)$ *becomes a quantum torsor with the torsor structure map* μ *and the Grunspan map* $\theta := S \circ T$*, where* T *is constructed analogously to* S *with the roles of* ω *and* ν *interchanged.*

Proof. We verify that $(\mathrm{cohom}(\nu,\omega),\mu,\theta)$ satisfies the five axioms of a quantum torsor in Definition 1.4.1. We have already noted that axiom 3) holds.

Axiom 1) will follow from the equality

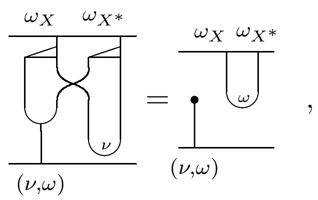


,

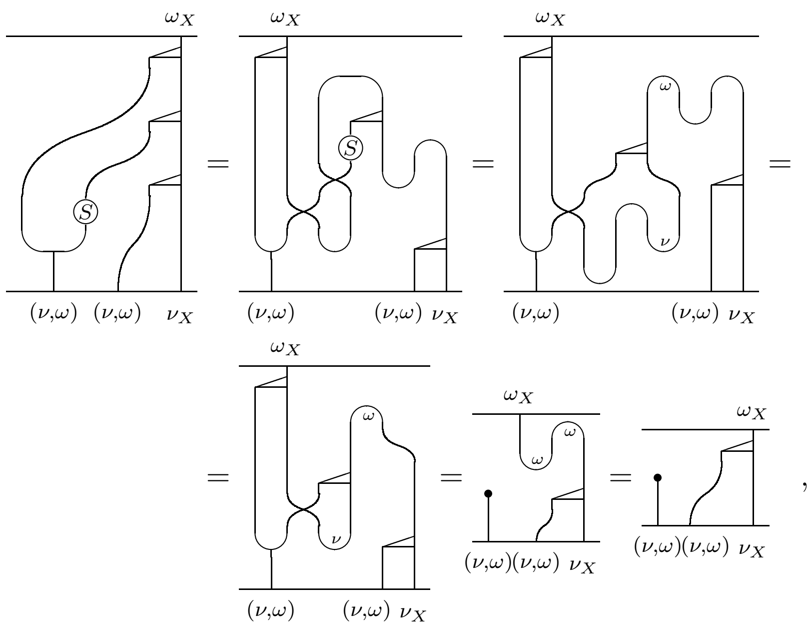
since these are the maps that induce, via the universal property of $\mathrm{cohom}(\nu,\omega)$, the maps $(\nabla \otimes \mathrm{id})\mu$ resp. $\eta \otimes \mathrm{id}$. By naturality of δ applied to the evaluation map, the diagram

$$\begin{array}{ccc}
\omega(X) \otimes \omega(X^*) & \xrightarrow{\mathrm{ev}_\omega} & I_{\mathcal{M}} \\
\downarrow{\scriptstyle \delta_X \otimes \delta_{X^*}} & & \\
\mathrm{cohom}(\nu,\omega) \otimes \mathrm{cohom}(\nu,\omega) \otimes \nu(X) \otimes \nu(X^*) & & \downarrow{\scriptstyle \delta_I = \eta} \\
\downarrow{\scriptstyle \nabla \otimes \mathrm{id} \otimes \mathrm{id}} & & \\
\mathrm{cohom}(\nu,\omega) \otimes \nu(X) \otimes \nu(X^*) & \xrightarrow{\mathrm{id} \otimes \mathrm{ev}_\nu} & \mathrm{cohom}(\nu,\omega)
\end{array}$$

commutes. This means that we have



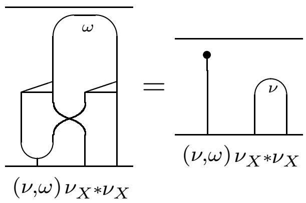
and thus



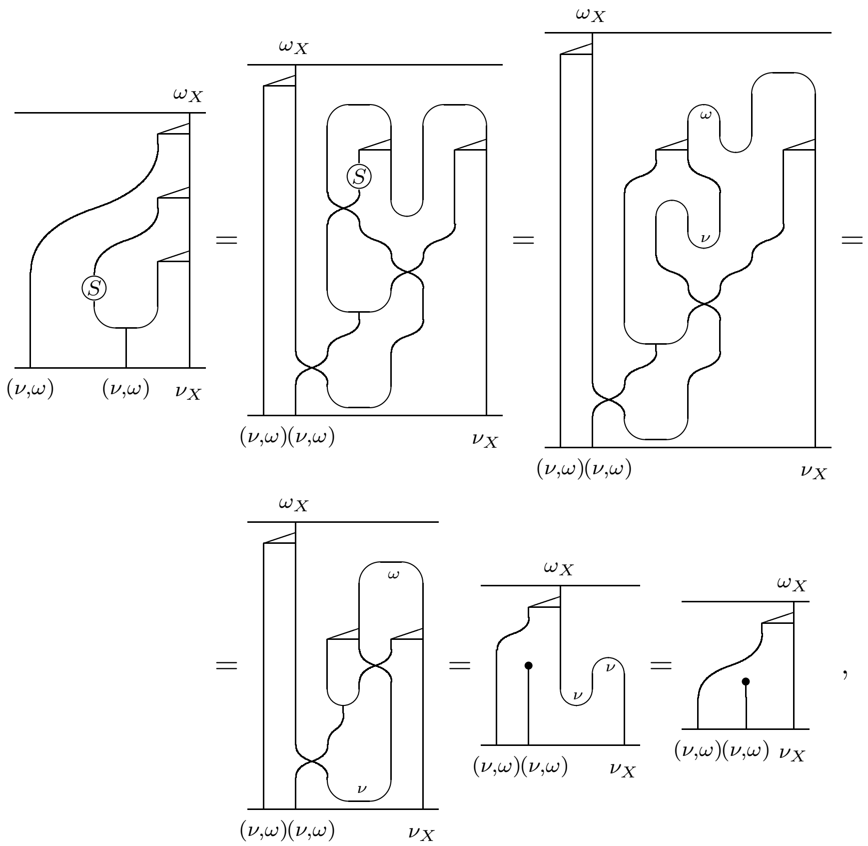
using the relation (2.5) in the second equality. The axiom 2) is proved similarly by using naturality of δ with respect to the dual basis:

$$\begin{array}{ccc}
I_{\mathcal{M}} & \xrightarrow{\quad \mathrm{db}_\omega \quad} & \omega(X^*) \otimes \omega(X) \\
 & & \big\downarrow {\scriptstyle \delta_{X^*} \otimes \delta_X} \\
{\scriptstyle \delta_I = \eta} \big\downarrow & & \mathrm{cohom}(\nu,\omega) \otimes \mathrm{cohom}(\nu,\omega) \otimes \nu(X)^* \otimes \nu(X) \\
 & & \big\downarrow {\scriptstyle \nabla \otimes \mathrm{id} \otimes \mathrm{id}} \\
\mathrm{cohom}(\nu,\omega) & \xrightarrow{\quad \mathrm{id} \otimes \mathrm{db}_\nu \quad} & \mathrm{cohom}(\nu,\omega) \otimes \nu(X^*) \otimes \nu(X)
\end{array}$$

The resulting equality



implies



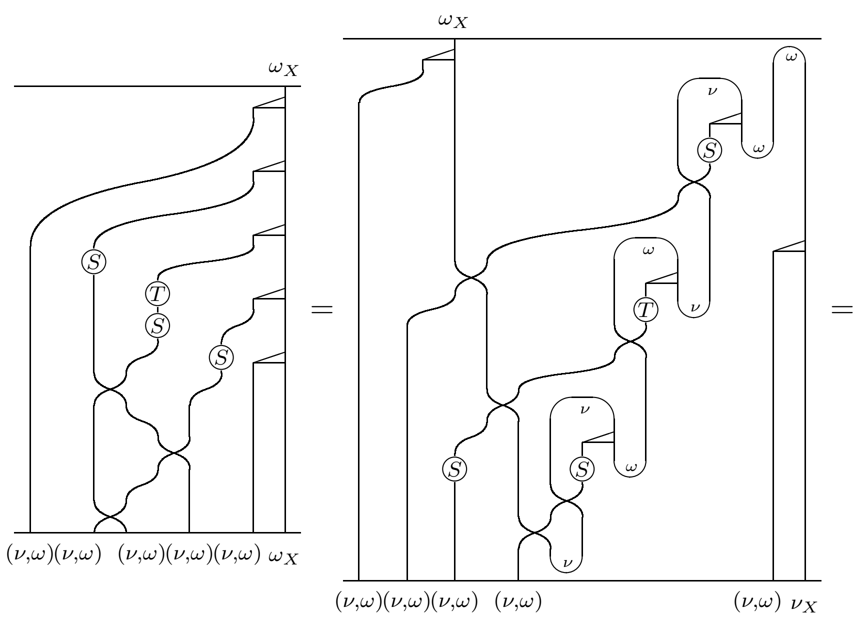
and this proves axiom 2).

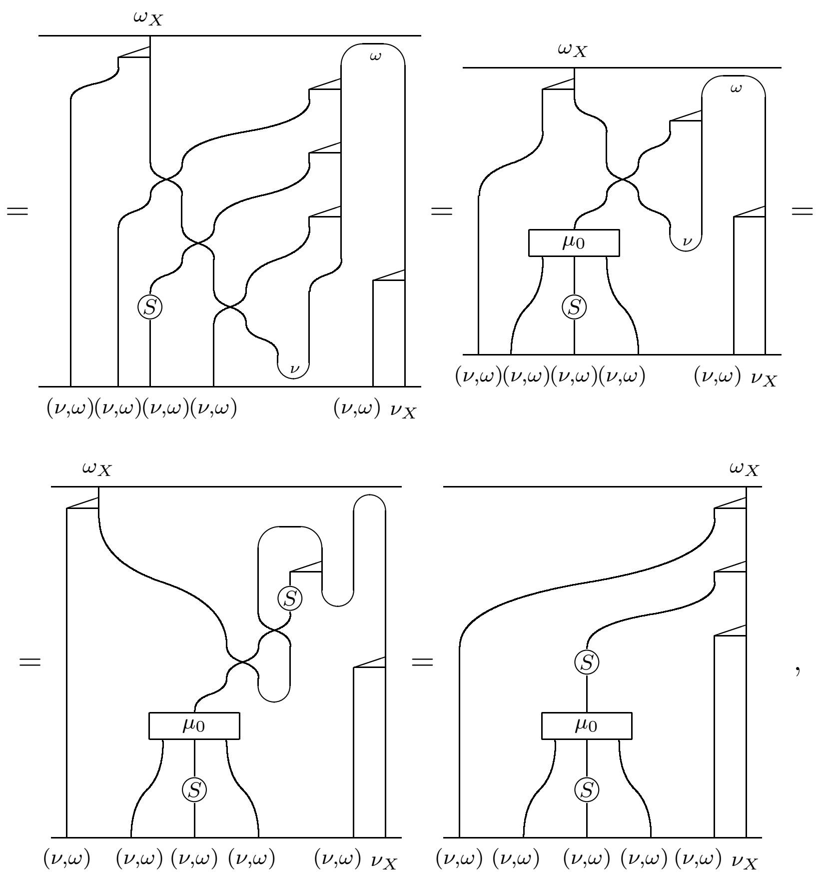
We claim that the Grunspan map for $\mathrm{cohom}(\nu,\omega)$ is given by $\theta := S \circ T$, where $T : \mathrm{cohom}(\nu,\omega) \longrightarrow \mathrm{cohom}(\omega,\nu)$ is constructed analogously to S with the roles of ν and ω interchanged.

The following diagram commutes by definition of μ_0 :

$$
\begin{array}{ccc}
\omega(X^*) & \xrightarrow{\quad \delta_{X^*} \quad} & (\nu,\omega)\otimes\nu(X^*) \\
\Big\downarrow{\scriptstyle \delta_{X^*}} & & \\
(\nu,\omega)\otimes\nu(X^*) & & \Big\downarrow{\scriptstyle \mu_0\otimes\mathrm{id}} \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes\delta_{X^*}} & & \\
(\nu,\omega)\otimes(\omega,\nu)\otimes\omega(X^*) & \xrightarrow{\ \mathrm{id}\otimes\mathrm{id}\otimes\delta_{X^*}\ } & (\nu,\omega)\otimes(\omega,\nu)\otimes(\nu,\omega)\otimes\omega(X^*)
\end{array}
$$

Therefore, we obtain





which shows that the maps inducing

$$(\mathrm{id} \otimes \sigma_{13} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \theta \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \mu)\mu$$

resp.

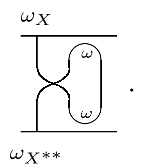
$$(\mathrm{id} \otimes \mu \otimes \mathrm{id})\mu$$

are equal. This means that axiom 4) holds.

For $X \in \mathcal{C}$ we denote by X^{**} the right dual of X^{*}. In the category $\mathcal{M}$, there is an isomorphism between $\omega(X)$ and $\omega(X^{**})$ given by

$$\omega(X) \xrightarrow{\mathrm{id} \otimes \mathrm{db}_{\omega}(X^{*})} \omega(X) \otimes \omega(X^{**}) \otimes \omega(X^{*}) \xrightarrow{\sigma \otimes \mathrm{id}}$$
$$\longrightarrow \omega(X^{**}) \otimes \omega(X) \otimes \omega(X^{*}) \xrightarrow{\mathrm{id} \otimes \mathrm{ev}_{\omega}(X)} \omega(X^{**}) \; ,$$

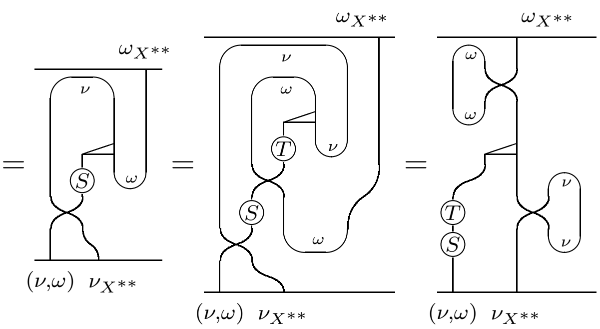
that is



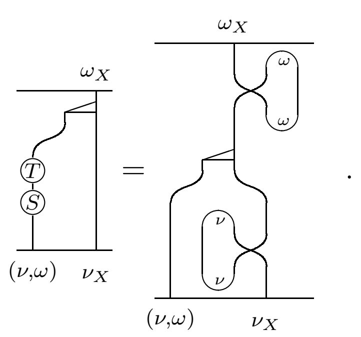
Of course, there is an analogous isomorphism $\nu(X) \cong \nu(X^{**})$. We note that such relations between an object X and its bi-dual X^{**} require the existence of a braiding in the respective monoidal category.

Now we observe that

$$\delta(X^{**}) =$$

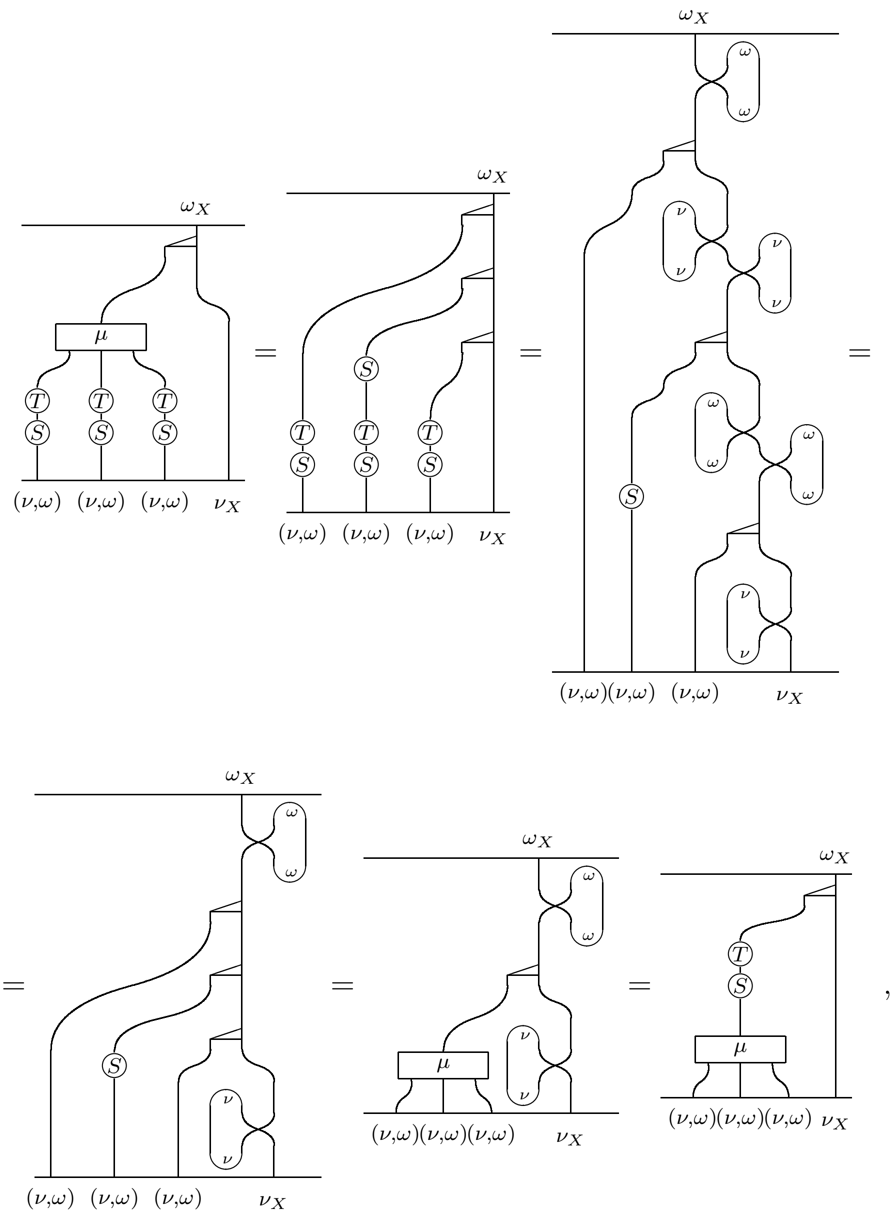


This means that we can express $\delta(X^{**})$ in terms of $\delta(X)$ by



(2.8)

Hence we have



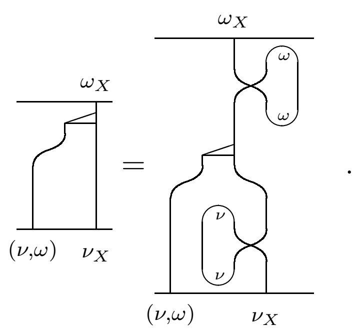
proving axiom 5).

Altogether we have seen that $\mathrm{cohom}(\nu, \omega)$ carries the structure of a quantum torsor with the Grunspan map $\theta = S \circ T$. Analogously, we can conclude that also $\mathrm{cohom}(\omega, \nu)$ becomes a quantum torsor with the Grunspan map $\theta = T \circ S$. □

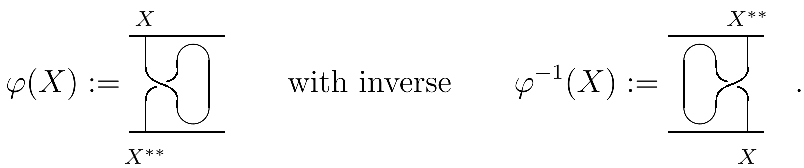
Remark 2.2.3 Applying the same method to the special case $\nu = \omega$ yields antipodes $S_\omega : \mathrm{coend}(\omega) \longrightarrow \mathrm{coend}(\omega)$ and $S_\nu : \mathrm{coend}(\nu) \longrightarrow \mathrm{coend}(\nu)$ for the bialgebras coend(ω) and coend(ν). This is Ulbrich's result in [56].
While constructing the quantum torsor structure on cohom(ν, ω) in the previous proposition, we have in fact shown that the map $S : \mathrm{cohom}(\omega, \nu) \longrightarrow \mathrm{cohom}(\nu, \omega)$makes $(\mathrm{coend}(\omega), \mathrm{coend}(\nu), \mathrm{cohom}(\nu, \omega), \mathrm{cohom}(\omega, \nu))$ into a Hopf-Galois system. This generalizes Bichon's result in [3].

We note that the torsor structure map for cohomomorphism objects from the previous proposition resembles, and in fact generalizes, the trivial torsor structure of a Hopf algebra in Example 1.4.2. So in "nature", Hopf algebras really appear as a special case of quantum torsors (namely when $\nu = \omega$). The Grunspan map θ can thus be interpreted as a generalized square of the antipode.

Remark 2.2.4 We address the question, whether there are situations in which $\theta = S \circ T = \mathrm{id}$ holds. By the formula (2.8) for $S \circ T$ in the proof of axiom 5) and the construction of S and T, we see that $\theta = \mathrm{id}$ is equivalent to

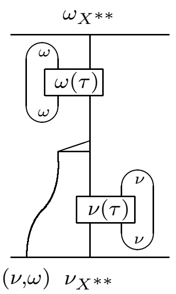


In order to have a connection between X and X^{**} in the category $\mathcal{C}$, we assume that the category $\mathcal{C}$ is braided with a braiding τ. Then each $X \in \mathcal{C}$ is isomorphic to X^{**} in $\mathcal{C}$ via the obvious natural isomorphism



By naturality of $\delta : \omega \longrightarrow \mathrm{cohom}(\nu, \omega) \otimes \nu$ we then have

$$\delta(X^{**}) =$$



.

So we could obviously derive the above equation if the diagram

$$\begin{array}{ccc} \omega(X \otimes Y) & \xrightarrow{\ \xi\ } & \omega(X) \otimes \omega(Y) \\ \Big\downarrow{\scriptstyle \omega(\tau)} & & \Big\downarrow{\scriptstyle \sigma} \\ \omega(Y \otimes X) & \xrightarrow{\ \xi\ } & \omega(Y) \otimes \omega(X) \end{array}$$

and an analogous one for the functor ν were commutative. This diagram says that the functors ω, ν have to map the braiding τ of $\mathcal{C}$ onto the symmetry σ in $\mathcal{M}$.

Obviously, the condition that ω and ν preserve braidings is a quite restrictive one, and not necessarily satisfied. We give an example for the case of fiber functors that we discussed in the previous section:

Example 2.2.5 Let $\mathbb{K}$ be a field and let H be a $\mathbb{K}$-Hopf algebra. Then the category $\mathcal{C} := {}^H\mathcal{M}_f$ of finite dimensional H-comodules is rigid and monoidal (see also the appendix). Let A be a finite dimensional H-Galois extension of $\mathbb{K}$. Then the fiber functor $\omega : {}^H\mathcal{M}_f \longrightarrow \mathbb{K}\text{-Vec}$, $M \mapsto A\Box_H M$ and the forgetful functor $\nu : {}^H\mathcal{M}_f \longrightarrow \mathbb{K}\text{-Vec}$ both take values in the category of finite dimensional $\mathbb{K}$-vector spaces, and satisfy our general conditions. Hence, by our results above, both $\mathrm{cohom}(\nu, \omega) \cong A$ and $\mathrm{cohom}(\omega, \nu) \cong A^{-1}$ are quantum torsors.
It is well-known, see for instance [28], that the category ${}^H\mathcal{M}$ is braided and monoidal with the tensor product from the underlying category $\mathbb{K}$-Vec iff H is a coquasitriangular Hopf algebra. The induced braiding in ${}^H\mathcal{M}$ is equal to the symmetry of $\mathbb{K}$-Vec if H is commutative. Then, of course, the forgetful functor ν maps it to the symmetry of $\mathcal{M}$. In this case, the braiding is also preserved by the fiber functor ω, and so we obtain $\theta = \mathrm{id}$ by what we have

shown above. But if $H \cong \text{coend}(\nu)$ is commutative, then the square of its antipode is the identity, and we can observe that $\theta = S \circ T$ inherits this behaviour. □

Chapter 3

Antipodes in Hopf-Galois Systems

3.1 Total Hopf-Galois Systems and their Properties

A Hopf-Galois system as defined by Bichon in [3], see Definition 1.3.1, carries exactly all the data needed for constructing a Hopf bi-Galois extension from it. Let L and H be k-flat bialgebras and let A be a faithfully flat L-H-bi-Galois extension. Out of this setting, Schauenburg [38] has constructed another algebra A^{-1} that is an H-L-bi-Galois extension of k. So the corresponding Hopf-Galois system (L, H, A, A^{-1}) seems to encode even more data, namely the structure maps of another Hopf-Galois system (H, L, A^{-1}, A). This connection became also obvious in the previous chapter, where the cohomomorphism objects $\mathrm{cohom}(\nu, \omega)$ and $\mathrm{cohom}(\omega, \nu)$ showed a similar behaviour.
Bichon's definition was extended by Grunspan in [18] to the notion of a total Hopf-Galois system. The latter contains the complete set of data that arises from the two Hopf-Galois systems that can be associated to an L-H-bi-Galois extension A.

Recall from the previous chapter that the natural quantum torsor structure on $\mathrm{cohom}(\nu, \omega)$ was really based on a bicomodule algebra structure. Then, under a rigidity assumption, we constructed maps $S : \mathrm{cohom}(\omega, \nu) \to \mathrm{cohom}(\nu, \omega)$ and $T : \mathrm{cohom}(\nu, \omega) \to \mathrm{cohom}(\omega, \nu)$ that allowed us to define the torsor structure maps on $\mathrm{cohom}(\nu, \omega)$.
We are going to put emphasis on this two-step construction that we also

know from Hopf algebras: A Hopf algebra is a bialgebra which possesses an antipode map. Keeping in mind that the map $S : Z \longrightarrow T$ in a Hopf-Galois system (L, H, T, Z) has properties that are similar to those of an antipode, or more concretely seem to generalize them, we start from a bicomodule algebra system. We define it such that it can be seen as the underlying object of a total Hopf-Galois system. Then we develop methods that uncover a striking similarity between Hopf algebras and total Hopf-Galois systems.

We mainly use graphical notations in this section. While all the definitions and results clearly also make sense in any symmetric monoidal category, we are mainly interested in the category $\mathcal{M}_k$ of modules over a commutative ring k.

Definition 3.1.1 A *bicomodule-algebra system* (L, H, T, Z) consists of

- bialgebras L and H with comultiplication Δ_L resp. Δ_H and counit ε_L resp ε_H.
- an (L, H)-bicomodule algebra T with comodule structure maps $\delta_{L,T}$ and $\delta_{T,H}$ and an (H, L)-bicomodule algebra Z with comodule structure maps $\delta_{H,Z}$ and $\delta_{Z,L}$.
- algebra morphisms $\rho_L : L \longrightarrow T \otimes Z$ and $\rho_H : H \longrightarrow Z \otimes T$,

such that the following diagrams commute:

$$\begin{array}{ccc} T & \xrightarrow{\delta_{L,T}} & L \otimes T \\ \downarrow{\scriptstyle \delta_{T,H}} & (B1) & \downarrow{\scriptstyle \rho_L \otimes \mathrm{id}} \\ T \otimes H & \xrightarrow[\mathrm{id} \otimes \rho_H]{} & T \otimes Z \otimes T \end{array} \qquad \begin{array}{ccc} Z & \xrightarrow{\delta_{H,Z}} & H \otimes Z \\ \downarrow{\scriptstyle \delta_{Z,L}} & (B2) & \downarrow{\scriptstyle \rho_H \otimes \mathrm{id}} \\ Z \otimes L & \xrightarrow[\mathrm{id} \otimes \rho_L]{} & Z \otimes T \otimes Z \end{array}$$

$$\begin{array}{ccc} L & \xrightarrow{\Delta_L} & L \otimes L \\ \downarrow{\scriptstyle \rho_L} & (B3) & \downarrow{\scriptstyle \mathrm{id} \otimes \rho_L} \\ T \otimes Z & \xrightarrow[\delta_{L,T} \otimes \mathrm{id}]{} & L \otimes T \otimes Z \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{\Delta_H} & H \otimes H \\ \downarrow{\scriptstyle \rho_H} & (B4) & \downarrow{\scriptstyle \mathrm{id} \otimes \rho_H} \\ Z \otimes T & \xrightarrow[\delta_{H,Z} \otimes \mathrm{id}]{} & H \otimes Z \otimes T \end{array}$$

$$\begin{array}{ccc} H & \xrightarrow{\Delta_H} & H \otimes H \\ \downarrow{\scriptstyle \rho_H} & (B5) & \downarrow{\scriptstyle \rho_H \otimes \mathrm{id}} \\ Z \otimes T & \xrightarrow[\mathrm{id} \otimes \delta_{T,H}]{} & Z \otimes T \otimes H \end{array} \qquad \begin{array}{ccc} L & \xrightarrow{\Delta_L} & L \otimes L \\ \downarrow{\scriptstyle \rho_L} & (B6) & \downarrow{\scriptstyle \rho_L \otimes \mathrm{id}} \\ T \otimes Z & \xrightarrow[\mathrm{id} \otimes \delta_{Z,L}]{} & T \otimes Z \otimes L \end{array}$$

We note that the axioms $(B1)$, $(B3)$, and $(B5)$ are also a part of Bichon's Hopf-Galois system in Definition 1.3.1. The three diagrams in the right column are their counterparts with respect to the algebra Z.

For calculations in $\mathcal{M}_k$, we introduce the notation

$$\rho_L(\ell) =: \sum \ell^T \otimes \ell^Z =: \ell^T \otimes \ell^Z \in T \otimes Z$$

for all $\ell \in L$, and analogously

$$\rho_H(h) =: h^Z \otimes h^T \in Z \otimes T$$

for $h \in H$.
Then for instance, the diagrams (B3) and (B6) read as

$$\ell^T{}_{(-1)} \otimes \ell^T{}_{(0)} \otimes \ell^Z = \ell_{(1)} \otimes \ell_{(2)}{}^T \otimes \ell_{(2)}{}^Z$$

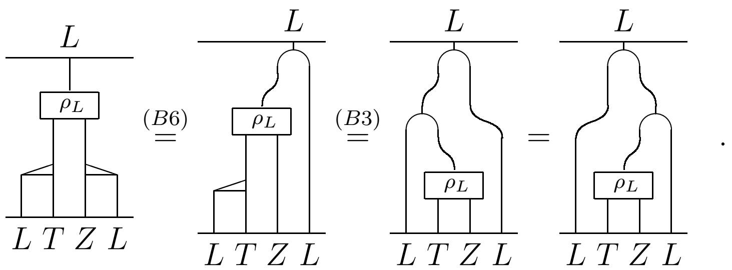
and

$$\ell^T \otimes \ell^Z{}_{(0)} \otimes \ell^Z{}_{(1)} = \ell_{(1)}{}^T \otimes \ell_{(1)}{}^Z \otimes \ell_{(2)} \ ,$$

which implies the equation

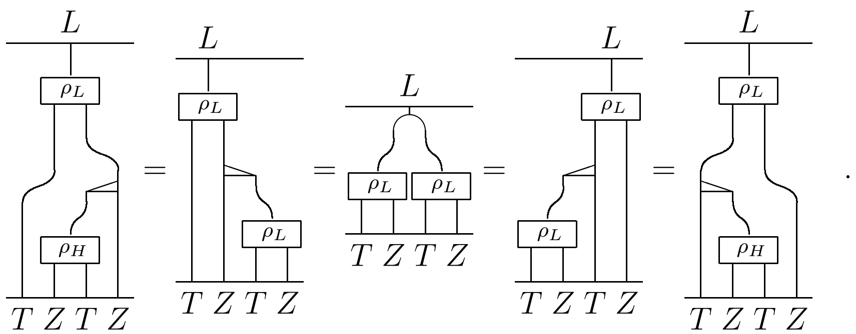
$$\ell^T{}_{(-1)} \otimes \ell^T{}_{(0)} \otimes \ell^Z{}_{(0)} \otimes \ell^Z{}_{(1)} = \ell_{(1)} \otimes \ell_{(2)}{}^T \otimes \ell_{(2)}{}^Z \otimes \ell_{(3)}$$

for all $\ell \in L$. This can also be derived via graphical calculus as



(3.1)

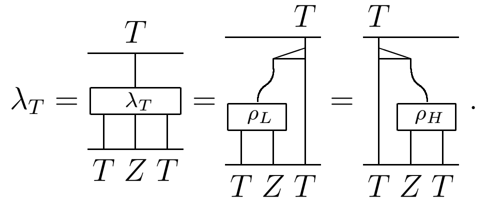
We deduce some more equalities for further applications: By (B2), (B6), (B3) and (B1) we get



(3.2)

The axiom (B1) yields an algebra morphism $\lambda_T : T \longrightarrow T \otimes Z \otimes T$

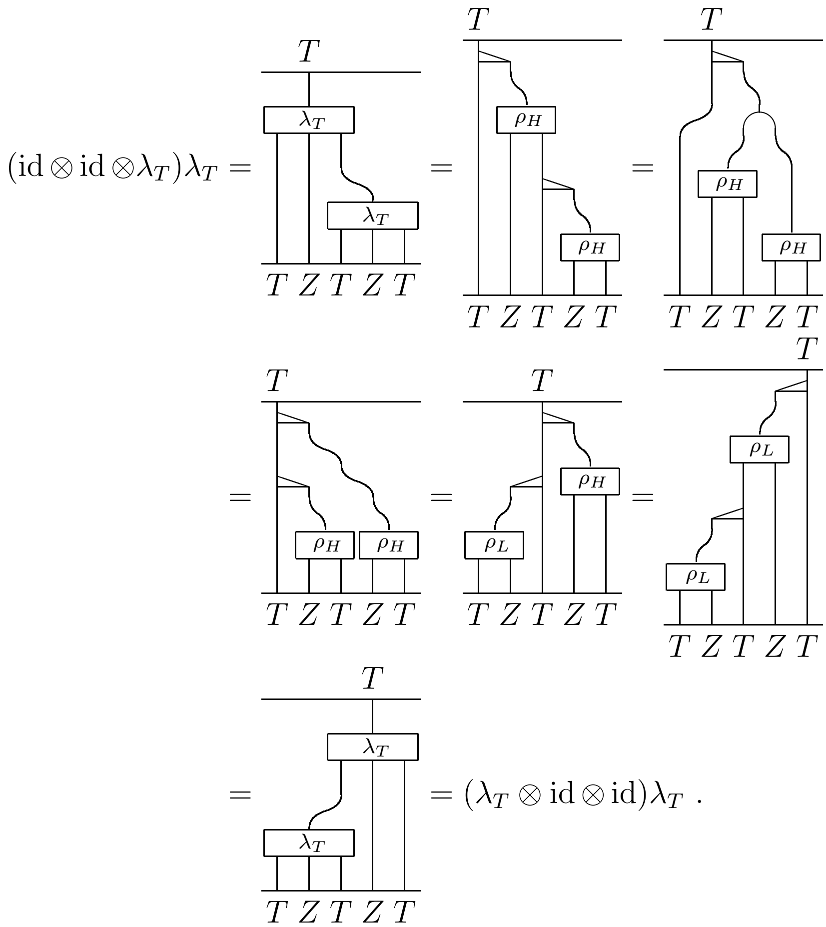
$$\lambda_T := (\rho_L \otimes \mathrm{id})\delta_{L,T} = (\mathrm{id} \otimes \rho_H)\delta_{T,H} \ ,$$



It satisfies

$$(\mathrm{id} \otimes \mathrm{id} \otimes \lambda_T)\lambda_T = (\lambda_T \otimes \mathrm{id} \otimes \mathrm{id})\lambda_T$$

by the axioms (B5) and (B1) :



Analogously, replacing T by the algebra Z, we arrive at an algebra morphism $\lambda_Z : Z \longrightarrow Z \otimes T \otimes Z$, that is also coassociative in the above sense.

Now we define morphisms of bicomodule algebra systems. Of course, they have to be compatible with the axioms in Definition 3.1.1.

Definition 3.1.2 Let (L, H, T, Z) and (L', H', T', Z') be two bicomodule-algebra systems. A *morphism of bicomodule-algebra systems*

$$(\ell, h, f, g) : (L, H, T, Z) \longrightarrow (L', H', T', Z')$$

consists of bialgebra morphisms $\ell : L \longrightarrow L'$, $h : H \longrightarrow H'$ and algebra morphisms $f : T \longrightarrow T'$, $g : Z \longrightarrow Z'$ such that the following diagrams commute:

$$\begin{array}{ccc} L & \xrightarrow{\ell} & L' \\ \downarrow{\scriptstyle \rho_L} & (M1) & \downarrow{\scriptstyle \rho_{L'}} \\ T \otimes Z & \xrightarrow{f \otimes g} & T' \otimes Z' \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{h} & H' \\ \downarrow{\scriptstyle \rho_H} & (M2) & \downarrow{\scriptstyle \rho_{H'}} \\ Z \otimes T & \xrightarrow{g \otimes f} & Z' \otimes T' \end{array}$$

$$\begin{array}{ccc} T & \xrightarrow{f} & T' \\ \downarrow{\scriptstyle \delta_{L,T}} & (M3) & \downarrow{\scriptstyle \delta_{L',T'}} \\ L \otimes T & \xrightarrow{\ell \otimes f} & L' \otimes T' \end{array} \qquad \begin{array}{ccc} Z & \xrightarrow{g} & Z' \\ \downarrow{\scriptstyle \delta_{H,Z}} & (M4) & \downarrow{\scriptstyle \delta_{H',Z'}} \\ H \otimes Z & \xrightarrow{h \otimes g} & H' \otimes Z' \end{array}$$

$$\begin{array}{ccc} T & \xrightarrow{f} & T' \\ \downarrow{\scriptstyle \delta_{T,H}} & (M5) & \downarrow{\scriptstyle \delta_{T',H'}} \\ T \otimes H & \xrightarrow{f \otimes \ell} & T' \otimes H' \end{array} \qquad \begin{array}{ccc} Z & \xrightarrow{g} & Z' \\ \downarrow{\scriptstyle \delta_{Z,L}} & (M6) & \downarrow{\scriptstyle \delta_{Z',L'}} \\ Z \otimes L & \xrightarrow{g \otimes h} & Z' \otimes L' \end{array}$$

Now that we have defined our base category, we can write down generalized antipode axioms.
An antipode S for a bialgebra H is defined as the convolution inverse of the identity in $\mathrm{Hom}(H, H)$. This can be expressed by requiring that S satisfy the two equations

$$S(h_{(1)})h_{(2)} = \eta\varepsilon(h) \quad , \quad h_{(1)}S(h_{(2)}) = \eta\varepsilon(h)$$

for all $h \in H$.
Since there is no obvious convolution algebra for a bicomodule algebra system, we define our generalized antipodes in terms of similar equations. This leads to Grunspan's definition of a total Hopf-Galois system from [18].

Definition 3.1.3 A *total Hopf-Galois system*

$$(L, H, T, Z, S_L, S_H, S_T, S_Z)$$

is a total bicomodule-algebra system (L, H, T, Z) endowed with the following additional data:

- L and H are Hopf algebras with antipodes S_L resp. S_H.
- there exist maps $S_T : T \longrightarrow Z$ and $S_Z \longrightarrow T$, that make the following diagrams commute:

$$\begin{CD} L @>\varepsilon_L>> I @>\eta_T>> T \\ @V\rho_L VV (A1) @AA\nabla_T A \\ T \otimes Z @>>\mathrm{id}\otimes S_Z> @. T \otimes T \end{CD} \qquad \begin{CD} H @>\varepsilon_H>> I @>\eta_Z>> Z \\ @V\rho_H VV (A2) @AA\nabla_Z A \\ Z \otimes T @>>\mathrm{id}\otimes S_T> @. Z \otimes Z \end{CD}$$

$$\begin{CD} H @>\varepsilon_H>> I @>\eta_T>> T \\ @V\rho_H VV (A3) @AA\nabla_T A \\ Z \otimes T @>>S_Z\otimes \mathrm{id}> @. T \otimes T \end{CD} \qquad \begin{CD} L @>\varepsilon_L>> I @>\eta_Z>> Z \\ @V\rho_L VV (A4) @AA\nabla_Z A \\ T \otimes Z @>>S_T\otimes \mathrm{id}> @. Z \otimes Z \end{CD}$$

  We call S_T and T_Z the *generalized antipodes* of the total Hopf-Galois system.

We define a *morphism of Hopf-Galois systems* to be a morphism of the underlying bicomodule-algebra systems.

Remark 3.1.4 Given a Hopf-Galois system $(L, H, T, Z, S_L, S_H, S_T, S_Z)$, we can endow L and H with their opposite coopposite bialgebra structures to obtain the Hopf algebras L^{opcop} and H^{opcop}. The opposite algebras T^{op} and Z^{op} can be considered as (H^{opcop}, L^{opcop})- resp. (L^{opcop}, H^{opcop})-bicomodule algebras with the comodule structures $\sigma \circ \delta$. We denote these bicomodule algebras as T^{opcop} resp. Z^{opcop}. Then the algebra morphisms

$$\rho_{L^{opcop}} : L^{opcop} \xrightarrow{\rho_L} T^{opcop} \otimes Z^{opcop} \xrightarrow{\sigma} Z^{opcop} \otimes T^{opcop}$$

$$\rho_{H^{opcop}} : H^{opcop} \xrightarrow{\rho_H} Z^{opcop} \otimes T^{opcop} \xrightarrow{\sigma} T^{opcop} \otimes Z^{opcop} ,$$

make $(L^{opcop}, H^{opcop}, Z^{opcop}, T^{opcop}, S_H, S_L, S_Z, S_T)$ into a total Hopf-Galois system.

Proof. The axioms for $(L^{opcop}, H^{opcop}, Z^{opcop}, T^{opcop})$ can all be deduced from the properties of the Hopf-Galois system (L, H, T, Z). For instance, the axiom $(B1)$ follows from the commutative diagram

$$
\begin{array}{ccccc}
Z & \xrightarrow{\delta_{Z,L}} & Z \otimes L & \xrightarrow{\sigma} & L \otimes Z \\
{\scriptstyle \delta_{H,Z}}\Big\downarrow & (B2) & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\rho_L} & & \Big\downarrow{\scriptstyle \rho_L\otimes\mathrm{id}} \\
H \otimes Z & \xrightarrow{\rho_H\otimes\mathrm{id}} & Z \otimes T \otimes Z & \xrightarrow{\sigma_{Z,T\otimes Z}} & T \otimes Z \otimes Z \\
{\scriptstyle \sigma}\Big\downarrow & & \Big\downarrow{\scriptstyle \sigma_{Z\otimes T,Z}} & & \Big\downarrow{\scriptstyle \sigma\otimes\mathrm{id}} \\
Z \otimes H & \xrightarrow[\mathrm{id}\otimes\rho_H]{} & Z \otimes Z \otimes T & \xrightarrow[\mathrm{id}\otimes\sigma]{} & Z \otimes T \otimes Z
\end{array}
$$

and $(B6)$ follows from

$$
\begin{array}{ccccc}
L & \xrightarrow{\Delta_L} & L \otimes L & \xrightarrow{\sigma} & L \otimes L \\
{\scriptstyle \rho_L}\Big\downarrow & (B6) & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\rho_L} & & \Big\downarrow{\scriptstyle \rho_L\otimes\mathrm{id}} \\
T \otimes Z & \xrightarrow{\delta_{L,T}\otimes\mathrm{id}} & L \otimes T \otimes Z & \xrightarrow{\sigma_{L,T\otimes Z}} & T \otimes Z \otimes L \\
{\scriptstyle \sigma}\Big\downarrow & & \Big\downarrow{\scriptstyle \sigma_{L\otimes T,Z}} & & \Big\downarrow{\scriptstyle \sigma\otimes\mathrm{id}} \\
Z \otimes T & \xrightarrow[\mathrm{id}\otimes\delta_{L,T}]{} & Z \otimes L \otimes T & \xrightarrow[\mathrm{id}\otimes\sigma]{} & Z \otimes T \otimes L
\end{array}
$$

The remaining axioms for a bicomodule-algebra system can be shown similarly. The generalized antipode axioms can be shown via diagrams of the following kind:

$$
\begin{array}{ccccc}
L & \xrightarrow{\varepsilon_L} & I & \xrightarrow{\eta_Z} & Z \\
{\scriptstyle \rho_L}\Big\downarrow & & (A4) & & \Big\uparrow{\scriptstyle \nabla_Z} \\
T \otimes Z & & \xrightarrow[S_T\otimes\mathrm{id}]{} & & Z \otimes Z \\
{\scriptstyle \sigma}\Big\downarrow & & & & \Big\uparrow{\scriptstyle \sigma} \\
Z \otimes T & & \xrightarrow[\mathrm{id}\otimes S_T]{} & & Z \otimes Z
\end{array}
$$

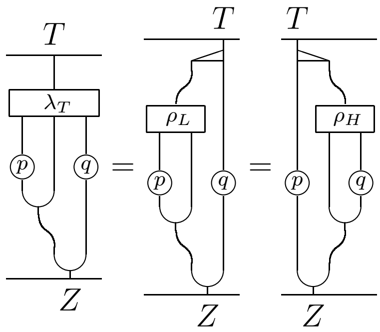
□

The antipode of a Hopf algebra H is defined as the convolution inverse of the identity on H. It is therefore uniquely determined by the structure morphisms of the underlying bialgebra. Since the generalized antipodes of a total Hopf-Galois system are not defined through such a property, but via axioms that generalize those of an antipode, it is not clear whether they are uniquely determined.
We are going to show that a generalized antipode always plays the role of the unit in a suitably chosen associative multiplication. A unit element in a monoid is uniquely determined by the monoid structure. This implies that generalized antipodes are unique.

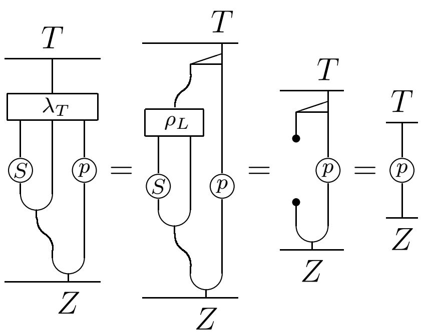
Proposition 3.1.5 *The generalized antipodes of a Hopf-Galois system are uniquely determined by the structure morphisms of the underlying bicomodule-algebra system.*

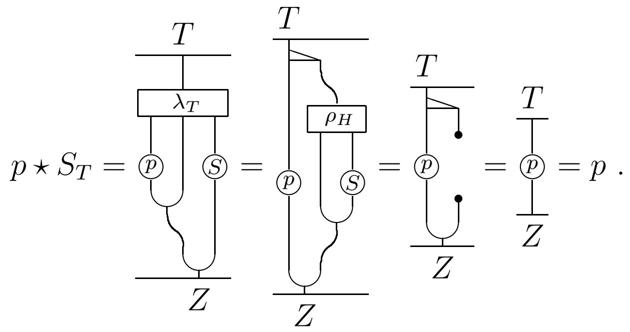
Proof. Let $(L, H, T, Z, S_L, S_H, S_T, S_Z)$ be a total Hopf-Galois system. We define an associative multiplication on the set $\mathrm{Hom}(T, Z)$ by

$$\star : \mathrm{Hom}(T,Z) \otimes \mathrm{Hom}(T,Z) \longrightarrow \mathrm{Hom}(T,Z)\ ,\ p \otimes q \mapsto p \star q\ ,$$

$p \star q :=$  .

Associativity of this composition follows from coassociativity of λ_T and associativity of ∇_Z. We show that $S_T : T \longrightarrow Z$ is a left and right unit for this multiplication. Let $p \in \mathrm{Hom}(T, Z)$. Then we obtain

$S_T \star p =$  $= p$, using the generalized antipode axiom (A4). Similarly, we get by axiom (A2) that



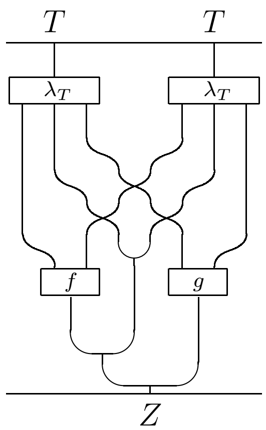
So it follows by uniqueness of the unit element in a monoid that the generalized antipode S_T is uniquely determined. The same follows for S_Z by applying the above multiplication on $\mathrm{Hom}(Z,T)$. □

Recall that the antipode of a Hopf algebra is an algebra anti-morphism. The proof of this result is based on the fact that the antipode is a convolution inverse. Keeping this in mind, we can apply our technique to show that the generalized antipodes of a total Hopf-Galois system have an analogous property:

Proposition 3.1.6 *Let* $(L,H,T,Z,S_L,S_H,S_T,S_Z)$ *be a total Hopf-Galois system. Then the generalized antipodes are algebra morphisms* $S_T : T \longrightarrow Z^{op}$ *and* $S_Z : Z \longrightarrow T^{op}$.

Proof. We introduce an associative multiplication on $\mathrm{Hom}(T\otimes T, Z)$ by

$$* : \mathrm{Hom}(T\otimes T,Z)\otimes \mathrm{Hom}(T\otimes T,Z) \longrightarrow \mathrm{Hom}(T\otimes T,Z)\ ,\ f\otimes g \mapsto f * g\ ,$$

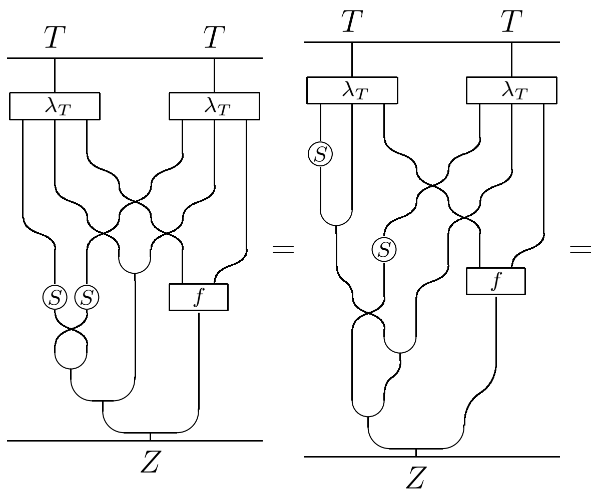
$f * g :=$  .

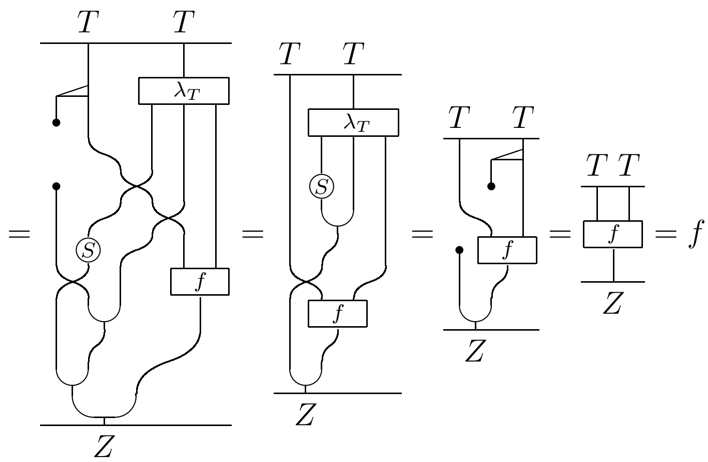
Associativity of $*$ follows from coassociativity of λ_T and associativity of the multiplication on Z. A unit for this multiplication is given by

$$\nabla_{Z^{op}}(S_T \otimes S_T) = \nabla_Z \circ \sigma(S_T \otimes S_T) \in \mathrm{Hom}(T \otimes T, Z) \ ,$$

since we have for $f \in \mathrm{Hom}(T \otimes T, Z)$

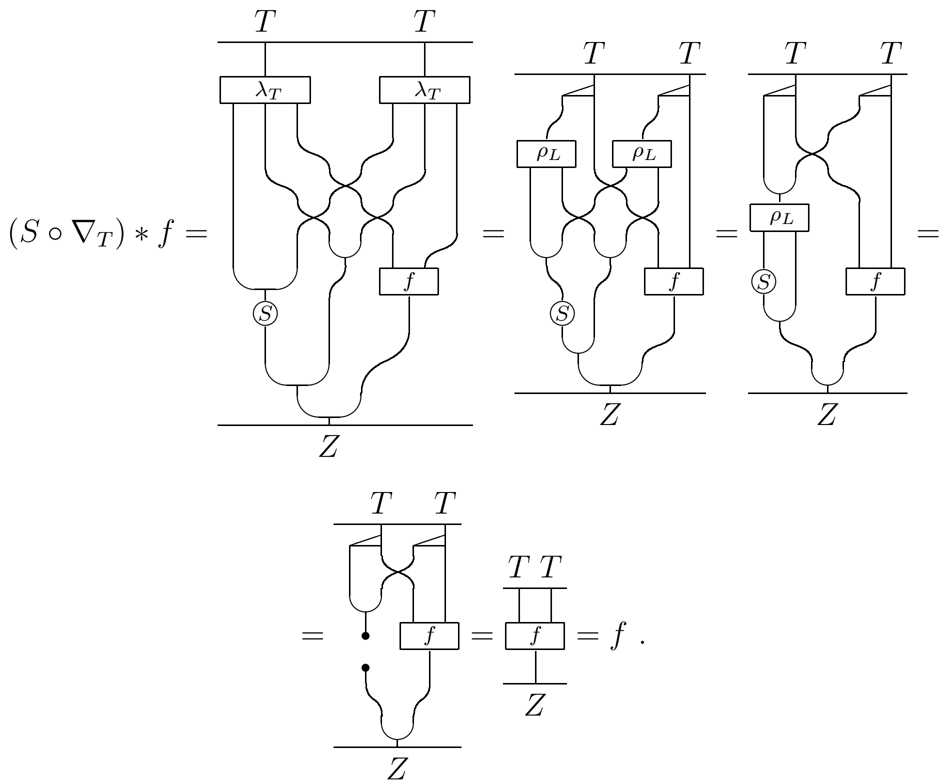
$$(\nabla_Z \circ \sigma(S_T \otimes S_T)) * f =$$





by axiom (A4). Similarly, we obtain also $f * (\nabla_{Z^{op}}(S_T \otimes S_T)) = f$. On the other hand, using that ρ_H is an algebra morphism, we can show that $S_T \circ \nabla_T$

also acts as a unit:



Now uniqueness of the unit element implies that $S_T \circ \nabla_T = \nabla_{Z^{op}}(S_T \otimes S_T)$, which means that S_T is an algebra anti-morphism. The proof for the generalized antipode $S_Z : Z \longrightarrow T^{op}$ is analogous. □

The antipode S of a Hopf algebra H is being preserved by bialgebra morphisms. This means that a morphism of Hopf algebras can be defined as a morphism of the underlying bialgebras. We have already defined what a morphism of total Hopf-Galois systems should be. Now we show, that such a morphism is really compatible with the generalized antipodes:

Proposition 3.1.7 *Let* $(\ell, h, f, g) : (L, H, T, Z) \longrightarrow (L', H', T', Z')$ *be a morphism of total Hopf-Galois systems. Then the following diagrams commute:*

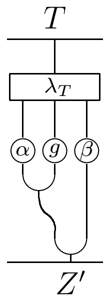
$$\begin{array}{ccc} T & \xrightarrow{S_T} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ T' & \xrightarrow[S_{T'}]{} & Z' \end{array} \qquad \begin{array}{ccc} Z & \xrightarrow{S_Z} & T \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow[S_{Z'}]{} & T' \end{array} \qquad (3.3)$$

This means that the generalized antipodes are preserved by morphisms of bicomodule-algebra systems.

Proof. We define an associative multiplication on $\mathrm{Hom}(T, Z')$ by

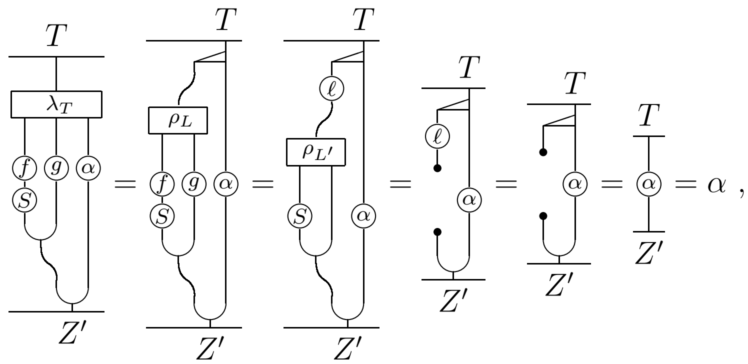
$$\times : \mathrm{Hom}(T, Z') \otimes \mathrm{Hom}(T, Z') \longrightarrow \mathrm{Hom}(T, Z')\ ,\ \ \alpha \otimes \beta \mapsto \alpha \times \beta\ ,$$

$\alpha \times \beta :=$



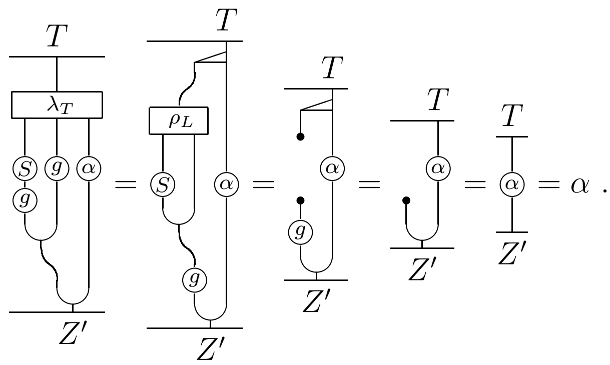
where the algebra morphism $g : Z \longrightarrow Z'$ comes from the given morphism of Hopf-Galois systems. It is clear that this multiplication is associative. The unit is given by $S_{T'} \circ f$, since we get for $\alpha \in \mathrm{Hom}(T, Z')$

$(S_{T'} \circ f) \times \alpha =$



using property (M1). On the other hand, we see that $g \circ S_T$ also acts as a unit since g is an algebra morphism:

$(g \circ S_T) \times \alpha =$



Thus, we obtain $g \circ S_T = S_{T'} \circ f$ by uniqueness of the unit element. This shows that the first diagram commutes. Commutativity of the second diagram can be proved analogously. □

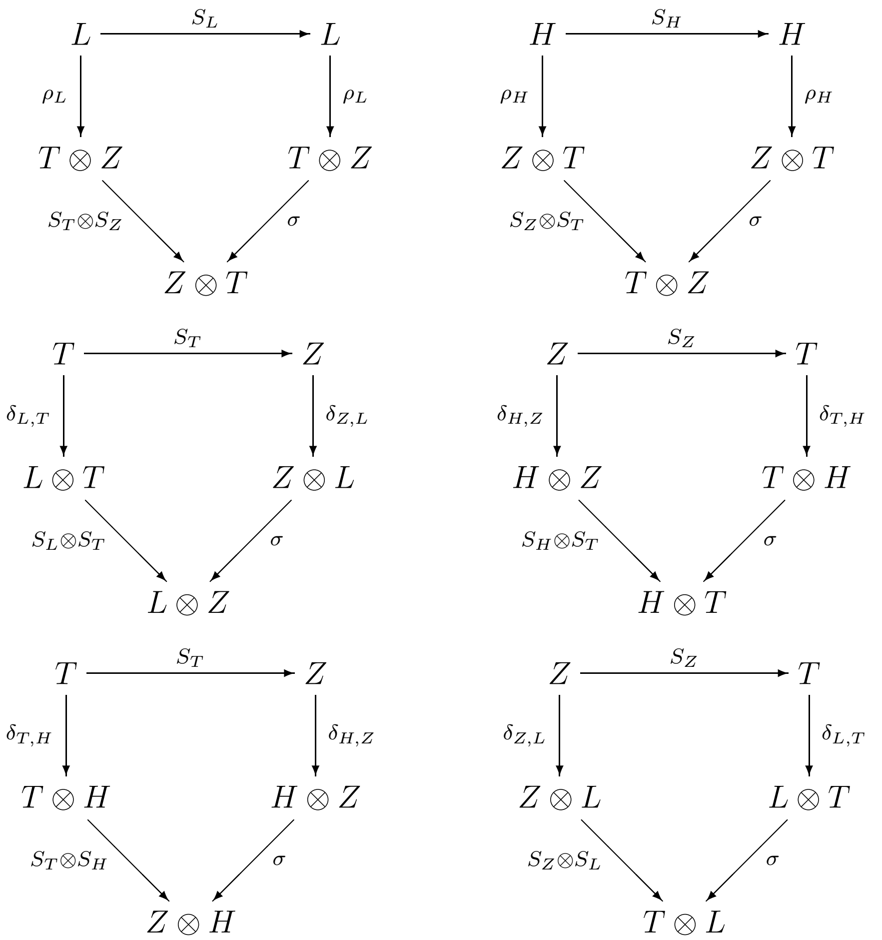
The antipode S of a Hopf algebra H is initially just defined as a linear map $S : H \longrightarrow H$. Then its properties imply that it is really an algebra anti-morphism and a coalgebra anti-morphism, that is, a bialgebra morphism $S : H \longrightarrow H^{opcop}$. This fact has a generalization for total Hopf-Galois systems:

The antipodes and generalized antipodes of a Hopf-Galois system (L, H, T, Z) give rise to a morphism of Hopf-Galois systems

$$(S_L, S_H, S_T, S_Z) : (L, H, T, Z) \longrightarrow (L^{opcop}, H^{opcop}, Z^{opcop}, T^{opcop}) .$$

This follows from the next proposition.

Proposition 3.1.8 *The antipodes and generalized antipodes of a total Hopf-Galois system* $(L, H, T, Z, S_L, S_H, S_T, S_Z)$ *are related in the following way:*



(3.4)

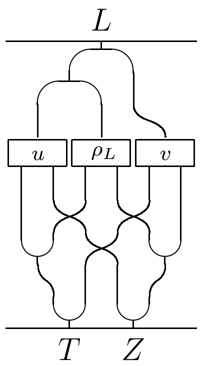
(3.5)

(3.6)

Proof. For the first diagram, we define an associative multiplication $*$ on the set $\mathrm{Hom}(L, T \otimes Z)$ by

$$* : \mathrm{Hom}(L, T \otimes Z) \otimes \mathrm{Hom}(L, T \otimes Z) \longrightarrow \mathrm{Hom}(L, T \otimes Z)\ ,\ u \otimes v \mapsto u * v$$

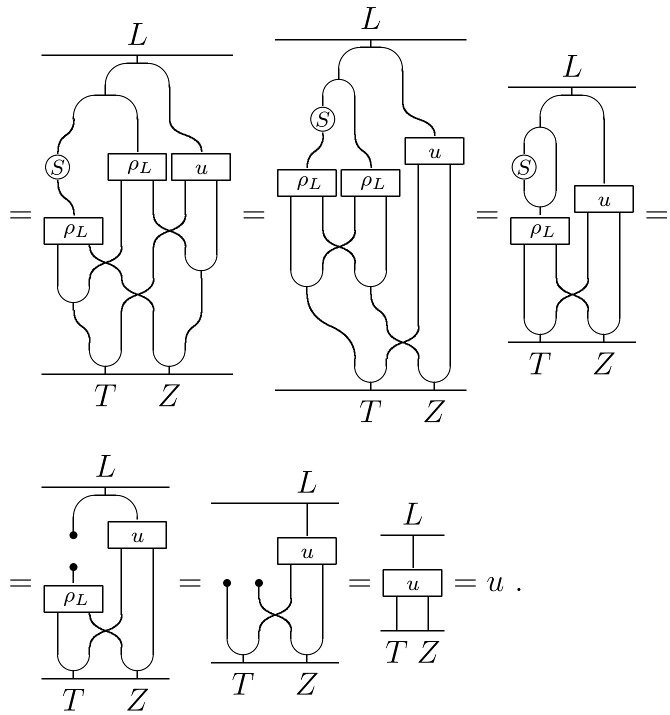
$u * v :=$



.

The unit for this multiplication is given by $\rho_L \circ S_L$, since ρ_L is an algebra morphism:

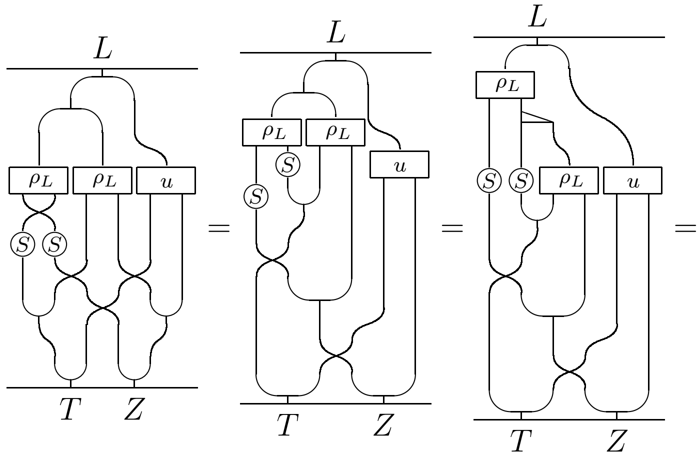
$(\rho_L \circ S_L) * u =$

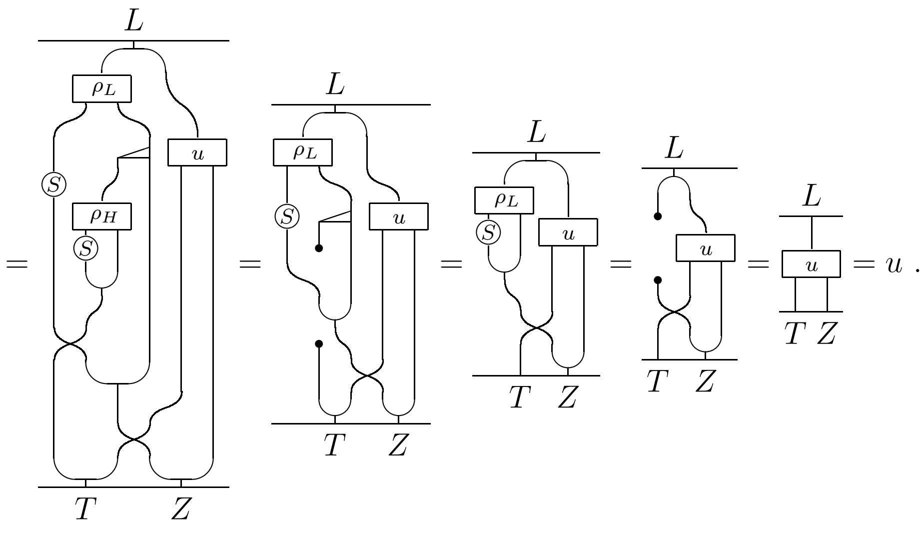


$= u$.

But $(S_Z \otimes S_T) \circ \sigma \circ \rho_L$ also acts as a unit, as follows from the equation (3.2) and axioms (B2), (A3) and (A4) :

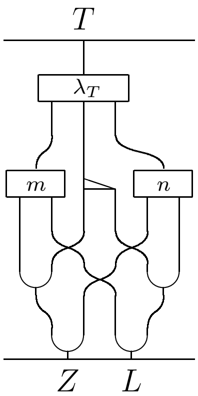
$((S_Z \otimes S_T)\sigma\rho_L) * u =$



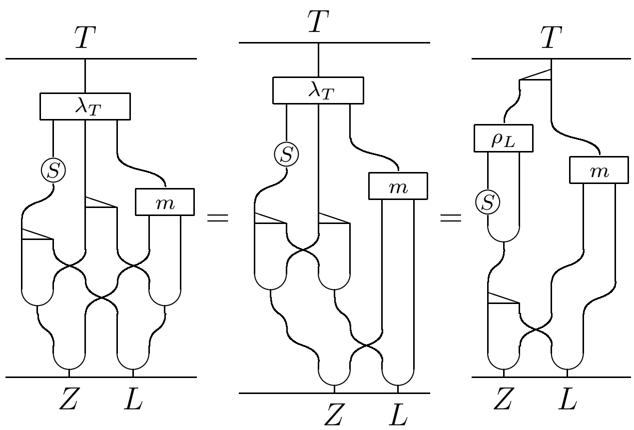


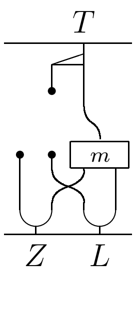
So it follows by uniqueness of the unit element that the maps $\sigma \circ \rho_L \circ S_L$ and $(S_Z \otimes S_T) \circ \rho_L$ are equal. This means that the first diagram commutes.

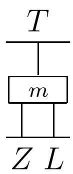
In order to prove the equality $\sigma \circ \delta_{Z,L} \circ S_T = (S_L \otimes S_T)\delta_{L,T}$, we define an associative multiplication $\star$ on $\mathrm{Hom}(T, Z \otimes L)$ by

$m \star n :=$ 

and show that both $\delta_{Z,L} \circ S_T$ and $\sigma(S_L \otimes S_T)\delta_{L,T}$ are units for $\star$. We have

$(\delta_{Z,L} \circ S_T) \star m =$ 

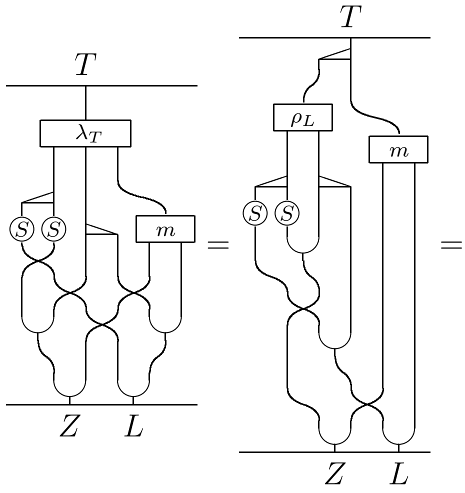
$=$ 

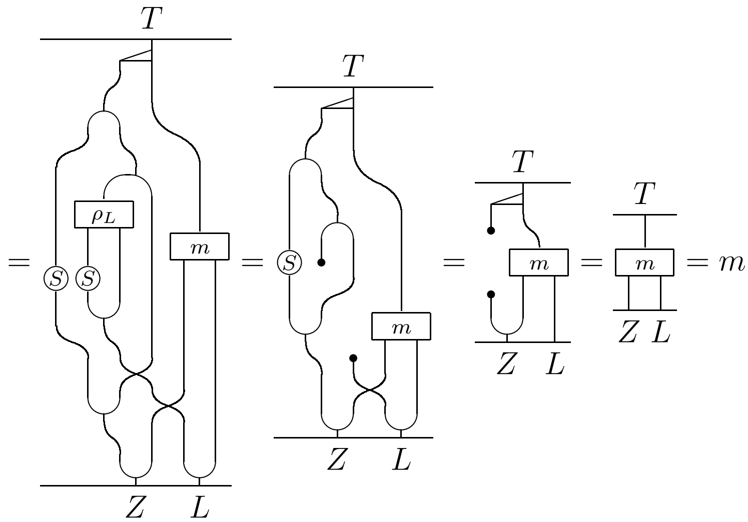
$=$ 

$= m$,

using that Z is an L-comodule algebra, and

$$(\sigma(S_L \otimes S_T)\delta_{L,T}) \star m =$$





by equation (3.1).
Commutativity of the remaining diagrams can be proved similarly by considering suitable multiplications. □

Remark 3.1.9 It is straightforward to show that a total Hopf-Galois system (L, H, T, Z) gives rise to a quantum torsor structure on both T and Z. The structure maps are given by $\mu_T := (T \otimes S_Z \otimes T)\lambda_T$ and $\mu_Z := (Z \otimes S_T \otimes Z)\lambda_Z$ respectively. Similar to the construction for cohomomorphism objects, the Grunspan maps are given by $\theta_T := S_Z \circ S_T$ resp. $\theta_Z := S_T \circ S_Z$. The fact that they are morphisms of quantum torsors follows from the properties of the generalized antipodes in Proposition 3.1.8.
Since S_T and S_Z have properties that generalize those of an antipode, the

Grunspan map θ can be interpreted as a generalized square of the antipode. This is consistent with the fact that the torsor arising from a Hopf algebra in Example 1.4.2 has the Grunspan map S^2.

Examples for bicomodule algebra systems can be found easily in "nature". As we have already mentioned, we have implicitly considered the bicomodule algebra system $(\mathrm{coend}(\omega), \mathrm{coend}(\nu), \mathrm{cohom}(\nu,\omega)\,\mathrm{cohom}(\omega,\nu))$ in the previous chapter.
To give a concrete example, we compute all the objects and structure maps of the total Hopf-Galois system that arises from a faithfully flat Hopf-Galois extension. For this, we recall from Section 1.2 the notation "$h^{[1]} \otimes h^{[2]}$" for the inverse of the Galois map.

Proposition 3.1.10 *Let H be a k-Hopf algebra and let A be a right H-Galois extension that is faithfully flat over k. This gives rise to a total Hopf-Galois system (L, H, A, Z), whose structure maps are given as follows:*

- *$L := (A \otimes A)^{coH}$ is a subalgebra of $A \otimes A^{op}$ and a Hopf algebra by*

$$\Delta_L(\sum x \otimes y_i) = \sum x_{i(0)} \otimes {x_{i(1)}}^{[1]} \otimes {x_{i(1)}}^{[2]} \otimes y_i$$
$$\varepsilon_L(\sum x_i \otimes y_i) = \sum x_i y_i \ \in A^{coH} = k$$
$$S_L(\sum x_i \otimes y_i) = \sum y_{i(0)} \otimes {y_{i(1)}}^{[1]} x_i {y_{i(1)}}^{[2]}$$

- *A is a left L-comodule algebra by*

$$\delta_{L,A}(a) = a_{(0)} \otimes {a_{(1)}}^{[1]} \otimes {a_{(1)}}^{[2]}$$

- *$Z := A^{-1} := (H \otimes A)^{coH}$ is a left H-subcomodule algebra of $H \otimes A^{op}$ with the (H,L)-bicomodule algebra structure given by*

$$\delta_{H,Z}(\sum h_i \otimes a_i) = \sum h_{i(1)} \otimes h_{i(2)} \otimes a_i$$
$$\delta_{Z,L}(\sum h_i \otimes a_i) = h_{i(1)} \otimes {h_{i(2)}}^{[1]} \otimes {h_{i(2)}}^{[2]} \otimes a_i$$

- *The algebra morphisms ρ_L and ρ_H are given by*

$$\rho_L : L \longrightarrow A \otimes Z \ , \ \sum x_i \otimes y_i \mapsto \sum x_{i(0)} \otimes x_{i(1)} \otimes y_i$$
$$\rho_H : H \longrightarrow Z \otimes A \ , \ h \mapsto h_{(1)} \otimes {h_{(2)}}^{[1]} \otimes {h_{(2)}}^{[2]}$$

- *The generalized antipodes are given by*

$$S_A : A \longrightarrow Z \ , \ a \mapsto a_{(0)} S_H(a_{(1)})^{[2]} S_H(a_{(2)}) \otimes S_H(a_{(1)})^{[1]}$$
$$S_Z : Z \longrightarrow A \ , \ \sum h_i \otimes a_i \mapsto \sum \varepsilon_H(h_i) a_i$$

- *If the antipode S_H of H is invertible, then both generalized antipodes S_A and S_Z are invertible with inverses*

$$S_A^{-1} : Z \longrightarrow A \ , \ \sum h_i \otimes a_i \mapsto \sum \varepsilon(h_i) S_H^{-2}(a_{i(1)})^{[1]} a_{i(0)} S_H^{-2}(a_{i(1)})^{[2]}$$
$$S_Z^{-1} : A \longrightarrow Z \ , \ a \mapsto S_H^{-1}(a_{(1)}) \otimes a_{(0)}$$

Proof. We know from [38] that L is a Hopf algebra and that A becomes an L-H-bi-Galois extension via $\delta_{L,A}$. There, it was also proved that there exists a right L-comodule structure on $A^{-1} =: Z$ turning it into an H-L-bi-Galois extension. Moreover, the isomorphism of left L-comodule algebras

$$\phi : A \square_H (H \otimes A)^{coH} \longrightarrow L \ , \ \sum a_i \otimes \sum h_{ij} \otimes b_{ij} \mapsto \sum a_i \otimes \varepsilon(h_{ij}) b_{ij}$$

from [38] is such that $A \square_H Z$ is an L-L-bi-Galois extension isomorphic to the trivial L-L-bi-Galois extension L. The inverse of ϕ is given by $\phi^{-1}(\sum a_i \otimes b_i) = \sum a_{i(0)} \otimes a_{i(1)} \otimes b_i$. So we can determine the right L-comodule structure on $A \square_H Z$, which is induced by Z, via the commutative diagram

$$\begin{array}{ccc} A \square_H Z & \xrightarrow{\ \phi\ } & L \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \Delta_L} \\ A \square_H Z \otimes L & \xleftarrow[\ \phi^{-1} \otimes L\ ]{} & L \otimes L \end{array}$$

as

$$\begin{aligned} \delta(\sum a_i \otimes \sum h_{ij} \otimes b_{ij}) &= (\phi^{-1} \otimes L) \circ \Delta_L \circ \phi(\sum a_i \otimes \sum h_{ij} \otimes b_{ij}) \\ &= (\phi^{-1} \otimes L)(\sum a_{i(0)} \varepsilon(h_{ij}) \otimes a_{i(1)}{}^{[1]} \otimes a_{i(1)}{}^{[2]} \otimes b_{ij}) \\ &= \sum a_{i(0)} \varepsilon(h_{ij}) \otimes a_{i(1)} \otimes a_{i(2)}{}^{[1]} \otimes a_{i(2)}{}^{[2]} \otimes b_{ij}) \\ &= \sum a_i \otimes h_{ij(1)} \otimes h_{ij(2)}{}^{[1]} \otimes h_{ij(2)}{}^{[2]} \otimes b_{ij} \ . \end{aligned}$$

The last equality is deduced from the property of the cotensor product $A \square_H Z$. So the comodule structure must be

$$\delta_{Z,L} : Z \longrightarrow Z \otimes L \ , \ \sum h_i \otimes a_i \mapsto \sum h_{i(1)} \otimes h_{i(2)}{}^{[1]} \otimes h_{i(2)}{}^{[2]} \otimes a_i \ ,$$

and it is straightforward to see that this map is well-defined, and makes Z into an L-comodule algebra.

The map $\rho_L : L \longrightarrow A \otimes Z$ in the statement of the proposition is given by the same formula as the inverse of $\phi : A\Box_H Z \longrightarrow L$. Thus, it follows that ρ_L is an algebra morphism, and we moreover have $\rho_L(L) \subseteq A\Box_H Z$.

In order to show that the map ρ_H is well-defined, we recall that for two right H-modules V and W the space $(V \otimes W)^{coH}$ consists of those $\sum v_i \otimes w_i \in V \otimes W$ that have the property

$$\sum v_i \otimes w_{i(0)} \otimes w_{i(1)} = \sum v_{i(0)} \otimes w_i \otimes S(v_{i(1)}) \ . \tag{3.7}$$

Then we see that the map ρ_H is well defined, since we have for $h \in H$

$$\begin{aligned}(\delta_{H\otimes A,H} \otimes \mathrm{id})\rho_H(h) &= (\delta_{H\otimes A,H} \otimes \mathrm{id})(h_{(1)} \otimes {h_{(2)}}^{[1]} \otimes {h_{(2)}}^{[2]}) \\ &= h_{(1)} \otimes {{h_{(3)}}^{[1]}}_{(0)} \otimes h_{(2)}{{h_{(3)}}^{[1]}}_{(1)} \otimes {h_{(3)}}^{[2]} \\ &= h_{(1)} \otimes {h_{(4)}}^{[1]} \otimes h_{(2)}S(h_{(3)}) \otimes {h_{(4)}}^{[2]} \\ &= h_{(1)} \otimes {h_{(2)}}^{[1]} \otimes 1 \otimes {h_{(2)}}^{[2]}\end{aligned}$$

by property (1.5), which implies $\rho_H(H) \subseteq (H \otimes A)^{coH} \otimes A = Z \otimes A$ by faithful flatness of A. It is clearly an algebra map, and we moreover have $\rho_H(H) \subseteq Z\Box_L A$, since

$$\begin{aligned}(Z \otimes \delta_{L,A})\rho_H(h) &= (Z \otimes \delta_{L,A})(h_{(1)} \otimes {h_{(2)}}^{[1]} \otimes {h_{(2)}}^{[2]}) \\ &= h_{(1)} \otimes {h_{(2)}}^{[1]} \otimes {{h_{(2)}}^{[2]}}_{(0)} \otimes {{{h_{(2)}}^{[2]}}_{(1)}}^{[1]} \otimes {{{h_{(2)}}^{[2]}}_{(1)}}^{[2]} \\ &= h_{(1)} \otimes {h_{(2)}}^{[1]} \otimes {h_{(2)}}^{[2]} \otimes {h_{(3)}}^{[1]} \otimes {h_{(3)}}^{[2]} \\ &= (\delta_{Z,L} \otimes A)(h_{(1)} \otimes {h_{(2)}}^{[1]} \otimes {h_{(2)}}^{[2]}) \\ &= (\delta_{Z,L} \otimes A)\rho_H(h)\end{aligned}$$

for all $h \in H$ by (1.4).

It can now be easily be seen that (L, H, A, Z) satisfies the axioms of a bicomodule algebra system from Definition 3.1.1.

One can easily check that the generalized antipode maps S_A and S_Z are well-defined. We finally show that they satisfy the axioms for a total Hopf-Galois system in Definition 3.1.3: Axioms $(A1)$ and $(A3)$ follow from

$$\begin{aligned}\nabla_A(A \otimes S_Z)\rho_L(\sum x_i \otimes y_i) = \sum x_{i(0)}\varepsilon_H(x_{i(1)})y_i = \sum x_i y_i = \\ = \eta_A\varepsilon_L(\sum x_i \otimes y_i)\end{aligned}$$

and

$$\nabla_A(S_Z \otimes A)\rho_H(h) = \varepsilon_H(h_{(1)}){h_{(2)}}^{[1]}{h_{(2)}}^{[2]} = \eta_A\varepsilon_H(h)$$

for $\sum x_i \otimes y_i \in L$ and $h \in H$ by (1.6). We use properties (1.4) and (1.7) for axiom $(A2)$

$$\begin{aligned}
&\nabla_Z(Z \otimes S_A)\rho_H(h) = \\
&\quad = h_{(1)}{h_{(2)}^{[2]}}_{(0)}S({h_{(2)}^{[2]}}_{(1)})^{[2]}S({h_{(2)}^{[2]}}_{(2)}) \otimes S({h_{(2)}^{[2]}}_{(1)})^{[1]}{h_{(2)}}^{[1]} = \\
&\qquad = h_{(1)}{h_{(2)}}^{[2]}S(h_{(3)})^{[2]}S(h_{(4)}) \otimes S(h_{(3)})^{[1]}{h_{(2)}}^{[1]} = \\
&\qquad = h_{(1)}(h_{(2)}S(h_{(3)}))^{[2]}S(h_{(4)}) \otimes (h_{(2)}S(h_{(3)}))^{[1]} = \\
&\qquad\qquad = h_{(1)}S(h_{(2)}) \otimes 1 = \eta_Z\varepsilon_H(h)\ .
\end{aligned}$$

The axiom $(A4)$ follows by applying (3.7) and (1.10):

$$\begin{aligned}
\nabla_Z(S_A \otimes Z)\rho_L(\sum x_i \otimes y_i) &= \sum x_{i(0)}S(x_{i(1)})^{[2]}S(x_{i(2)})x_{i(3)} \otimes y_iS(x_{i(1)})^{[1]} \\
&= \sum x_{i(0)}S(x_{i(1)})^{[2]} \otimes y_iS(x_{i(1)})^{[1]} \\
&= \sum x_i{y_{i(1)}}^{[2]} \otimes y_{i(0)}{y_{i(1)}}^{[1]} \\
&= \sum x_iy_i \otimes 1 \\
&= \eta_Z\varepsilon_L(\sum x_i \otimes y_i)\ .
\end{aligned}$$

□

3.2 Universal Hopf-Galois Systems

Takeuchi has shown in [51] that every bialgebra B can be mapped into a universal Hopf algebra $\mathcal{H}(B)$. This can be done such that each bialgebra morphism from B to an arbitrary Hopf algebra H factors uniquely over $\mathcal{H}(B)$. By the analogy between bialgebras and Hopf algebras on one side and bicomodule algebra systems and total Hopf-Galois systems on the other, we can hope to obtain a similar result for bicomodule algebra systems. This would answer the question whether one can "adjoin" antipodes and generalized antipodes to a bicomodule algebra system to obtain a total Hopf algebra system from it. Such a result would be particularly interesting, since we have seen in the previous chapter that bicomodule algebra systems appear naturally as quadruples $(\mathrm{coend}(\omega), \mathrm{coend}(\nu), \mathrm{cohom}(\omega,\nu), \mathrm{cohom}(\nu,\omega))$ for functors $\nu, \omega : \mathcal{C} \longrightarrow \mathcal{M}$ (without having to require that the category $\mathcal{C}$ be rigid).

We have seen in the previous section that a total Hopf-Galois system contains just the data needed for encoding two "inverse" bi-Galois extensions inside it. If we take a closer look at the six diagrams in Definition 3.1.1, we realize that two particular diagrams one might hope for are missing. We can in fact formulate two more compatibility constraints as follows:

Let (L, H, T, Z) be a bicomodule algebra system. Then we define the properties $(B7)$ and $(B8)$ as

$$\begin{CD} L @>{\rho_L}>> T \otimes Z \\ @V{\rho_L}VV (B7) @VV{\mathrm{id} \otimes \delta_{H,Z}}V \\ T \otimes Z @>>{\delta_{T,H} \otimes \mathrm{id}}> T \otimes H \otimes Z \end{CD} \qquad \begin{CD} H @>{\rho_H}>> Z \otimes T \\ @V{\rho_H}VV (B8) @VV{\mathrm{id} \otimes \delta_{L,T}}V \\ Z \otimes T @>>{\delta_{Z,L} \otimes \mathrm{id}}> Z \otimes L \otimes T \end{CD}$$

Commutativity of diagram $(B7)$ is clearly equivalent to the image of ρ_L lying in the cotensor product of T and Z, i.e. $\rho_L(L) \subset T \Box_L Z$. If $(B8)$ holds, then we have $\rho_H(H) \subset Z \Box_H T$, and this implies that the equation

$$h^Z{}_{(0)} \otimes h^Z{}_{(1)} \otimes h^T{}_{(-1)} \otimes h^T{}_{(0)} = h^Z{}_{(0)} \otimes h^Z{}_{(1)} \otimes h^Z{}_{(2)} \otimes h^T \tag{3.8}$$

holds for all $h \in H$.

It follows from the coassociativity axiom of a coalgebra that the trivial Hopf-Galois system (H, H, H, H) that arises from a Hopf algebra H satisfies the properties (B7) and (B8).
The same holds for the bicomodule algebra system $(\mathrm{coend}(\omega), \mathrm{coend}(\nu), \mathrm{cohom}(\omega, \nu), \mathrm{cohom}(\nu, \omega))$. This follows directly by construction of the respective comultiplications in Proposition D.3.
Surprisingly, also the total Hopf-Galois system arising from a faithfully flat Hopf-Galois extensions has this additional property, as we have seen in the proof of Proposition 3.1.10.

These examples indicate that, although we do not include these properties in the definition of a Hopf-Galois system, it still makes sense to consider them. If we assume that a bicomodule algebra system satisfies these additional properties, then we can show that it admits a universal Hopf-Galois system, generalizing Takeuchi's result for bialgebras. So then it is possible to "adjoin" generalized antipodes as in the sense of the following theorem.

In this section, we work in the base category $\mathbb{K}$-Vec.

Theorem 3.2.1 *Let (L, H, T, Z) be a bicomodule-algebra system that has the additional properties (B7) and (B8). Then there exists a Hopf-Galois system $(\mathcal{H}(L), \mathcal{H}(H), \mathcal{H}(T), \mathcal{H}(Z))$ and a morphism of bicomodule-algebra systems*

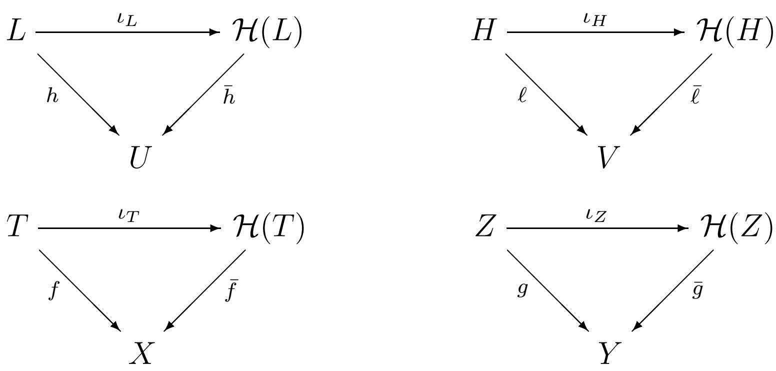
$$(\iota_L, \iota_H, \iota_T, \iota_Z) : (L, H, T, Z) \longrightarrow (\mathcal{H}(L), \mathcal{H}(H), \mathcal{H}(T), \mathcal{H}(Z))$$

such that the following universal property holds:

For every Hopf-Galois system (U, V, X, Y) and every morphism of bicomodule-algebra systems $(\ell, h, f, g) : (L, H, T, Z) \longrightarrow (U, V, X, Y)$ there exists a unique morphism of Hopf-Galois systems

$$(\bar{\ell}, \bar{h}, \bar{f}, \bar{g}) : (\mathcal{H}(L), \mathcal{H}(H), \mathcal{H}(T), \mathcal{H}(Z)) \longrightarrow (U, V, X, Y)$$

such that the following diagrams simultaneously commute:



The proof of this theorem requires the following two results from [33].

Lemma 3.2.2 *([33]) The category* $\mathbb{K}$-Alg *of* $\mathbb{K}$*-algebras has arbitrary coproducts.*

Lemma 3.2.3 *([33]) The category* $\mathbb{K}$-Bialg *of* $\mathbb{K}$*-bialgebras has arbitrary coproducts.*

We also need the following lemma:

Lemma 3.2.4 *Let B and C be $\mathbb{K}$-bialgebras. Then the category of (B, C)-bicomodule algebras has arbitrary coproducts.*

Proof. Let $(P_i)_{i \in \mathbb{N}}$ be a family of (B, C)-bicomodule algebras and let P be their coproduct in $\mathbb{K}$-Alg with the injections $\iota_i : P_i \longrightarrow P$. By the universal property of the coproduct there exist algebra morphisms $\delta_B : P \longrightarrow B \otimes P$ and $\delta_C : P \longrightarrow P \otimes C$ such that the diagrams

$$\begin{array}{ccc} P_i & \xrightarrow{\iota_i} & P \\ {\scriptstyle \delta_i}\downarrow & & \downarrow{\scriptstyle \delta_B} \\ B \otimes P_i & \xrightarrow[\mathrm{id} \otimes \iota_i]{} & B \otimes P \end{array} \qquad \begin{array}{ccc} P_i & \xrightarrow{\iota_i} & P \\ {\scriptstyle \delta_i}\downarrow & & \downarrow{\scriptstyle \delta_C} \\ P_i \otimes C & \xrightarrow[\mathrm{id} \otimes \iota_i]{} & P \otimes C \end{array}$$

commute. Now it is straightforward to see that δ_B and δ_C define a bicomodule algebra structure on P. □

Proof of Theorem 3.2.1. We consider the following families of bialgebras resp. comodule algebras taken from the given bicomodule-algebra system (L, H, T, Z): For $i \in \mathbb{N}_0$ let

$$\begin{array}{ll} F_{2i} := L & F_{2i+1} := L^{opcop} \\ G_{2i} := H & G_{2i+1} := H^{opcop} \\ A_{2i} := T & A_{2i+1} := Z^{opcop} \\ B_{2i} := Z & B_{2i+1} := T^{opcop} \end{array}$$

where $(L^{opcop}, H^{opcop}, Z^{opcop}, T^{opcop})$ is the opposite bicomodule-algebra system described in Remark 3.1.4.

Let $F := \coprod_i F_i$ be the coproduct of the family $(F_i)_{i \in \mathbb{N}_0}$ with the injections $\iota_i : F_i \longrightarrow F$. By Lemma 3.2.3, F is a bialgebra, and the same holds for $G := \coprod_i G_i$.

The following maps define an (F, G)-bicomodule algebra structure on each of the algebras A_i, $i \in \mathbb{N}_0$:

$$A_{2i} = T \xrightarrow{\delta_{L,T}} L \otimes T \xrightarrow{\iota_{2i} \otimes \mathrm{id}} F \otimes A_{2i}$$

$$A_{2i} = T \xrightarrow{\delta_{T,H}} T \otimes H \xrightarrow{\mathrm{id} \otimes \iota_{2i}} A_{2i} \otimes G$$

$$A_{2i+1} = Z^{opcop} \xrightarrow{\sigma \circ \delta_{Z,L}} L^{opcop} \otimes Z^{opcop} \xrightarrow{\iota_{2i+1} \otimes \mathrm{id}} F \otimes A_{2i+1}$$

$$A_{2i+1} = Z^{opcop} \xrightarrow{\sigma \circ \delta_{H,Z}} Z^{opcop} \otimes H^{opcop} \xrightarrow{id \otimes \iota_{2i+1}} A_{2i+1} \otimes G$$

Hence, when defining $A := \coprod_i A_i$ in the category of $\mathbb{K}$-algebras, we get by Lemma 3.2.4 that A is an (F, G)-bicomodule algebra. In the same way we obtain a (G, F)-bicomodule algebra structure on $B := \coprod_i B_i$.

Proposition 3.2.5 *(F, G, A, B) is a bicomodule-algebra system.*

Proof. The family of algebra morphisms

$$\rho_{F_{2i}} : F_{2i} = L \xrightarrow{\rho_L} T \otimes Z = A_{2i} \otimes B_{2i}$$

$$\rho_{F_{2i+1}} : F_{2i+1} = L^{opcop} \xrightarrow{\sigma \circ \rho_L} Z^{opcop} \otimes T^{opcop} = A_{2i+1} \otimes B_{2i+1}$$

for all $i \in \mathbb{N}_0$, gives rise to an an algebra morphism $\rho_F : F \longrightarrow A \otimes B$ such that the diagram

$$\begin{array}{ccc} F_i & \xrightarrow{\iota_i} & F \\ \downarrow{\scriptstyle \rho_{F_i}} & & \downarrow{\scriptstyle \rho_F} \\ A_i \otimes B_i & \xrightarrow[\iota_i \otimes \iota_i]{} & A \otimes B \end{array}$$

commutes. Similarly, we construct an algebra morphism $\rho_G : G \longrightarrow B \otimes A$. In order to show that ρ_F and ρ_G together with the previously constructed structure morphisms make (F, G, A, B) into a bicomodule-algebra system, it suffices to show that this holds for each component (F_i, G_i, A_i, B_i). But for even indices, this is just the given bicomodule-algebra system, and for odd indices the opposite bicomodule-algebra system from Remark 3.1.4. □

In order to construct antipodes and generalized antipodes, we define the following four maps through the universal properties of the respective coproducts:

$$\begin{array}{ccc} F_i & \xrightarrow{\iota_i} & F \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle S'_F} \\ (F_i^{opcop})^{opcop} = F_{i+1}^{opcop} & \xrightarrow[\iota_{i+1}]{} & F^{opcop} \end{array} \tag{3.9}$$

$$\begin{array}{ccc} G_i & \xrightarrow{\iota_i} & G \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle S'_G} \\ (G_i^{opcop})^{opcop} = G_{i+1}^{opcop} & \xrightarrow[\iota_{i+1}]{} & G^{opcop} \end{array} \tag{3.10}$$

$$\begin{array}{ccc} A_i & \xrightarrow{\iota_i} & A \\ \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle S'_A} \\ (A_i^{opcop})^{opcop} = B_{i+1}^{opcop} & \xrightarrow[\iota_{i+1}]{} & B^{opcop} \end{array} \tag{3.11}$$

$$\begin{array}{ccc} B_i & \xrightarrow{\iota_i} & B \\ \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle S'_B} \\ (B_i^{opcop})^{opcop} = A_{i+1}^{opcop} & \xrightarrow[\iota_{i+1}]{} & A^{opcop} \end{array} \tag{3.12}$$

Let I_F be the two-sided ideal of F generated by the set

$$\{(S'_F * \mathrm{id} - \eta\varepsilon)(x_i)\ ,\ (\mathrm{id} * S'_F - \eta\varepsilon)(x_i) \mid x_i \in \iota_i(F_i), i \in \mathbb{N}_0\}\ ,$$

where $*$ denotes the convolution in $\mathrm{Hom}(F, F)$. It is shown in [33] that I_F is a biideal of F. In the same way we get a biideal I_G of G. Then the quotient bialgebras $\mathcal{H}(L) := F/I_F$ and $\mathcal{H}(H) := G/I_G$ become Hopf algebras with antipodes $S_{\mathcal{H}(L)}$ and $S_{\mathcal{H}(H)}$ which are given as the obvious factorizations of S'_F resp. S'_G.

It is clear that A becomes an $(\mathcal{H}(L), \mathcal{H}(H))$-bicomodule algebra via

$$\delta_{\mathcal{H}(L),A} : A \xrightarrow{\delta_{F,A}} F \otimes A \xrightarrow{\pi\otimes\mathrm{id}} \mathcal{H}(L) \otimes A$$

and

$$\delta_{A,\mathcal{H}(H)} : A \xrightarrow{\delta_{A,G}} A \otimes G \xrightarrow{\mathrm{id}\otimes\pi} A \otimes \mathcal{H}(H)\ ,$$

where we denote both residue class morphisms $F \longrightarrow \mathcal{H}(L)$ and $G \longrightarrow \mathcal{H}(H)$ as π. In the same way, B becomes an $(\mathcal{H}(H), \mathcal{H}(L))$-bicomodule algebra.

Proposition 3.2.6 *Let I_A be the two-sided ideal in A generated by the set*

$$\Big\{(\nabla_A(\mathrm{id}\otimes S'_B)\rho_F - \eta_A\varepsilon_F)(x_i)\ ,\ (\nabla_A(S'_B \otimes \mathrm{id})\rho_G - \eta_A\varepsilon_G)(y_i) \mid$$
$$\mid x_i \in \iota_i(F_i), y_i \in \iota_i(G_i), i \in \mathbb{N}_0\Big\}\ .$$

Then $\delta_{\mathcal{H}(L),A}(I_A) \subseteq \mathcal{H}(L) \otimes I_A$ and $\delta_{A,\mathcal{H}(H)}(I_A) \subseteq I_A \otimes \mathcal{H}(H)$, which means that I_A is a sub-bicomodule of A.

Proof. The following two diagrams commute by the definition of the morphisms S' above:

$$\begin{array}{ccc}
B_i & \xrightarrow{\iota_i} & B \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle S'_B} \\
A_{i+1}^{opcop} & \xrightarrow[\iota_{i+1}]{} & A^{opcop} \\
\downarrow{\scriptstyle \delta_{i+1}} & & \downarrow{\scriptstyle \delta_{F,A}} \\
F_{i+1}^{opcop} \otimes A_{i+1}^{opcop} & \xrightarrow[\iota_{i+1}\otimes\iota_{i+1}]{} & F^{opcop} \otimes A^{opcop}
\end{array}$$

$$\begin{array}{ccc}
B_i & \xrightarrow{\iota_i} & B \\
\downarrow{\scriptstyle \delta_i} & & \downarrow{\scriptstyle \delta_{B,F}} \\
B_i \otimes F_i & \xrightarrow[\iota_i]{} & B \otimes F \\
\downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{id}} & & \downarrow{\scriptstyle S_B\otimes S_F} \\
A_{i+1}^{opcop} \otimes F_{i+1}^{opcop} & \xrightarrow[\iota_{i+1}\otimes\iota_{i+1}]{} & A^{opcop} \otimes F^{opcop}
\end{array}$$

Since the maps on the left hand sides are equal up to a twist σ, it follows by the universal property of the coproduct B that

$$\delta_{F,A} \circ S'_B = \sigma(S'_B \otimes S'_F)\delta_{B,F} \ .$$

So we get

$$\begin{aligned}
&(\delta_{F,A} \otimes \delta_{F,A})(\mathrm{id} \otimes S'_B)\rho_F \circ \iota_i(x) = \\
&\quad = (\delta_{F,A} \otimes \sigma(S'_B \otimes S'_F)\delta_{B,F})(\iota_i \otimes \iota_i)\rho_{F_i}(x) = \\
&\quad = (\mathrm{id} \otimes \mathrm{id} \otimes \sigma(S'_B \otimes S'_F))(\delta_{F,A} \otimes \delta_{B,F})(\iota_i(x^{A_i}) \otimes \iota_i(x^{B_i})) = \\
&\quad = (\mathrm{id} \otimes \mathrm{id} \otimes \sigma(S'_B \otimes S'_F))(\iota_i(x^{A_i})_{(-1)} \otimes \iota_i(x^{A_i})_{(0)} \otimes \iota_i(x^{B_i})_{(0)} \otimes \iota_i(x^{B_i})_{(1)}) = \\
&\quad = (\mathrm{id} \otimes \mathrm{id} \otimes \sigma(S'_B \otimes S'_F))(\iota_i({x^{A_i}}_{(-1)}) \otimes \iota_i({x^{A_i}}_{(0)}) \otimes \iota_i({x^{B_i}}_{(0)}) \otimes \iota_i({x^{B_i}}_{(1)})) = \\
&\quad = (\mathrm{id} \otimes \mathrm{id} \otimes \sigma(S'_B \otimes S'_F))(\iota_i(x_{(1)}) \otimes \iota_i({x_{(2)}}^{A_i}) \otimes \iota_i({x_{(2)}}^{B_i}) \otimes \iota_i(x_{(3)}))
\end{aligned}$$

for $x \in F_i$, applying the equation (3.1) for $h \in L$

$${h^T}_{(-1)} \otimes {h^T}_{(0)} \otimes {h^Z}_{(0)} \otimes {h^Z}_{(1)} = h_{(1)} \otimes {h_{(2)}}^T \otimes {h_{(2)}}^Z \otimes h_{(3)} \ .$$

This yields for $x \in F_i$

$$\begin{aligned}
&\delta_{F,A}(\nabla_A(\mathrm{id}\otimes S'_B)\rho_F)\iota_i(x) = \\
&\quad = (\nabla_F \otimes \nabla_A)(\mathrm{id}\otimes\sigma\otimes\mathrm{id})(\delta_{F,A}\otimes\delta_{F,A})(\mathrm{id}\otimes S'_B)\rho_F\circ\iota_i(x) = \\
&\quad = (\nabla_F \otimes \nabla_A)(\mathrm{id}\otimes\sigma\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes\sigma(S'_B\otimes S'_F)) \\
&\qquad (\iota_i(x_{(1)})\otimes\iota_i({x_{(2)}}^{A_i})\otimes\iota_i({x_{(2)}}^{B_i})\otimes\iota_i(x_{(3)})) = \\
&\quad = \iota_i(x_{(1)})S'_F(\iota_i(x_{(3)}))\otimes\iota_i({x_{(2)}}^{A_i})S'_B(\iota_i({x_{(2)}}^{B_i})) = \\
&\quad = \iota_i(x_{(1)})S'_F\iota_i(x_{(3)})\otimes(\nabla_A(\mathrm{id}\otimes S'_B)\rho_F)\iota_i(x_{(2)}) \ ,
\end{aligned}$$

which implies

$$\begin{aligned}
&\delta_{F,A}(\nabla_A(\mathrm{id}\otimes S'_B)\rho_F - \eta_A\varepsilon_F)\iota_i(x) = \\
&\quad = \iota_i(x_{(1)})S'_F\iota_i(x_{(3)})\otimes(\nabla_A(\mathrm{id}\otimes S'_B)\rho_F)\iota_i(x_{(2)}) - \delta_{F,A}\eta_A\varepsilon_F\iota_i(x) = \\
&\quad = \iota_i(x_{(1)})S'_F\iota_i(x_{(3)})\otimes(\nabla_A(\mathrm{id}\otimes S'_B)\rho_F - \eta_A\varepsilon_F)\iota_i(x_{(2)}) + \\
&\qquad + \iota_i(x_{(1)})S'_F\iota_i(x_{(3)})\otimes\eta_A\varepsilon_F\iota_i(x_{(2)}) - \delta_{F,A}\eta_A\varepsilon_F\iota_i(x) = \\
&\quad = \iota_i(x_{(1)})S'_F\iota_i(x_{(3)})\otimes(\nabla_A(\mathrm{id}\otimes S'_B)\rho_F - \eta_A\varepsilon_F)\iota_i(x_{(2)}) + \\
&\qquad + \iota_i(x_{(1)})S'_F\iota_i(x_{(2)})\otimes\eta_A(1) - \eta_F\varepsilon_F\iota_i(x)\otimes\eta_A(1) = \\
&\quad = \iota_i(x_{(1)})S'_F\iota_i(x_{(3)})\otimes(\nabla_A(\mathrm{id}\otimes S'_B)\rho_F - \eta_A\varepsilon_F)\iota_i(x_{(2)}) + \\
&\qquad + (\mathrm{id}*S'_F - \eta_F\varepsilon_F)\iota_i(x)\otimes\eta_A(1) \\
&\qquad \subseteq F\otimes I_A + I_F\otimes A \ .
\end{aligned}$$

So we obtain that

$$\delta_{\mathcal{H}(L),A}(\nabla_A(\mathrm{id}\otimes S'_B)\rho_F - \eta_A\varepsilon_F)\iota_i(x) \subseteq \mathcal{H}(L)\otimes I_A \ .$$

To prove that $\delta_{\mathcal{H}(L),A}$ maps also elements of the form $(\nabla_A(S'_B\otimes\mathrm{id})\rho_G - \eta_A\varepsilon_G)(y_i)$ to $\mathcal{H}(L)\otimes I_A$, we first calculate for $y\in G_i$

$$\begin{aligned}
&(\delta_{F,A}\otimes\delta_{F,A})(S'_B\otimes\mathrm{id})\rho_G\circ\iota_i(y) = \\
&\quad = (\sigma(S'_B\otimes S'_F)\otimes\mathrm{id}\otimes\mathrm{id})(\delta_{B,F}\otimes\delta_{F,A})\rho_{G_i}(y) = \\
&\quad = (\sigma(S'_B\otimes S'_F)\otimes\mathrm{id}\otimes\mathrm{id})(\iota_i(y^{B_i})_{(0)}\otimes\iota_i(y^{B_i})_{(1)}\otimes\iota_i(y^{A_i})_{(-1)}\otimes\iota_i(y^{A_i})_{(0)}) = \\
&\quad = (\sigma(S'_B\otimes S'_F)\otimes\mathrm{id}\otimes\mathrm{id})(\iota_i(y^{B_i})_{(0)}\otimes\iota_i(y^{B_i})_{(1)}\otimes\iota_i(y^{B_i})_{(2)}\otimes\iota_i(y^{A_i})) \ ,
\end{aligned}$$

where the last equality is deduced from (3.8) and hence requires the additional property $(B8)$ of the bicomodule-algebra system (H, L, T, Z). With this formula we get

$$
\begin{aligned}
&\delta_{F,A}(\nabla_A(S'_B \otimes \mathrm{id})\rho_G)\iota_i(y) = \\
&\quad = (\nabla_F \otimes \nabla_A)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\delta_{F,A} \otimes \delta_{F,A})(S'_B \otimes \mathrm{id})\rho_G \iota_i(y) = \\
&\quad = (\nabla_F \otimes \nabla_A)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\sigma(S'_B \otimes S'_F) \otimes \mathrm{id} \otimes \mathrm{id}) \\
&\qquad (\iota_i(y^{B_i})_{(0)} \otimes \iota_i(y^{B_i})_{(1)} \otimes \iota_i(y^{B_i})_{(2)} \otimes \iota_i(y^{A_i})) = \\
&\quad = S'_F(\iota_i(y^{B_i})_{(1)})\iota_i(y^{B_i})_{(2)} \otimes S'_B(\iota_i(y^{B_i})_{(0)})\iota_i(y^{A_i}) = \\
&\quad = (S'_F * \mathrm{id})(\iota_i(y^{B_i})_{(1)}) \otimes S'_B(\iota_i(y^{B_i})_{(0)})\iota_i(y^{A_i}) \ .
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
&\delta_{F,A}(\nabla_A(S'_B \otimes \mathrm{id})\rho_G - \eta_A\varepsilon_G)\iota_i(y) = \\
&\quad = (S'_F * \mathrm{id})(\iota_i(y^{B_i})_{(1)}) \otimes S'_B(\iota_i(y^{B_i})_{(0)})\iota_i(y^{A_i}) - \delta_{F,A}(\eta_A\varepsilon_G\iota_i(y)) = \\
&\quad = (S'_F * \mathrm{id} - \eta_F\varepsilon_F)(\iota_i(y^{B_i})_{(1)}) \otimes S'_B(\iota_i(y^{B_i})_{(0)})\iota_i(y^{A_i}) + \\
&\qquad + \eta_F\varepsilon_F\iota_i(y^{B_i})_{(1)} \otimes S'_B(\iota_i(y^{B_i})_{(0)})\iota_i(y^{A_i}) - \delta_{F,A}(\eta_A\varepsilon_G\iota_i(y)) = \\
&\quad = (S'_F * \mathrm{id} - \eta_F\varepsilon_F)(\iota_i(y^{B_i})_{(1)}) \otimes S'_B(\iota_i(y^{B_i})_{(0)})\iota_i(y^{A_i}) + \\
&\qquad + \eta_F(1) \otimes S'_B(\iota_i(y^{B_i}))\iota_i(y^{A_i}) - \eta_F(1) \otimes \eta_A\varepsilon_G\iota_i(y) = \\
&\quad = (S'_F * \mathrm{id} - \eta_F\varepsilon_F)(\iota_i(y^{B_i})_{(1)}) \otimes S'_B(\iota_i(y^{B_i})_{(0)})\iota_i(y^{A_i}) + \\
&\qquad + \eta_F(1) \otimes (\nabla_A(S'_B \otimes \mathrm{id})\rho_G - \eta_A\varepsilon_G)\iota_i(y) \\
&\qquad \subseteq I_F \otimes A + F \otimes I_A \ ,
\end{aligned}
$$

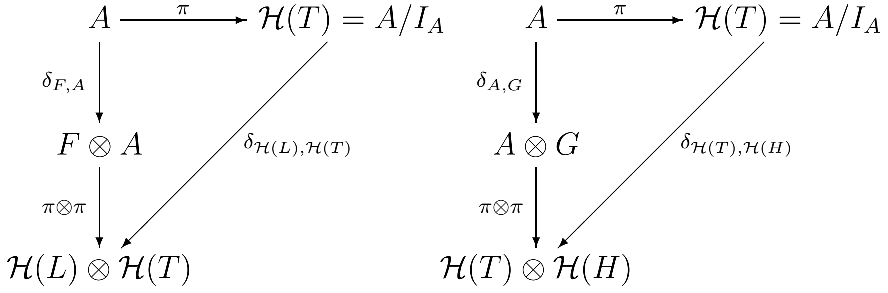
and thus

$$\delta_{\mathcal{H}(L),A}(\nabla_A(S'_B \otimes \mathrm{id})\rho_G - \eta_A\varepsilon_G)\iota_i(y) \subseteq \mathcal{H}(L) \otimes I_A \ .$$

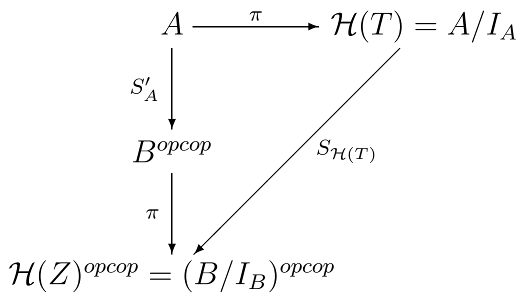
The proof for the right comodule structure $\delta_{A,\mathcal{H}(H)}$ is similar. □

Of course, the statement in Proposition 3.2.6 holds also for the analogously defined biideal I_B of B. For this we need the property $(B7)$.

Now we can define the quotient algebras $\mathcal{H}(T) := A/I_A$ and $\mathcal{H}(Z) := B/I_B$, and because of $I_A \subseteq \mathrm{Ke}((\pi\otimes\pi)\delta_{F,A})$ and $I_A \subseteq \mathrm{Ke}((\pi\otimes\pi)\delta_{A,G})$, which we just proved, we get the following bicomodule algebra structure maps for $\mathcal{H}(T)$,



and for $\mathcal{H}(Z)$ respectively.
It is clear that $(\mathcal{H}(L), \mathcal{H}(H), \mathcal{H}(T), \mathcal{H}(Z))$ is a bicomodule algebra system.
We show that S'_A can be factored over $\mathcal{H}(T)$ as



i.e. that $I_A \subseteq \mathrm{Ke}(\pi S'_A)$. Since $\mathrm{Ke}(\pi) = I_B$, we just have to show that $S'_A(I_A) \subseteq I_B$. Let $x \in F_i$. Then we have

$$\begin{aligned} S'_A(\nabla_A(\mathrm{id} \otimes S'_B)\rho_F)\iota_i(x) &= \nabla_B \sigma(S'_A \otimes S'_A S'_B)(\iota_i \otimes \iota_i)\rho_{F_i}(x) \\ &= \nabla_B \sigma(\mathrm{id} \otimes S'_A)(\iota_{i+1} \otimes \iota_{i+1})\rho_{F_i}(x) \\ &= \nabla_B (S'_A \otimes \mathrm{id})(\iota_{i+1} \otimes \iota_{i+1})\sigma\rho_{F_i}(x) \\ &= \nabla_B (S'_A \otimes \mathrm{id})(\iota_{i+1} \otimes \iota_{i+1})\rho_{F_{i+1}}(x) \\ &= \nabla_B (S'_A \otimes \mathrm{id})\rho_F \iota_{i+1}(x) \end{aligned}$$

and

$$S'_A(\eta_A \varepsilon_F \iota_i(x)) = S'_A(1)\varepsilon_i(x) = \eta_B(1)\varepsilon_{i+1}(x) = \eta_B \varepsilon_F \iota_{i+1}(x) \ ,$$

which implies

$$S'_A(\nabla_A(\mathrm{id} \otimes S'_B)\rho_F - \eta_A\varepsilon_F)\iota_i(x) = (\nabla_B(S'_A \otimes \mathrm{id})\rho_F - \eta_B\varepsilon_F)\iota_{i+1}(x) \subseteq I_B \ .$$

Together with similar computations we obtain the desired factorization.

Proposition 3.2.7 *The morphism $S_{\mathcal{H}(T)}$ and the analogously constructed morphism $S_{\mathcal{H}(Z)} : \mathcal{H}(Z) \longrightarrow \mathcal{H}(T)$ are generalized antipodes for the bicomodule-algebra system $(\mathcal{H}(L), \mathcal{H}(H), \mathcal{H}(T), \mathcal{H}(Z))$.*

Proof. By a generalization of [23], Proposition III.3.6, it obviously suffices to test the antipode property on algebra generators. For instance, we get for $x \in F_i$

$$\begin{aligned} \nabla_{\mathcal{H}(T)}(\mathrm{id} \otimes S_{\mathcal{H}(Z)})\rho_{\mathcal{H}(L)}\pi\iota_i(x) &= \nabla_{\mathcal{H}(T)}(\pi \otimes S_{\mathcal{H}(Z)}\pi)\rho_F\iota_i(x) \\ &= \nabla_{\mathcal{H}(T)}(\pi \otimes \pi)(\mathrm{id} \otimes S'_B)\rho_F\iota_i(x) \\ &= \pi\nabla_A(\mathrm{id} \otimes S'_B)\rho_F\iota_i(x) \\ &= \pi\eta_A\varepsilon_F\iota_i(x) \\ &= \eta_{\mathcal{H}(T)}\varepsilon_{\mathcal{H}(L)}\pi\iota_i(x) \ , \end{aligned}$$

and the other equalities can be shown similarly. □

So far, we have constructed a total Hopf-Galois system

$$(\mathcal{H}(L), \mathcal{H}(H), \mathcal{H}(T), \mathcal{H}(Z))$$

such that setting $(\iota_F, \iota_G, \iota_A, \iota_B) := (\pi\iota_0, \pi\iota_0, \pi\iota_0, \pi\iota_0)$, we obtain a morphism of bicomodule-algebra systems

$$(\iota_F, \iota_G, \iota_A, \iota_B) : (L, H, T, Z) \longrightarrow (\mathcal{H}(L), \mathcal{H}(H), \mathcal{H}(T), \mathcal{H}(Z)) \; .$$

It remains to be shown that the universal property is satisfied.

So let (U, V, X, Y) be a total Hopf-Galois system and let

$$(\ell, h, f, g) : (L, H, T, Z) \longrightarrow (U, V, X, Y)$$

be a morphism of bicomodule-algebra systems. We define recursively the following families of bialgebra resp. algebra morphisms for all $i \in \mathbb{N}$:

$$\begin{aligned} \ell_0 &:= \ell \qquad & \ell_{i+1} &:= S_U \ell_i \\ h_0 &:= h \qquad & h_{i+1} &:= S_V h_i \end{aligned}$$

$$f_0 : A_0 = T \xrightarrow{f} X \qquad\qquad g_0 : B_0 = Z \xrightarrow{g} Y$$

$$f_1 : A_1 = Z^{opcop} \xrightarrow{g} Y^{opcop} \xrightarrow{S_Y} X$$

$$g_1 : B_1 = T^{opcop} \xrightarrow{f} X^{opcop} \xrightarrow{S_X} Y$$

$$f_{i+1} := S_Y g_i \qquad\qquad g_{i+1} := S_X f_i$$

This means in fact that we have morphisms of bicomodule-algebra systems $(\ell_i, h_i, f_i, g_i) : (F_i, G_i, A_i, B_i) \longrightarrow (U, V, X, Y)$ for each $i \in \mathbb{N}_0$. The reason for this is that (S_U, S_V, S_X, S_Y) is a morphism of Hopf-Galois systems by Proposition 3.1.8 and thus each (ℓ_i, h_i, f_i, g_i) is a composition of bicomodule-algebra morphisms.

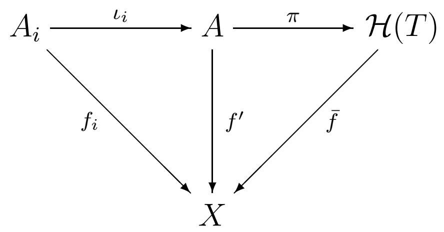
By the universal property of the coproduct, the above families define bialgebra morphisms $\ell' : F \longrightarrow U$ with $\ell'\iota_i = \ell_i$, $h' : G \longrightarrow V$ with $h'\iota_i = h_i$ and algebra morphisms $f' : A \longrightarrow X$ with $f'\iota_i = f_i$, $g' : B \longrightarrow Y$ with $g'\iota_i = g_i$ for all $i \in \mathbb{N}_0$.

Now we see that $I_A \subseteq \mathrm{Ke}(f')$, since we get for $x \in F_i$

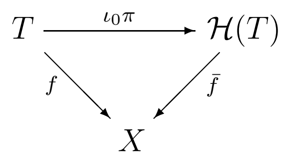
$$\begin{aligned} f'(\nabla_A(\mathrm{id} \otimes S'_B)\rho_F)\iota_i(x) &= f'(\nabla_A(\mathrm{id} \otimes S'_B)(\iota_i \otimes \iota_i)\rho_{F_i}(x)) \\ &= f'\nabla_A(\iota_i \otimes S'_B\iota_i)\rho_{F_i}(x) \\ &= f'\nabla_A(\iota_i \otimes \iota_{i+1})\rho_{F_i}(x) \\ &= \nabla_X(f'\iota_i \otimes f'\iota_{i+1})\rho_{F_i}(x) \\ &= \nabla_X(f_i \otimes f_{i+1})\rho_{F_i}(x) \\ &= \nabla_X(f_i \otimes S_Y g_i)\rho_{F_i}(x) \\ &= \nabla_X(\mathrm{id} \otimes S_Y)(f_i \otimes g_i)\rho_{F_i}(x) \\ &= \nabla_X(\mathrm{id} \otimes S_Y)\rho_U \ell_i(x) \\ &= \eta_X \varepsilon_U \ell_i(x) \\ &= \eta_X \varepsilon_{L_i}(x) \\ &= f'(\eta_A \varepsilon_L \iota_i(x)) \end{aligned}$$

using the properties of a morphism of bicomodule algebra systems. By a similar calculation we have the same result for the other part of I_A .

Hence, f' factors over $\mathcal{H}(T)$, i.e. there exists a unique algebra morphism $\bar{f} : \mathcal{H}(T) \longrightarrow X$ such that the diagram



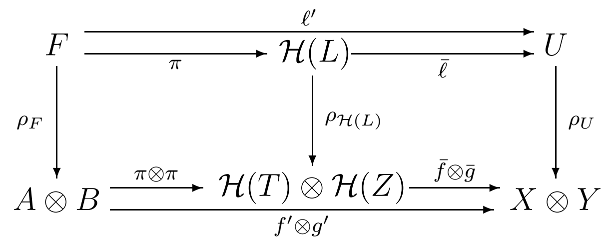
commutes for for all $i \in \mathbb{N}_0$, and so in particular the diagram



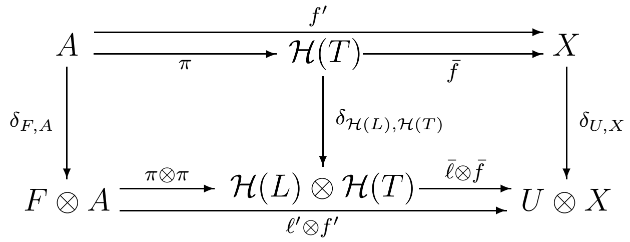
commutes. As before, we get the corresponding results for the other three morphisms, such that there exists a map

$$(\bar{\ell}, \bar{h}, \bar{f}, \bar{g}) : (\mathcal{H}(L), \mathcal{H}(H), \mathcal{H}(T), \mathcal{H}(Z)) \longrightarrow (U, V, X, Z) \ ,$$

that makes the four diagrams in the statement of the proposition commute. We still have to verify that $(\bar{\ell}, \bar{h}, \bar{f}, \bar{g})$ is in fact a morphism of Hopf-Galois systems. For the axiom (M1) we consider the diagram



that commutes with the possible exception of the right hand square. But π is surjective and so the last square must commute, too. The axiom (M3) can be deduced in the same way from



as well as the other diagrams.

Finally, the uniqueness of the morphism $(\bar{\ell}, \bar{h}, \bar{f}, \bar{g})$ follows from the fact, that $\bar{\ell}$, $\bar{h}$, $\bar{f}$ and $\bar{g}$ are already uniquely determined as morphisms of algebras by the universal property of the coproduct in $\mathbb{K}$-Alg. □

Chapter 4

Bialgebroids and Hopf Algebroids for B-Torsors

4.1 Bialgebroids and $\times_A$-Bialgebras

The following is a well-known fact in the theory of Hopf algebras: The category of modules over a k-algebra H is monoidal with the tensor product of the underlying category of k-modules, if and only if H has a bialgebra structure. This means that the forgetful functor ${}_H\mathcal{M} \longrightarrow {}_k\mathcal{M}$ is monoidal.

Let A be a noncommutative ring. For two A-bimodules $M, N \in {}_A\mathcal{M}_A$, one can consider the tensor product $M \otimes_A N$ over A. Then, as was shown in [39], an algebra whose modules form a monoidal category with the tensor product $\otimes_A$ has the structure of a $\times_A$-bialgebra. The definition of a $\times_A$-bialgebra, and that of the equivalent notion of A-bialgebroid, is based on a new tensor product "$\times_A$", and generalizes the notion of a bialgebra over a commutative ring k.

The notion of $\times_A$-bialgebra was introduced by Sweedler [50], and later generalized by Takeuchi [52]. Motivated by studies of Poisson groupoids, the same objects were introduced again under the name of bialgebroids by Lu in [26]. There exists a side-reversed version to these bialgebroids, called right bialgebroid in [22].

In [26] Lu also defined an antipode for bialgebroids, which was supposed to generalize antipodes for Hopf algebras. This definition requires the existence resp. choice of a section for the canonical projection of the tensor product over k to the tensor product over A. To avoid this quite technical and unnatural approach, several other authors have given other definitions of Hopf algebroids resp. $\times_A$-Hopf algebras based on categorical properties. One of them is Schauenburg's concept of $\times_A$-Hopf algebra [43].

We are going to review the definitions of $\times_A$-bialgebra and A-bialgebroid. Both left and right bialgebroids arise naturally with the generalized versions of quantum torsors that we are going study in the next section. We give a definition of $\times^A$-bialgebras. These are right bialgebroids in the terminology of $\times_A$-bialgebras. This translation of axioms allows us then to give a definition of $\times^A$-Hopf algebras based on categorical properties analogous to those of Schauenburg's $\times_A$-Hopf algebras in [43]. We show that $\times_A$-bialgebras that admit a Hopf-Galois extension are $\times_A$-Hopf algebras.

We follow Takeuchi's and Schauenburg's descriptions of $\times_A$-bialgebras in [52] resp. [43]:

Let A be a k-algebra. We denote the opposite algebra by $\bar{A} := A^{op}$ and let $A \ni a \mapsto \bar{a} \in \bar{A}$ be the obvious anti-isomorphism of k-algebras. We abbreviate $A^e := A \otimes \bar{A}$ for the enveloping algebra, where A and $\bar{A}$ are considered as subalgebras in the obvious way.
Let $M, N \in {}_{A^e}\mathcal{M}_{A^e}$. Using a notation due to MacLane, we let

$$\int_a {}_{\bar{a}}M \otimes {}_aN$$

be the quotient submodule

$$M \otimes N / < \bar{a}m \otimes n - m \otimes an \mid a \in A, m \in M, n \in N >,$$

and

$$\int^a M_{\bar{a}} \otimes N_a$$

be the submodule of $M \otimes N$ consisting of

$$\{\sum m_i \otimes n_i \subset M \otimes N \mid \sum m_i\bar{a} \otimes n_i = \sum m_i \otimes n_i a \ \forall\ a \in A\} \ .$$

For two left A^e-modules M, N we set

$$M \diamond N := \int_a {}_{\bar{a}}M \otimes {}_aN \ .$$

This is again a left A^e-module with the left $\bar{A}$-module structure induced by the structure of N, and the left A-module structure induced by the structure of M. It is easy to see that the category $({}_{A^e}\mathcal{M}, \diamond)$ is monoidal and naturally isomorphic to the category $({}_A\mathcal{M}_A, \otimes_A)$.

For two A^e-bimodules M, N we set

$$M \times_A N := \int^b \int_a {}_{\bar{a}}M_{\bar{b}} \otimes {}_aN_b \ ,$$

which is the submodule of $M \diamond N$ consisting of

$$\{\sum m_i \otimes n_i \subset M \diamond N \mid \sum m_i\bar{a} \otimes n_i = \sum m_i \otimes n_i a \ \forall \ a \in A\} \ .$$

It is again an A^e-bimodule with A acting on the left and $\bar{A}$ acting on the right tensor factor as $(a \otimes \bar{b})(m \otimes n)(c \otimes \bar{d}) := amc \otimes \bar{b}n\bar{d}$ for $a, b, c, d \in A$ and $m \otimes n \in M \times_A N$. Thus, $\times_A$ defines a product on the category ${}_{A^e}\mathcal{M}_{A^e}$, which, however, must neither be associative nor unitary.

A triple $\times_A$-product for $M, N, P \in {}_{A^e}\mathcal{M}_{A^e}$ is defined as

$$M \times_A P \times_A N := \int^{c,d} \int_{a,b} {}_{\bar{a}}M_{\bar{c}} \otimes {}_{a,\bar{b}}P_{c,\bar{d}} \otimes {}_bN_d \ .$$

There exist associativity maps

$$\alpha : (M \times_A N) \times_A P \longrightarrow M \times_A N \times_A P$$
$$\alpha' : M \times_A (N \times_A P) \longrightarrow M \times_A N \times_A P$$

that are both given by $m \otimes n \otimes p \mapsto m \otimes n \otimes p$ on elements. They are A^e-bimodule maps, but need not be isomorphisms.

The endomorphism ring $\mathrm{End}(A)$ inherits the A^e-bimodule structure from A. For every $M \in {}_{A^e}\mathcal{M}_{A^e}$, the following are A^e-bimodule maps:

$$\vartheta : M \times_A \mathrm{End}(A) \longrightarrow M \ , \qquad m \otimes f \mapsto \overline{f(1)}m$$
$$\vartheta' : \mathrm{End}(A) \times_A M \longrightarrow M \ , \qquad f \otimes m \mapsto f(1)m$$

Now we are ready to introduce the notion of $\times_A$-coalgebra.

Definition 4.1.1 A $\times_A$-*coalgebra* is an A^e-bimodule L together with a comultiplication $\Delta : L \longrightarrow L \times_A L$ and a counit $\varepsilon : L \longrightarrow \mathrm{End}(A)$, both of which are A^e-bimodule maps, such that the following diagrams commute:

$$\begin{array}{ccc} L & \xrightarrow{\quad\Delta\quad} & L \times_A L \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle L\times_A \Delta} \\ L \times_A L & & L \times_A (L \times_A L) \\ \downarrow{\scriptstyle \Delta\times_A L} & & \downarrow{\scriptstyle \alpha'} \\ (L \times_A L) \times_A L & \xrightarrow[\quad\alpha\quad]{} & L \times_A L \times_A L \end{array}$$

$$\begin{array}{ccc} L & \xrightarrow{\quad\Delta\quad} & L \times_A L \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle L\times_A \varepsilon} \\ L & \xleftarrow[\quad\vartheta\quad]{} & L \times_A \mathrm{End}(A) \end{array} \qquad \begin{array}{ccc} L & \xrightarrow{\quad\Delta\quad} & L \times_A L \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varepsilon\times_A L} \\ L & \xleftarrow[\quad\vartheta'\quad]{} & \mathrm{End}(A) \times_A L \end{array}$$

We use generalized Sweedler notation to denote the comultiplication map of a $\times_A$-bialgebra L by $\Delta(\ell) =: \ell_{(1)} \otimes \ell_{(2)} \in L \times_A L$. Note that one has to be very careful when applying the coassociativity axiom on this notation, since both α and α' need not be isomorphisms (see [43] for details).

An A^e-ring (or an algebra over A^e) is defined as an algebra L equipped with an algebra map $i_L : A^e \longrightarrow L$. A map of A^e-rings is an algebra map that commutes with the respective maps from A^e.
An A^e-ring L becomes an A^e-bimodule in a natural way, namely by left and right multiplication with the image of i_L. For two A^e-rings L and K, $L \times_A K$ is an algebra with componentwise multiplication and an A^e-ring via $i_{L\times_A K} : A \otimes \bar{A} \longrightarrow L \times_A K \ , \ a \otimes \bar{b} \mapsto i_L(a) \otimes i_K(\bar{b})$.

Definition 4.1.2 A $\times_A$-*bialgebra* L is an A^e-ring that has the structure of a $\times_A$-coalgebra such that the comultiplication Δ and the counit ε are maps of A^e-rings.

The category of left modules over a $\times_A$-bialgebra L is a monoidal category: The tensor product $M \diamond N$ of two left L-modules $M, N \in {}_L\mathcal{M}$ becomes a left L-module with the diagonal module structure

$$\ell \triangleright (m \otimes n) = \ell_{(1)} \triangleright m \otimes \ell_{(2)} \triangleright n \ ,$$

and the unit object A^e becomes an L-module via the A^e-bimodule map $\varepsilon : L \longrightarrow \mathrm{End}(A)$. This means that the underlying functor ${}_L\mathcal{M} \longrightarrow {}_{A^e}\mathcal{M}$ is monoidal.

Remark 4.1.3 Let L be a $\times_A$-bialgebra with the A^e-ring structure $i_L : A^e = A \otimes \bar{A} \longrightarrow L$. Then L can be naturally considered as an $\bar{A}^e = \bar{A} \otimes A$-ring via the algebra morphism $j_L : \bar{A} \otimes A \longrightarrow L$, $j_L(\bar{a} \otimes b) := i_L(b \otimes \bar{a})$. The resulting A-resp. $\bar{A}$-module structures on L are such that they correspond to the original $\bar{A}$-resp. A-module structures given by i_L. This means that we have

$$L \times_{\bar{A}} L = \int^b \int_a {}_aL_b \otimes {}_{\bar{a}}L_{\bar{b}} \ .$$

Given two L-modules $M, N \in {}_L\mathcal{M}$, the symmetry $M \otimes N \longrightarrow N \otimes M$ in $\mathcal{M}_k$ induces an invertible map

$$\tau_{M,N} : M \diamond N \longrightarrow \int_a {}_aN \otimes {}_{\bar{a}}M \ .$$

Hence, composing the comultiplication map $\Delta : L \longrightarrow L \times_A L \subset L \diamond L$, $\ell \mapsto \ell_{(1)} \otimes \ell_{(2)}$ with $\tau_{L,L}$ is well-defined and yields a map

$$\Delta^{cop} := \tau_{L,L} \circ \Delta : L \longrightarrow \int^b \int_a {}_aL_b \otimes {}_{\bar{a}}L_{\bar{b}} = L \times_{\bar{A}} L \ .$$

We denote by $\varepsilon^{cop} : L \longrightarrow \mathrm{End}(\bar{A})$ the obvious $\bar{A}^e$-ring map induced by the counit $\varepsilon : L \longrightarrow \mathrm{End}(A)$. Now it is straightforward to see that the $\bar{A}^e$-ring L together with the maps Δ^{cop} and ε^{cop} satisfies the axioms of a $\times_{\bar{A}}$-bialgebra. We call this $\times_{\bar{A}}$-bialgebra the coopposite bialgebra of L, and denote it by L^{cop}.

The category ${}_{L^{cop}}\mathcal{M}$ of left L^{cop}-modules is monoidal with the tensor product of the underlying category ${}_{\bar{A}}\mathcal{M}_{\bar{A}}$ given by

$$M \bar{\diamond} N := \int_a {}_aM \otimes {}_{\bar{a}}N$$

for $M, N \in {}_{L^{cop}}\mathcal{M}$. The L^{cop}-module structure on $M \bar{\diamond} N$ is given by $\ell \triangleright (m \otimes n) = \ell_{(2)} \triangleright m \otimes \ell_{(1)} \triangleright n$.

Hopf algebras over a commutative ring k can be characterized through the following property: A k-bialgebra H is a Hopf algebra if and only if the category of its finitely generated projective comodules is rigid [56]. Schauenburg has given a definition of $\times_A$-Hopf algebra in [43] that demands $\times_A$-Hopf algebras to do something similar for their categories of modules. It is shown in [43] that a $\times_A$-bialgebra L satisfies the property in the following definition if and only if the underlying functor ${}_L\mathcal{M} \longrightarrow {}_{A^e}\mathcal{M}$ preserves right inner hom-functors.

Definition 4.1.4 A $\times_A$-Hopf algebra L is a $\times_A$-bialgebra such that the map

$$\beta : L \otimes_{\bar{A}} L \longrightarrow L \diamond L \ , \ \ell \otimes m \mapsto \ell_{(1)} \otimes \ell_{(2)} m$$

is a bijection.

We now state the definition of Lu's bialgebroid [26], which we shall call left bialgebroid. Actually, the axioms in Lu's original definition look somewhat different but were proved to be equivalent to those in the following definition by Xu in [59], see also [7].

Definition 4.1.5 A *left bialgebroid* H over R consists of:

- k-algebras H and R, called the *total* resp. *base ring*
- algebra morphisms $s : R \longrightarrow H$, $t : R^{op} \longrightarrow H$, called the *source* resp. *target map* such that
$$s(r)t(r') = t(r')s(r)$$
for all $r, r' \in R$, giving rise to an (R, R)-bimodule structure on H by
$$r \cdot x \cdot r' := s(r)t(r')x$$
- (R, R)-bimodule maps $\Delta : H \longrightarrow H \otimes_R H$ and $\varepsilon : H \longrightarrow R$ such that (H, Δ, ε) is an R-coring, that is a comonoid in ${}_R\mathcal{M}_R$.
- the following identities for $r \in R$ and $x, y \in H$:
  1) $\Delta(x)(1 \otimes s(r)) = \Delta(x)(t(r) \otimes 1)$
  2) $\Delta(xy) = \Delta(x)\Delta(y)$
  3) $\Delta(1) = 1 \otimes 1$
  4) $\varepsilon(xs(\varepsilon(y))) = \varepsilon(xy) = \varepsilon(xt(\varepsilon(y)))$
  5) $\varepsilon(1) = 1_R$

We note that $H \otimes_R H$ does not necessarily possess a well-defined algebra structure. Thus, the properties 1) and 2) can not be expressed by demanding that Δ be an algebra map.

It is shown in [7] that the notions of $\times_R$-bialgebra and left R-bialgebroid are equivalent. In the above definition, one can define an alternative bimodule structure on H by multiplying with the source and target maps on the right. This leads to the notion of right bialgebroid as given in [22]:

Definition 4.1.6 A *right bialgebroid* H over R consists of:

- k-algebras H and R, called the *total* resp. *base ring*
- algebra morphisms $s : R \longrightarrow H$, $t : R^{op} \longrightarrow H$, called the *source* resp. *target map* such that
$$s(r)t(r') = t(r')s(r)$$
for all $r, r' \in R$, giving rise to an (R, R)-bimodule structure on H by
$$r \cdot x \cdot r' := xs(r')t(r)$$
- (R, R)-bimodule maps $\Delta : H \longrightarrow H \otimes_R H$ and $\varepsilon : H \longrightarrow R$ such that (H, Δ, ε) is an R-coring.
- the following identities for $r \in R$ and $x, y \in H$:
  1) $(s(r) \otimes 1)\Delta(x) = (1 \otimes t(r))\Delta(x)$
  2) $\Delta(xy) = \Delta(x)\Delta(y)$
  3) $\Delta(1) = 1 \otimes 1$
  4) $\varepsilon(s(\varepsilon(x))y) = \varepsilon(xy) = \varepsilon(t(\varepsilon(x))y)$
  5) $\varepsilon(1) = 1_R$

Remark 4.1.7 Given a left bialgebroid $(H, R, s, t, \Delta, \varepsilon)$ with total ring H, the opposite algebra H^{op} becomes a right bialgebroid over R with the structure maps $s^{op} := t : R \longrightarrow H^{op}$, $t^{op} := s : R^{op} \longrightarrow H^{op}$ and $\Delta^{op} := \Delta$, $\varepsilon^{op} := \varepsilon$. This relation between left and right bialgebroids can be understood by passing from left H-modules to right H^{op}-modules and requiring that they form a monoidal category with the underlying functor to the category of R-bimodules.

Definition 4.1.8 Let H and K be two bialgebroids over the base ring R. An algebra morphism $f : H \longrightarrow K$ is called a *morphism of bialgebroids* if it is an R-bimodule morphism and compatible with both comonoid structures such that
$$\Delta_K \circ f = (f \otimes_R f) \circ \Delta_H \quad , \quad \varepsilon_K \circ f = \varepsilon_H$$

Lu has given the following definition of Hopf algebroid in [26]. It is not equivalent to Schauenburg's definition of a $\times_A$-Hopf algebra.

Definition 4.1.9 A left bialgebroid H over R is called a *Hopf algebroid* if there exists an algebra anti-automorphism $\varsigma : H \longrightarrow H$, called the *antipode*, such that

- $\varsigma \circ t = s$
- $\nabla(\varsigma \otimes \mathrm{id})\Delta = t \circ \varepsilon \circ \varsigma$
- there exists a linear section $\gamma : H \otimes_R H \longrightarrow H \otimes H$ for the natural projection such that $\nabla(\mathrm{id} \otimes \varsigma)\gamma\Delta = s \circ \varepsilon$.

Note that writing down the second property is well-defined because of the first one. The section γ is really needed for the third property, since an expression like $\nabla(\mathrm{id} \otimes \varsigma)\Delta$ might not be well-defined.

Some interesting examples of Hopf algebroids are discussed in [26], along with a construction method for Hopf algebroids that we will study later on. The following example from [26] is called the trivial Hopf algebroid:

Example 4.1.10 Let A be a k-algebra. Then $H := A \otimes A^{op}$ is a left bialgebroid over A with the source map $s : A \longrightarrow A \otimes A^{op}\ ,\ a \mapsto a \otimes 1$ and the target map $t : A \longrightarrow A \otimes A^{op}\ ,\ a \mapsto 1 \otimes a$. The A-ring structure on H is given by $\Delta : A \otimes A^{op} \longrightarrow A \otimes A^{op} \otimes_A A \otimes A^{op}\ ,\ a \otimes b \mapsto a \otimes 1 \otimes 1 \otimes b$ and $\varepsilon : A \otimes A^{op} \longrightarrow A\ ,\ a \otimes b \mapsto ab$.
The bialgebroid H becomes a Hopf algebroid with the antipode $\varsigma : H \longrightarrow H\ ,\ a \otimes b \mapsto b \otimes a$, where the section $\gamma : H \otimes_A H \cong A \otimes A^{op} \otimes A^{op} \longrightarrow H \otimes H = A \otimes A^{op} \otimes A \otimes A^{op}$ can be chosen as $a \otimes b \otimes c \mapsto a \otimes b \otimes 1 \otimes c$.

Lu's definition of Hopf algebroid has one disadvantage. In cases where the structure maps of the bialgebroid do not have such a simple form as in the previous example, it can be quite difficult to construct a suitable section. Nevertheless, Lu's Hopf algebroids will turn out to be useful for us in the next section. They arise naturally with B-torsors and provide a generalized Grunspan map for them.

Lu has shown in [26] that an antipode for a left Hopf algebroid H over R always comes along with a particular endomorphism of R:

Proposition 4.1.11 *([26]) Let H be a Hopf algebroid over R. There exists an algebra morphism $\vartheta : R \longrightarrow R$ such that $\varsigma \circ s = t \circ \vartheta$.*

Proof. Since Δ is an (R, R)-bimodule morphism, we have $\Delta(s(r)) = r \cdot \Delta(1) = s(r) \otimes 1$ for $r \in R$. Applying the identity $\nabla(\mathrm{id} \otimes \varsigma)\Delta = t \circ \varepsilon \circ \varsigma$ to

$s(r)$ yields $t \circ \varepsilon \circ \varsigma \circ s(r) = \nabla(\varsigma \otimes \mathrm{id})(s(r) \otimes 1) = \varsigma \circ s(r)$. Now we see that the map $\vartheta := \varepsilon \circ \varsigma \circ s$ satisfies the above property. $\square$

It is shown in [7] that the notion of $\times_A$-bialgebra is equivalent to the notion of left bialgebroid. So it is clear that the properties of a right bialgebroid can be expressed by a structure that is modelled analogously to that of a $\times_A$-bialgebra. Later we will encounter these right versions of $\times_A$-bialgebras, arising naturally from torsor structures. Therefore, we have to work out their structure explicitly. It is based on a new product "$\times^A$" that we define as follows:

We denote by $A_e := \bar{A} \otimes A$ the opposite algebra of A^e and define for $M, N \in {}_{A_e}\mathcal{M}_{A_e}$

$$M \vartriangle N := \int_a M_a \otimes N_{\bar{a}}$$

and

$$M \times^A N := \int^b \int_a {}_bM_a \otimes {}_{\bar{b}}N_{\bar{a}} \ .$$

For $M, N \in \mathcal{M}_{A_e}$, $M \vartriangle N$ is again a right A_e-module by letting $\bar{A}$ act on the first tensorand and A on the second. It is obvious that the category $(\mathcal{M}_{A_e}, \vartriangle)$ is monoidal and isomorphic to the category $({}_A\mathcal{M}_A, \otimes_A)$. Since we just reversed sides and replaced a's by $\bar{a}$'s, it is clear that the product $\times^A$ has properties analogous to those we have listed above for $\times_A$. We denote the obvious associativity morphisms by α resp. α'. In particular, the product $H \times^A K$ of two A_e-rings H and K is again an algebra by componentwise multiplication and an A_e-ring by $i_{H\times^A K} : \bar{A} \otimes A \longrightarrow H \times^A K$, $\bar{a} \otimes b \mapsto i_H(\bar{a}) \otimes i_K(b)$.

The endomorphism ring of A becomes an A_e-bimodule via the right A_e-module structure on A, that is $((\bar{a} \otimes b) \cdot f)(\alpha) := f(a\alpha b)$ and $(f \cdot (\bar{a} \otimes b))(\alpha) := af(\alpha)b$ for $f \in \mathrm{End}(A)$ and $\alpha \in A$. This means that the opposite algebra $\mathrm{End}(A)^{op}$ is an A_e-ring via $i_{\mathrm{End}(A)^{op}} : \bar{A} \otimes A \longrightarrow \mathrm{End}(A)$, $\bar{a} \otimes b \mapsto (\alpha \mapsto a\alpha b)$. We define two A_e-bimodule maps for each $M \in {}_{A_e}\mathcal{M}_{A_e}$ by

$$\begin{aligned} \zeta : M \times^A \mathrm{End}(A) \longrightarrow M \ , \quad & m \otimes f \ \mapsto mf(1) \\ \zeta' : \mathrm{End}(A) \times^A M \longrightarrow M \ , \quad & f \otimes m \ \mapsto m\overline{f(1)} \ . \end{aligned}$$

Definition 4.1.12 A $\times^A$-*coalgebra* is an A_e-bimodule H together with a comultiplication $\Delta : H \longrightarrow H \times^A H$ and a counit $\varepsilon : H \longrightarrow \mathrm{End}(A)$, both of which are A_e-bimodule maps, such that the following diagrams commute:

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \times^A H \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle H \times^A \Delta} \\ H \times^A H & & H \times^A (H \times^A H) \\ {\scriptstyle \Delta \times^A H}\downarrow & & \downarrow{\scriptstyle \alpha'} \\ (H \times^A H) \times^A H & \xrightarrow[\alpha]{} & H \times^A H \times^A H \end{array}$$

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \times^A H \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle H \times^A \varepsilon} \\ H & \xleftarrow[\zeta]{} & H \times^A \mathrm{End}(A) \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{\Delta} & H \times^A H \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \varepsilon \times^A H} \\ H & \xleftarrow[\zeta']{} & \mathrm{End}(A) \times^A H \end{array}$$

Definition 4.1.13 A $\times^A$-*bialgebra* H is an A_e-ring that has the structure of a $\times^A$-coalgebra such that the comultiplication $\Delta : H \longrightarrow H \times^A H$ and counit $\varepsilon : H \longrightarrow \mathrm{End}(A)^{op}$ are maps of A_e-rings.

In order to show that this is the correct definition, we prove that the notions of right bialgebroid and $\times^A$-bialgebra are equivalent:

Theorem 4.1.14 *For an algebra H the following are equivalent:*

1) A right A-bialgebroid structure $(H, s, t, \Delta, \varepsilon)$.

2) A $\times^A$-bialgebra structure (H, Δ, ε).

Proof. Let $(H, s, t, \Delta, \varepsilon)$ be a right bialgebroid. The A-bimodule structure on H imposed by the definition is equivalent to a right A_e-module structure $h \cdot (\bar{a} \otimes b) := ht(a)s(b)$. The algebra H becomes an A_e-ring via the algebra map $i_H : A_e \longrightarrow H$, $i_H(\bar{a} \otimes b) := t(a)s(b)$, and the corresponding A_e-bimodule structure on H is given by the right A_e-module structure above and the left A_e-module structure $(\bar{a} \otimes b) \cdot h := i_H(\bar{a} \otimes b)h = t(a)s(b)h$.

With these structures we have $H \vartriangle H = (H \otimes H)/ < h \cdot a \otimes g - h \otimes g \cdot \bar{a} >= (H \otimes H)/ < hs(a) \otimes g - h \otimes gt(a) >= H \otimes_A H$, and thus $H \times^A H = \{h \otimes g \in H \vartriangle H \mid a \cdot h \otimes g = h \otimes \bar{a} \cdot g\} = \{h \otimes g \in H \otimes_A H \mid s(a)h \otimes g = h \otimes t(a)g\} =$

$\{h\otimes g \in H\otimes_A H \mid (s(a)\otimes 1)(h\otimes g) = (1\otimes t(a))(h\otimes g)\}$. Hence, the identities 1), 2) and 3) in Definition 4.1.6 imply that Δ induces an A_e-ring map $\Delta' : H \longrightarrow H \times^A H$. It is clear that coassociativity of Δ implies coassociativity of Δ' in the sense of Definition 4.1.12.
We use the counit $\varepsilon : H \longrightarrow A$ to define a map $\varepsilon' : H \longrightarrow \mathrm{End}(A)$ by $\varepsilon'(h)(\alpha) := \varepsilon(s(\alpha)h) = \varepsilon(t(\alpha)h)$ for $h \in H$ and $\alpha \in A$. The last equality holds, since we have $\varepsilon(s(\alpha)) = \alpha = \varepsilon(t(\alpha))$ by identity 5) and right A_e-linearity of ε, and this implies $\varepsilon(s(\alpha)h) = \varepsilon(s\varepsilon s(\alpha)h) = \varepsilon(t\varepsilon s(\alpha)h) = \varepsilon(t(\alpha)h)$ by identity 4). Obviously, the map ε' is right A_e-linear since ε is, and it is left A_e-linear since $\varepsilon'(a\cdot h)(\alpha) = \varepsilon(s(\alpha)s(a)h) = \varepsilon(s(\alpha a)h) = \varepsilon'(h)(\alpha a) = a\cdot(\varepsilon'(h))(\alpha)$ and $\varepsilon'(\bar{a}\cdot h)(\alpha) = \varepsilon(t(\alpha)t(a)h) = \varepsilon(t(a\alpha)(h) = \varepsilon'(h)(a\alpha) = \bar{a}\cdot\varepsilon'(h)(\alpha)$ by definition of the A_e-bimodule structure on $\mathrm{End}(A)$.
By identity 4) we get $\varepsilon'(g)\varepsilon'(h)(\alpha) = \varepsilon'(g)(\varepsilon(s(\alpha)h)) = \varepsilon(s(\varepsilon(s(\alpha)h))g) = \varepsilon(s(\alpha)hg) = \varepsilon'(gh)(\alpha)$, which shows that $\varepsilon' : H \longrightarrow \mathrm{End}(A)^{op}$ is an A_e-ring map.
Finally, we have $\zeta \circ (H \times^A \varepsilon') \circ \Delta'(h) = h_{(1)}\varepsilon'(h_{(2)})(1) = h_{(1)}\varepsilon(h_{(2)}) = h$ and $\zeta' \circ (\varepsilon' \times^A H) \circ \Delta'(h) = h_{(2)} \cdot \overline{\varepsilon'(h_{(1)})(1)} = \varepsilon(h_{(1)})h_{(2)} = h$ for all $h \in H$.

Conversely, let H be a $\times^A$-bialgebra with comultiplication Δ and counit ε. We define the source map by $s : A \longrightarrow H\ ,\ a \mapsto i_H(1\otimes a)$ and the target map by $t : \bar{A} \longrightarrow H\ ,\ a \mapsto i_H(\bar{a}\otimes 1)$, where $i_H : A_e \longrightarrow H$ is the algebra map that gives the A^e-ring structure on H. Obviously, the images of s and t commute in H and the induced A-bimodule structure is $a\cdot h\cdot b = hs(b)t(a) = hi_H(\bar{a}\otimes b)$, which corresponds to the natural right A_e-module structure on H.

We have $H\otimes_A H = (H\otimes H)/ < hs(a)\otimes g - h\otimes gt(a) >$, and since $hs(a)\otimes g = hi_H(1\otimes a)\otimes g = ha\otimes g$ and $h\otimes gt(a) = h\otimes gi_H(\bar{a}\otimes 1) = h\otimes g\bar{a}$, it follows that $H \vartriangle H = H\otimes_A H$. Hence, the comultiplication $\Delta : H \longrightarrow H\times^A H$ induces a map $\Delta' : H \longrightarrow H \vartriangle H = H\otimes_A H$ in $\mathcal{M}_{A_e}$. The counit $\varepsilon : H \longrightarrow \mathrm{End}(A)$ induces a map $\varepsilon' : H \longrightarrow A\ ,\ h \mapsto \varepsilon(h)(1)$ in $\mathcal{M}_{A_e}$. It is clear that Δ' and ε' make H into an A-coring.
Moreover, identity 1) for a right bialgebroid corresponds to the fact that $\Delta'(H) \subset H\times^A H$, as follows from $(s(a)\otimes 1)\Delta'(h) = (i_H(1\otimes a)\otimes 1)(h_{(1)}\otimes h_{(2)}) = ah_{(1)}\otimes h_{(2)}$ and $(1\otimes t(a))\Delta'(h) = (1\otimes i_H(\bar{a}\otimes 1))(h_{(1)}\otimes h_{(2)}) = h_{(1)}\otimes \bar{a}h_{(2)}$. The identities 2) and 3) hold because $\Delta : H \longrightarrow H\times^A H$ is an A_e-ring map. The counit satisfies 4) and 5), since $\varepsilon : H \longrightarrow \mathrm{End}(A)^{op}$ is an A_e-ring map and hence $\varepsilon'(s(\varepsilon'(g))h) = \varepsilon(s(\varepsilon(g)(1))h)(1) = (\varepsilon(g)(1))\cdot\varepsilon(h)(1) = \varepsilon(h)(\varepsilon(g)(1)) = \varepsilon(gh)(1) = \varepsilon'(gh)$ and $\varepsilon'(t(\varepsilon'(g))h) = \varepsilon(t(\varepsilon(g)(1))h)(1) = \overline{\varepsilon(g)(1)}\varepsilon(h)(1) = \varepsilon(h)(\varepsilon(g)(1)) = \varepsilon(gh)(1) = \varepsilon'(gh)$. □

Let H be a $\times^A$-bialgebra. For $M, N \in \mathcal{M}_H$, we define a right H-module structure on $M \vartriangle N$ by $(m \otimes n) \triangleleft h := m \triangleleft h_{(1)} \otimes n \triangleleft h_{(2)}$ for $m \in M, n \in N$ and $h \in H$. It is well-known (see [39]) that an A^e-ring L is a $\times_A$-bialgebra if and only if the category $({}_L\mathcal{M}, \diamond)$ is monoidal such that the underlying functor ${}_L\mathcal{M} \longrightarrow {}_{A^e}\mathcal{M}$ is strictly monoidal. Of course, a corresponding property holds for right modules over $\times^A$-bialgebras:

Theorem 4.1.15 *The following are equivalent for an A_e-ring H:*

1) *A $\times^A$-bialgebra structure on H.*
2) *The structure of a monoidal category on $\mathcal{M}_H$ such that the underlying functor $\mathcal{M}_H \longrightarrow M_{A_e}$ is strictly monoidal.*

This theorem explains why the counit of a $\times^A$-bialgebra H has to be an A_e-ring map $\varepsilon : H \longrightarrow \mathrm{End}(A)^{op}$: The category $(\mathcal{M}_{A_e}, \vartriangle)$ is monoidal with unit object A. So for the underlying functor $\mathcal{M}_H \longrightarrow \mathcal{M}_{A_e}$ to be monoidal, A has to be made into a right H-module. This is done via an algebra morphism $H^{op} \longrightarrow \mathrm{End}(A)$.

Now we are looking for the correct categorical definition of $\times^A$-Hopf algebra. Recall that a $\times_A$-bialgebra L is called a $\times_A$-Hopf algebra if the underlying functor ${}_L\mathcal{M} \longrightarrow {}_{A^e}\mathcal{M}$ preserves right inner hom-functors (see the appendix for a definition of inner hom-functors). Since we are considering right modules over $\times^A$-bialgebras, we have to switch to left inner hom-functors:

Proposition 4.1.16 *Let H be a $\times^A$-bialgebra. The category $(\mathcal{M}_H, \vartriangle)$ is left closed with inner hom-functor* $\mathrm{hom}_{\mathcal{M}_H}(N, P) = \mathrm{Hom}_{-H}(N \vartriangle H, P)$.

Proof. Let N be a right H-module. There is an H-bimodule structure on $N \vartriangle H$ with the left H-module structure induced by H. We have for all $M \in \mathcal{M}_H$ a well-defined isomorphism

$$N \vartriangle M \xrightarrow{\cong} M \otimes_H (N \vartriangle H)\ ,\ n \otimes m \mapsto m \otimes n \otimes 1$$

with inverse $m \otimes (n \otimes h) \mapsto n \otimes m \cdot h$. So the hom-tensor adjunction yields

$$\begin{aligned}\mathrm{Hom}_{-H}(N \vartriangle M, P) &\cong \mathrm{Hom}_{-H}(M \otimes_H (N \vartriangle H), P)\\ &\cong \mathrm{Hom}_{-H}(M, \mathrm{Hom}_{-H}(N \vartriangle H, P))\ .\end{aligned}$$

This shows that the functor $P \mapsto \mathrm{Hom}_{-H}(N \vartriangle H, P)$ is right adjoint to the functor $M \mapsto N \vartriangle M$. □

The category $(\mathcal{M}_{A_e}, \vartriangle)$ is left closed, since the category $({}_A\mathcal{M}_A, \otimes_A)$ is closed with left inner hom-functor $\hom_{{}_A\mathcal{M}_A}(N,P) = \mathrm{Hom}_A(N,P)$. This means that the left inner hom-functor in $\mathcal{M}_{A_e}$ is given as $\hom_{\mathcal{M}_{A_e}}(N,P) = \mathrm{Hom}_{-\bar{A}}(N,P)$ with the right A_e-module structure $(f \cdot (\bar{a} \otimes b))(n) = f(na)b$.

Now we can prove, similarly to [43]:

Theorem and Definition 4.1.17 *Let H be a $\times^A$-bialgebra. Then the following are equivalent:*

1) *The underlying functor $\mathcal{M}_H \longrightarrow \mathcal{M}_{A_e}$ preserves left inner hom-functors.*
2) *The map $\gamma : H \otimes_{\bar{A}} H \longrightarrow H \vartriangle H$, $h \otimes g \mapsto hg_{(1)} \otimes g_{(2)}$ is a bijection.*

If these equivalent conditions hold, we call H a $\times^A$-Hopf algebra.

Proof. We note that if γ is a bijection, then so is the map

$$\gamma_N : N \otimes_{\bar{A}} H \longrightarrow N \vartriangle H \ , \ n \otimes h \mapsto n \cdot h_{(1)} \otimes h_{(2)}$$

for all $N \in \mathcal{M}_H$, since γ_N can be identified with $N \otimes_H \gamma$.

Let $N, P \in \mathcal{M}_H$. The evaluation map for the left inner hom-functor in $\mathcal{M}_H$ is given by

$$\mathrm{ev} : N \vartriangle \mathrm{Hom}_{-H}(N \vartriangle H, P) \longrightarrow P \ , \ \mathrm{ev}(n \otimes f) := f(n \otimes 1) \ .$$

The left inner hom-functor $\mathrm{Hom}_{-\bar{A}}(N,P)$ in $\mathcal{M}_{A_e}$ can be identified with $\mathrm{Hom}_{-H}(N \otimes_{\bar{A}} H, P)$ via $\mathrm{Hom}_{-H}(N \otimes_{\bar{A}} H, P) \ni f \mapsto (n \mapsto f(n \otimes 1)) \in \mathrm{Hom}_{-\bar{A}}(N,P)$. Under this identification the evaluation map is given by

$$\mathrm{ev}' : N \vartriangle \mathrm{Hom}_{-H}(N \otimes_{\bar{A}} H, P) \longrightarrow P \ , \ \mathrm{ev}'(n \otimes f) := f(n \otimes 1) \ .$$

Then the unique map

$$\varphi : \mathrm{Hom}_{-H}(N \vartriangle H, P) \longrightarrow \mathrm{Hom}_{-H}(N \otimes_{\bar{A}} H, P)$$

that satisfies $\mathrm{ev} = \mathrm{ev}'(N \vartriangle \varphi)$, and exists by the universal property of $\mathrm{Hom}_{-\bar{A}}(N,P)$, is given by

$$\begin{aligned} \varphi(f)(n \otimes h) &= \varphi(f)(n \otimes 1) \cdot h = f(n \otimes 1) \cdot h = f((n \otimes 1) \cdot h) \\ &= f(nh_{(1)} \otimes h_{(2)}) = f(\gamma_N(n \otimes h)) \ . \end{aligned}$$

Now we get by the Yoneda Lemma that φ is bijective for all P if and only if γ_N is bijective. □

Now that we have determined how $\times^A$-bialgebras are related to $\times_A$-bialgebras, it is clear that all the results that are known for $\times_A$-bialgebras hold in a similar way for $\times^A$-bialgebras. It is necessary to have both these concepts, though, especially if we want to consider left and right Hopf-Galois extensions.

Schauenburg has shown in [39] that $\times_A$-bialgebras occur naturally with Hopf-Galois extensions over non-trivial sets of coinvariants:
Let T be a left faithfully flat H-Galois extension of B and assume that T is flat as a k-module. Then there is a structure of $\times_B$-bialgebra on $L := \{\sum x_i \otimes y_i \in T \otimes T \mid \sum x_i \otimes y_{i(0)} \otimes y_{i(1)} = \sum x_{i(0)} \otimes y_i \otimes S(x_{i(1)})\}$. It makes T into a left L-comodule algebra in sense of the following definition:

Definition 4.1.18 Let L be a $\times_B$-coalgebra. A *left L-comodule* is a B-bimodule M together with a map $\delta : M \longrightarrow L \times_B M$ of B-bimodules such that the following diagrams commute:

$$\begin{array}{ccc}
M & \xrightarrow{\quad\delta\quad} & L \times_B M \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle L\times_B\delta} \\
L \times_B M & & L \times_B (L \times_B M) \\
\downarrow{\scriptstyle \Delta\times_B M} & & \downarrow{\scriptstyle \alpha'} \\
(L \times_B L) \times_B M & \xrightarrow[\quad\alpha\quad]{} & L \times_B L \times_B M
\end{array}$$

$$\begin{array}{ccc}
M & \xrightarrow{\quad\delta\quad} & L \times_B M \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varepsilon\times_B M} \\
M & \xleftarrow[\quad\vartheta'\quad]{} & \mathrm{End}(B) \times_B M
\end{array}$$

It is proved in [39] that the category ${}^L\mathcal{M}$ of left comodules over a $\times_B$-bialgebra is monoidal with the tensor product over B.

Definition 4.1.19 Let L be a $\times_B$-bialgebra and let T be a B-ring that is also a left L-comodule. Then we call T a *left L-comodule algebra* if the comodule structure map $\delta : T \longrightarrow L \times_B T$ is an algebra map.

Note that this definition makes sense, since we have seen that $L \times_B T$ really possesses an algebra structure.

For the above situation of a left faithfully flat H-Galois extension T of B, it was shown in [39] that the left action of the $\times_B$-bialgebra L is such that T becomes a left L-Galois extension of k. Hopf-Galois extensions with respect to $\times_A$-bialgebras are defined as follows, see [39]:

Definition 4.1.20 Let L be a $\times_B$-bialgebra and let T be a left L-comodule algebra. Let $C := {}^{coL}T := \{x \in T \mid \delta(x) = 1_L \otimes x\}$. Then T is called a *left L-Galois extension of C* if the map

$$\beta : T \otimes_C T \longrightarrow L \diamond T \ , \ x \otimes y \mapsto x_{(-1)} \otimes x_{(0)} y$$

is an isomorphism, and restricts to an isomorphism

$$T \otimes_C T^B \cong L \times_B T \ .$$

It is well-known that a k-bialgebra that admits a Hopf-Galois extension is a Hopf algebra. We prove that the same statement holds for $\times_B$-bialgebras. The following generalizes a proof of Takeuchi, see [42]:

Lemma 4.1.21 *Let L be a $\times_B$-bialgebra and let T be a left L-Galois extension of $C = {}^{coL}T$ that is faithfully flat as a left B-module. Then L is a $\times_B$-Hopf algebra.*

Proof. The $\times_B$-bialgebra L is $\times_B$-Hopf algebra if the map $\beta_L : L \otimes_{\bar{B}} L \longrightarrow L \diamond L \ , \ \ell \otimes m \mapsto \ell_{(1)} \otimes \ell_{(2)} m$ is bijective. By assumption, we have a bijection $\beta_T : T \otimes_C T \longrightarrow L \diamond T \ , \ x \otimes y \mapsto x_{(-1)} \otimes x_{(0)} y$. We consider the following diagram

$$\begin{array}{ccc}
T \otimes_C T \otimes_C \otimes T & \xrightarrow{\beta_T \otimes_C T} & L \diamond T \otimes_C T \\
\downarrow{\scriptstyle T \otimes_C \beta_T} & & \\
T \otimes_C (L \diamond .T) & & \downarrow{\scriptstyle L \diamond \beta_T} \\
\downarrow{\scriptstyle \beta_{13}} & & \\
(L \otimes_{\bar{B}} L) \diamond T & \xrightarrow[\beta_L \diamond T]{} & L \diamond L \diamond T
\end{array}$$

where the map $\beta_{13} : T \otimes_C (L \diamond .T) \longrightarrow (L \otimes_{\bar{B}} L) \diamond T$ is given by applying β_T to the first and third tensorand and leaving the middle tensorand untouched, that is $\beta_{13}(x \otimes \ell \otimes y) := x_{(-1)} \otimes \ell \otimes x_{(0)} y$. The left C-module structure on $L \diamond .T$ is induced by the left C-module structure of T.

In order to show that β_{13} is well-defined, we first consider the map ${\beta_{13}}^{00}$: $T \otimes (L \otimes T) \longrightarrow (L \otimes_{\bar{B}} L) \diamond T$, $x \otimes \ell \otimes y \mapsto x_{(-1)} \otimes \ell \otimes x_{(0)}y$. We have for $b \in B$

$$\begin{aligned} {\beta_{13}}^{00}(x \otimes \ell \otimes by) &= x_{(-1)} \otimes \ell \otimes x_{(0)}by \\ &= x_{(-1)}\bar{b} \otimes \ell \otimes x_{(0)}y \end{aligned}$$

since $x_{(-1)} \otimes x_{(0)} \in L \times_B T$, and

$$\begin{aligned} {\beta_{13}}^{00}(x \otimes \bar{b}\ell \otimes y) &= x_{(-1)} \otimes \bar{b}\ell \otimes x_{(0)}y \\ &= x_{(-1)}\bar{b} \otimes \ell \otimes x_{(0)}y \end{aligned}$$

in $(L \otimes_{\bar{B}} L) \diamond T$, which shows that there exists a factorization ${\beta_{13}}^{0} : T \otimes (L \diamond T) \longrightarrow (L \otimes_{\bar{B}} L) \diamond T$. Now we get for $c \in C = {}^{coL}T$

$$\begin{aligned} {\beta_{13}}^{0}(xc \otimes \ell \otimes y) &= (xc)_{(-1)} \otimes \ell \otimes (xc)_{(0)}y \\ &= x_{(-1)}c_{(-1)} \otimes \ell \otimes x_{(0)}c_{(0)}y \\ &= x_{(-1)} \otimes \ell \otimes x_{(0)}cy \\ &= {\beta_{13}}^{0}(x \otimes \ell \otimes cy) \ , \end{aligned}$$

and therefore ${\beta_{13}}^{0}$ factors over $T \otimes_C (L \diamond .T)$ as β_{13}. It is clear that β_{13} is an isomorphism. It is straightforward to see that the other maps in the diagram are also well-defined and bijective with the possible exception of the map $\beta_L \diamond T$. But the diagram commutes, since

$$\begin{aligned} (L \diamond \beta_T)(\beta_T \otimes_C T)(x \otimes y \otimes z) &= (L \diamond \beta_T)(x_{(-1)} \otimes x_{(0)}y \otimes z) \\ &= x_{(-2)} \otimes x_{(-1)}y_{(-1)} \otimes x_{(0)}y_{(0)}z \end{aligned}$$

and

$$\begin{aligned} (\beta_L \diamond T)\beta_{13}(T \otimes_C \beta_T)(x \otimes y \otimes z) &= (\beta_L \otimes_B T)\beta_{13}(x \otimes y_{(-1)} \otimes y_{(0)}z) \\ &= (\beta_L \otimes_B T)(x_{(-1)} \otimes y_{(-1)} \otimes x_{(0)}y_{(0)}z \\ &= x_{(-2)} \otimes x_{(-1)}y_{(-1)} \otimes x_{(0)}y_{(0)}z \end{aligned}$$

for all $x \otimes y \otimes z \in T \otimes_C T \otimes_C T$. We conclude that the map $\beta_L \diamond T$ must also be bijective. Since the left $\bar{B}$-module structures on both $(L \otimes_{\bar{B}} L)$ and $L \diamond L$ can be identified with a right B-module structure (and hence $\diamond$ can be interpreted as $\otimes_B$), we can conclude by left faithful flatness of T over B that the map β_L is bijective. □

It is straightforward to find the correct definition of right comodules over a $\times^A$-bialgebra H via the analogy in the definitions of $\times_A$-bialgebra and $\times^A$-bialgebra. Then one can consider H-comodule algebras and right H-Galois extensions. We conclude that the previous proposition has a $\times^A$-version for right H-Galois extensions, saying that a $\times^A$-bialgebra that admits a faithfully flat right Hopf-Galois extension is a $\times^A$-Hopf algebra.

4.2 B-Torsors and associated Hopf Algebroids

Let H be a k-Hopf algebra and let A be a right H-Galois extension of B such that B is not isomorphic to k. Then the model of quantum torsor as defined by Grunspan is obviously no longer sufficient to encode both the right coaction of H and the left coaction of the $\times_B$-bialgebra that we described in the previous section. Nevertheless, one can still find a way to "hide" both coactions in just one map. Such a structure was defined by Schauenburg in [42] under the name of B-torsor:

Definition 4.2.1 Let B be a k-algebra, and $B \subset T$ an algebra extension, with T a faithfully flat k-module.
The centralizer $(T \otimes_B T)^B$ of B in the (B, B)-bimodule $T \otimes_B T$ is an algebra by $(a \otimes b)(x \otimes y) = xa \otimes by$ for $a \otimes b, x \otimes y \in (T \otimes_B T)^B$.
A *B-torsor structure* on T is an algebra morphism $\mu : T \longrightarrow T \otimes (T \otimes_B T)^B$ satisfying the axioms

1) $x^{(1)}x^{(2)} \otimes x^{(3)} = 1 \otimes x \quad \in T \otimes_B T$

2) $x^{(1)} \otimes x^{(2)}x^{(3)} = x \otimes 1 \quad \in T \otimes T$

3) $\mu(b) = b \otimes 1 \otimes 1 \quad \forall\, b \in B$

4) $\mu(x^{(1)}) \otimes x^{(2)} \otimes x^{(3)} = x^{(1)} \otimes x^{(2)} \otimes \mu(x^{(3)}) \quad \in T \otimes T \otimes_B T \otimes T \otimes_B T$

where $\mu_0(x) := x^{(1)} \otimes x^{(2)} \otimes x^{(3)}$ is the notation for the induced map $\mu_0 : T \longrightarrow T \otimes T \otimes_B T$.
We note that the composition of maps in axiom 4) is well-defined because of axiom 3).

With the observation that also in this more general case, the torsor structure map μ defines a descent data from T to k on $T \otimes_B T$, Schauenburg proves the following theorem in [42]. It generalizes Grunspan's results on quantum torsors in [18] and establishes, under certain assumptions on faithful flatness, an equivalence between B-torsors and Hopf-Galois extensions of B.

In order not to become confused with too many sums and indices, we will abuse notations in this section and denote elements in a subset of a tensor product as if they were decomposable tensors.

Theorem 4.2.2 *Let T be a B-torsor, and assume that T is a faithfully flat right B-module.*
Then the subalgebra of $(T \otimes_B T)^B$

$$H := \{x \otimes y \in T \otimes_B T \mid xy^{(1)} \otimes y^{(2)} \otimes y^{(3)} = 1 \otimes x \otimes y\}$$

is a faithfully flat Hopf algebra with structure maps

$$\Delta(x \otimes y) = x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)}$$
$$\varepsilon(x \otimes y) = xy \ .$$

The algebra T is an H-Galois extension of B under the coaction $\delta : T \longrightarrow T \otimes H$, $\delta(x) = \mu(x) = x^{(1)} \otimes x^{(2)} \otimes x^{(3)}$. The Galois map is given by $\beta(x \otimes y) = xy^{(1)} \otimes y^{(2)} \otimes y^{(3)}$.

Let H be a faithfully flat Hopf algebra and T a right faithfully flat H-Galois extension of $B \subset T$. Then T is a B-torsor with

$$\mu : T \ni x \mapsto x_{(0)} \otimes {x_{(1)}}^{[1]} \otimes {x_{(1)}}^{[2]} \in T \otimes (T \otimes_B T)^B \ ,$$

where $h^{[1]} \otimes h^{[2]} := \beta^{-1}(1 \otimes h) \in T \otimes_B T$ for all $h \in H$ with respect to the Galois map $\beta : T \otimes_B T \longrightarrow T \otimes H$.

In case $B = k$, we know from [18] that the Grunspan map θ of a quantum torsor T is uniquely determined by the other structure maps. It has a property that links the torsor structure map of T with the structure map of its opposite torsor T^{op}. Using the theory of faithfully flat descent, it was shown in [41] that the existence of the map θ and its properties do not have to be included in the definition of a quantum torsor in case it is faithfully flat. We have also shown at the end of the first chapter that the parallelogram axiom, which was in fact responsible for the appearance of θ in the quantized case, is redundant. Nevertheless, a Grunspan map always exists, and starting from a faithfully flat H-Galois extension A of k, it is given by the formula (1.21).

In this section we will show that in the more general case of a B-torsor T, we can still recover a remainder of the Grunspan map as an endomorphism of the centralizer T^B. This is because a map that is given by the formula (1.22) appears in the context of a particular Hopf algebroid that arises with B-torsors.

We start by showing that we can associate a right T^B-bialgebroid to each B-torsor:

Proposition 4.2.3 *Let T be a B-torsor, and assume that the centralizer $(T \otimes_B T)^B$ is faithfully flat over k.*
Then $(T \otimes_B T)^B$ is a right bialgebroid over T^B with the following structure maps:

- *source map* $s : T^B \longrightarrow (T \otimes_B T)^B \ , \ r \mapsto 1 \otimes r$
- *target map* $t : T^B \longrightarrow (T \otimes_B T)^B \ , \ r \mapsto r \otimes 1$
- *comultiplication*

$$\Delta : (T \otimes_B T)^B \longrightarrow (T \otimes_B T)^B \otimes_{T^B} (T \otimes_B T)^B$$
$$x \otimes y \mapsto x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)}$$

- *counit* $\varepsilon : (T \otimes_B T)^B \longrightarrow T^B \ , \ x \otimes y \mapsto xy$

Proof. We show that the bialgebroid axioms in Definition 4.1.6 are satisfied. It is easy to see that the base ring T^B is a subalgebra of T. The total ring $(T \otimes_B T)^B$ becomes an algebra by $(a \otimes b)(x \otimes y) = xa \otimes by$ for $a \otimes b, x \otimes y \in (T \otimes_B T)^B$. With this algebra structure on $(T \otimes_B T)^B$, it follows that s is an algebra morphism and t is an algebra anti-morphism satisfying $s(r)t(r') = (1 \otimes r)(r' \otimes 1) = r' \otimes r = (r' \otimes 1)(1 \otimes r) = t(r')s(r)$.
The resulting T^B-bimodule structure on $(T \otimes_B T)^B$ is

$$r \cdot (x \otimes y) \cdot r' = (x \otimes y)s(r')t(r) = (x \otimes y)(1 \otimes r')(r \otimes 1) = rx \otimes yr' \ .$$

In order to show that Δ is well-defined, we first consider the map

$$\Delta_0 : T \otimes_B T \longrightarrow T \otimes_B T \otimes (T \otimes_B T)^B \ , \ u \otimes v \mapsto u \otimes v^{(1)} \otimes v^{(2)} \otimes v^{(3)} \ .$$

This map is well-defined, since we have $ub \otimes v^{(1)} \otimes v^{(2)} \otimes v^{(3)} = u \otimes bv^{(1)} \otimes v^{(2)} \otimes v^{(3)} = u \otimes (bv)^{(1)} \otimes (bv)^{(2)} \otimes (bv)^{(3)}$ for $b \in B$ and $u, v \in T$ by the B-torsor axiom 3). Now we can show that $\Delta_0((T \otimes_B T)^B) \subseteq (T \otimes_B T)^B \otimes (T \otimes_B T)^B$: Let $x \otimes y \in (T \otimes_B T)^B$ and $b \in B$. Then $\Delta_0(bx \otimes y) = \Delta_0(x \otimes yb)$ and hence

$$\begin{aligned} bx \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)} &= x \otimes (yb)^{(1)} \otimes (yb)^{(2)} \otimes (yb)^{(3)} \\ &= x \otimes y^{(1)} b^{(1)} \otimes b^{(2)} y^{(2)} \otimes y^{(3)} b^{(3)} \\ &= x \otimes y^{(1)} b \otimes y^{(2)} \otimes y^{(3)} \end{aligned}$$

by axiom 3). The claim follows by faithful flatness of $(T \otimes_B T)^B$ over k.
The map Δ is obtained by composing Δ_0 with the canonical residue class morphism $(T \otimes_B T)^B \otimes (T \otimes_B T)^B \longrightarrow (T \otimes_B T)^B \otimes_{T^B} (T \otimes_B T)^B$.

The counit map ε is well-defined, since for $x \otimes y \in (T \otimes_B T)^B, b \in B$ we have $bxy = xyb$ and thus $\varepsilon((T \otimes_B T)^B) \subseteq T^B$.

We check that Δ and ε are (T^B, T^B)-bimodule morphisms. Let $x \otimes y \in (T \otimes_B T)^B$ and $r \in T^B$. Then

$$\begin{aligned}\Delta(r \cdot (x \otimes y)) = \Delta(rx \otimes y) &= rx \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)} = \\ &= r \cdot (x \otimes y^{(1)}) \otimes (y^{(2)} \otimes y^{(3)}) = r \cdot \Delta(x \otimes y)\end{aligned}$$

and

$$\begin{aligned}\Delta((x \otimes y) \cdot r) = \Delta(x \otimes yr) &= x \otimes (yr)^{(1)} \otimes (yr)^{(2)} \otimes (yr)^{(3)} = \\ &= x \otimes y^{(1)} r^{(1)} \otimes r^{(2)} y^{(2)} \otimes y^{(3)} r^{(3)} = \\ &= x \otimes y^{(1)} \otimes r^{(1)} r^{(2)} y^{(2)} \otimes y^{(3)} r^{(3)} = \\ &= x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)} r = \Delta(x \otimes y) \cdot r \ ,\end{aligned}$$

where we used $\mu(r) \in T^B \otimes (T \otimes_B T)^B$, which follows from $\mu(br) = br^{(1)} \otimes r^{(2)} \otimes r^{(3)}$ and $\mu(rb) = r^1 b \otimes r^{(2)} \otimes r^{(3)}$ and faithful flatness of $(T \otimes_B T)^B$. For the counit we get $\varepsilon(rx \otimes y) = rxy = r \cdot \varepsilon(x \otimes y)$ and $\varepsilon(x \otimes yr) = xyr = \varepsilon(x \otimes y) \cdot r$.

Coassociativity of Δ follows from coassociativity of μ and the counit axioms follow from $(\varepsilon \otimes \mathrm{id})\Delta(x \otimes y) = xy^{(1)} \otimes y^{(2)} \otimes y^{(3)} = 1 \otimes xy^{(1)} y^{(2)} \otimes y^{(3)} = 1 \otimes x \otimes y \cong x \otimes y$, since $xy^{(1)} \otimes y^{(2)} \otimes y^{(3)} \in T^B \otimes (T \otimes_B T)^B$, and $(\mathrm{id} \otimes \varepsilon)\Delta$ $(x \otimes y) = x \otimes y^{(1)} \otimes y^{(2)} y^{(3)} = x \otimes y \otimes 1 \cong x \otimes y$. This means that Δ and ε make $(T \otimes_B T)^B$ into a comonoid in ${}_{T^B}\mathcal{M}_{T^B}$.

The proof of the first identity 1) requires the following lemma:

Lemma 4.2.4 *The map*

$$\begin{aligned}\psi : (T \otimes_B T)^B \otimes_{T^B} (T \otimes_B T)^B &\longrightarrow (T \otimes_B T \otimes_B T)^B \\ x \otimes y \otimes v \otimes w &\mapsto x \otimes yv \otimes w\end{aligned}$$

is bijective.

Proof. We claim that the inverse of ψ is given by

$$\begin{aligned}\phi : (T \otimes_B T \otimes_B T)^B &\longrightarrow (T \otimes_B T)^B \otimes_{T^B} (T \otimes_B T)^B \\ x \otimes y \otimes z &\mapsto x \otimes yz^{(1)} \otimes z^{(2)} \otimes z^{(3)}\end{aligned}$$

It is easy to see that both ψ and ϕ are well-defined. They are inverse to each other since

$$\begin{aligned}\phi \circ \psi(x \otimes y \otimes v \otimes w) = \phi(x \otimes yv \otimes w) &= x \otimes yvw^{(1)} \otimes w^{(2)} \otimes w^{(3)} \\ &= x \otimes y \otimes vw^{(1)} w^{(2)} \otimes w^{(3)} = x \otimes y \otimes v \otimes w \ ,\end{aligned}$$

where we used that $vw^{(1)} \otimes w^{(2)} \otimes w^{(3)} \in T^B \otimes (T \otimes_B T)^B$, and

$$\psi \circ \phi(x \otimes y \otimes z) = \psi(x \otimes yz^{(1)} \otimes z^{(2)} \otimes z^{(3)}) = x \otimes yz^{(1)} z^{(2)} \otimes z^{(3)} = x \otimes y \otimes z \ .$$

□

Now we obtain for $x \otimes y \in (T \otimes_B T)^B$ and $r \in T^B$ that

$$\begin{aligned}(s(r) \otimes 1 \otimes 1)\Delta(x \otimes y) &= (1 \otimes r \otimes 1 \otimes 1)(x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)}) \\ &= x \otimes ry^{(1)} \otimes y^{(2)} \otimes y^{(3)}\end{aligned}$$

$$\begin{aligned}(1 \otimes 1 \otimes t(r))\Delta(x \otimes y) &= (1 \otimes 1 \otimes r \otimes 1)(x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)}) \\ &= x \otimes y^{(1)} \otimes y^{(2)} r \otimes y^{(3)} \ .\end{aligned}$$

Applying the map ψ to both terms yields

$$\begin{aligned}\psi(x \otimes ry^{(1)} \otimes y^{(2)} \otimes y^{(3)}) &= x \otimes ry^{(1)}y^{(2)} \otimes y^{(3)} = x \otimes r \otimes y = \\ &= x \otimes y^{(1)}y^{(2)}r \otimes y^{(3)} = \psi(x \otimes y^{(1)} \otimes y^{(2)}r \otimes y^{(3)}) \ ,\end{aligned}$$

and so it follows by bijectivity of ψ that $(s(r) \otimes 1 \otimes 1)\Delta(x \otimes y) = (1 \otimes 1 \otimes t(r))\Delta(x \otimes y)$, which is 1).

The remaining identities are proved by direct calculation:

Identity 2) holds, since $\Delta((x \otimes y)(v \otimes w)) = \Delta(vx \otimes yw) = vx \otimes (yw)^{(1)} \otimes (yw)^{(2)} \otimes (yw)^{(3)} = vx \otimes y^{(1)}w^{(1)} \otimes w^{(2)}y^{(2)} \otimes y^{(3)}w^{(3)} = (x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)})(v \otimes w^{(1)} \otimes w^{(2)} \otimes w^{(3)}) = \Delta(x \otimes y)\Delta(v \otimes w)$, and we have 3) since μ is an algebra morphism.
Finally, also 4) and 5) hold, since $\varepsilon(s(\varepsilon(x \otimes y))(v \otimes w)) = \varepsilon(s(xy)(v \otimes w)) = \varepsilon((1 \otimes xy)(v \otimes w)) = \varepsilon(v \otimes xyw) = vxyw$ and $\varepsilon((x \otimes y)(v \otimes w)) = \varepsilon(vx \otimes yw) = vxyw$ and also $\varepsilon(t(\varepsilon(x \otimes y))(v \otimes w)) = \varepsilon(t(xy)(v \otimes w)) = \varepsilon((xy \otimes 1)(v \otimes w)) = \varepsilon(vxy \otimes w) = vxyw$, and of course $\varepsilon(1 \otimes 1) = 1$. □

From now on let T be a B-torsor that is right faithfully flat over B, and assume that $(T \otimes_B T)^B$ is a faithfully flat k-module. With these assumptions we have the results of Propositions 4.2.2 and 4.2.3 at hand.
We denote the right H-comdule structure on T by

$$\delta(t) := t_{(0)} \otimes t_{(1)} = t^{(1)} \otimes t^{(2)} \otimes t^{(3)} \subseteq T \otimes H \subseteq T \otimes (T \otimes_B T)^B \ .$$

Now we are going to derive a result that can be found in [20] for finite dimensional Hopf-Galois extensions.

We know by Theorem 4.2.2 that the Galois map for the B-torsor T is given by

$$\beta : T \otimes_B T \longrightarrow T \otimes H \ , \ x \otimes y \longrightarrow xy^{(1)} \otimes y^{(2)} \otimes y^{(3)} \ .$$

This is an isomorphism of (T, B)-bimodules, where the (T, B)-bimodule structure on $T \otimes H$ is induced by T and the structure on $T \otimes_B T$ comes from the obvious (T, T)-bimodule structure. By restriction, both $T \otimes_B T$ and $T \otimes H$ become (B, B)-bimodules, and hence the Galois map induces an isomorphism

$$\beta : (T \otimes_B T)^B \cong (T \otimes H)^B \cong T^B \otimes H \ .$$

So the algebra structure of the T^B-bialgebroid $(T \otimes_B T)^B$ from the previous proposition can be transfered to $T^B \otimes H$, turning $\beta : (T \otimes_B T)^B \longrightarrow T^B \otimes H$ into an algebra morphism.
Recall the properties of the expression $h^{[1]} \otimes h^{[2]} = \beta^{-1}(1 \otimes h)$ for $h \in H$ from Section 1.2. By (1.3) we have $\beta^{-1}(r \otimes h) = rh^{[1]} \otimes h^{[2]}$ for all $r \otimes h \in T^B \otimes H$. Let $r, s \in T^B$ and $h = h_1 \otimes h_2$, $g = g_1 \otimes g_2 \in H \subset T \otimes_B T$. We determine the induced algebra structure on $T^B \otimes H$ as

$$\begin{aligned}(r \otimes h) \bullet (s \otimes g) &= \beta(\beta^{-1}(r \otimes h)\beta^{-1}(s \otimes g)) \\ &= \beta((rh^{[1]} \otimes h^{[2]})(sg^{[1]} \otimes g^{[2]})) \\ &= \beta(sg^{[1]}rh^{[1]} \otimes h^{[2]}g^{[2]}) \\ &= sg^{[1]}rh^{[1]}(h^{[2]}g^{[2]})^{(1)} \otimes (h^{[2]}g^{[2]})^{(2)} \otimes (h^{[2]}g^{[2]})^{(3)} \\ &= sg^{[1]}rh^{[1]}h^{[2](1)}g^{[2](1)} \otimes g^{[2](2)}h^{[2](2)} \otimes h^{[2](3)}g^{[2](3)} \\ &= sg^{[1]}rg^{[2](1)} \otimes g^{[2](2)}h_1 \otimes h_2g^{[2](3)} \ ,\end{aligned}$$

since $\beta(h^{[1]} \otimes h^{[2]}) = h^{[1]}h^{[2](1)} \otimes h^{[2](2)} \otimes h^{[2](3)} = 1 \otimes h_1 \otimes h_2$.

Now H-colinearity of β implies that

$$\begin{aligned}{g_{(1)}}^{[1]} \otimes {g_{(1)}}^{[2]} \otimes g_{(2)} &= \beta^{-1}(1 \otimes g_{(1)}) \otimes g_{(2)} \\ &= (\mathrm{id} \otimes \delta)\beta^{-1}(1 \otimes g) \\ &= (\mathrm{id} \otimes \delta)(g^{[1]} \otimes g^{[2]}) \\ &= g^{[1]} \otimes g^{[2](1)} \otimes g^{[2](2)} \otimes g^{[2](3)} \ ,\end{aligned}$$

and hence

$$\begin{aligned}(r \otimes h) \bullet (s \otimes g) &= sg^{[1]}rg^{[2](1)} \otimes g^{[2](2)}h_1 \otimes h_2g^{[2](3)} \\ &= s{g_{(1)}}^{[1]}r{g_{(1)}}^{[2]} \otimes g_{(2)1}h_1 \otimes h_2g_{(2)2} \\ &= s{g_{(1)}}^{[1]}r{g_{(1)}}^{[2]} \otimes hg_{(2)} \\ &= s(r \triangleleft g_{(1)}) \otimes hg_{(2)} \ ,\end{aligned}$$

where $\triangleleft : T^B \otimes H \longrightarrow T^B,\ r \triangleleft h := h^{[1]} r h^{[2]}$ denotes the Miyashita-Ulbrich action of H on T^B that we introduced in Section 1.2.

Proposition 4.2.5 *The Galois map β induces a smash product algebra structure on $T^B \otimes H^{op}$ by*

$$(r\#h)(s\#g) := r(h_{(1)} \triangleright s)\#h_{(2)} \cdot g$$

where $\triangleright$ denotes the Miyashita-Ulbrich action of H^{op} on T^B from the left and $h \cdot g := gh$ indicates the multiplication in H^{op}.

Proof. Applying the above calculation on the opposite algebra structure of $(T \otimes_B T)^B$ yields the formula $(r \otimes h) \bullet_{op} (s \otimes g) = r(s \triangleleft h_{(1)}) \otimes gh_{(2)} \in T^B \otimes H$. The right Miyashita-Ulbrich action $\triangleleft : T^B \otimes H \longrightarrow T^B$ corresponds to a left action of H^{op} on T^B that we denote by $h \triangleright r := h^{[1]} r h^{[2]}$. Hence, in $T^B \otimes H^{op}$, the formula reads $(r \otimes h) \bullet_{op} (s \otimes g) = r(h_{(1)} \triangleright s) \otimes h_{(2)} \cdot g \in T^B \otimes H^{op}$, which shows that it is in fact a smash product multiplication. □

Remark 4.2.6 We know by Remark 4.1.7 that the structure of right T^B-bialgebroid on $(T \otimes_B T)^B$ leads to a structure of left T^B-bialgebroid on the opposite algebra $((T \otimes_B T)^B)^{op}$. The corresponding structure maps are

- source map $s : T^B \longrightarrow ((T \otimes_B T)^B)^{op}\ ,\ r \mapsto r \otimes 1$
- target map $t : T^B \longrightarrow ((T \otimes_B T)^B)^{op}\ ,\ r \mapsto 1 \otimes r$
- comultiplication

$$\Delta : ((T \otimes_B T)^B)^{op} \longrightarrow ((T \otimes_B T)^B)^{op} \otimes_{T^B} ((T \otimes_B T)^B)^{op}$$
$$x \otimes y \mapsto x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)}$$

- counit $\varepsilon : ((T \otimes_B T)^B)^{op} \longrightarrow T^B\ ,\ x \otimes y \mapsto xy$

Now the algebra isomorphism $\beta : ((T \otimes_B T)^B)^{op} \longrightarrow T^B\#H^{op}$ allows us to transfer the left T^B-bialgebroid structure of $((T \otimes_B T)^B)^{op}$ onto $T^B\#H^{op}$. This will make β into a morphism of left bialgebroids over T^B.

Proposition 4.2.7 *The smash product algebra $T^B\#H^{op}$ is a left bialgebroid over T^B with the structure maps*

- *source map* $\tilde{s} : T^B \longrightarrow T^B\#H^{op}\ ,\ r \mapsto r \otimes 1$
- *target map* $\tilde{t} : T^B \longrightarrow T^B\#H^{op}\ ,\ r \mapsto r^{(1)} \otimes r^{(2)} \otimes r^{(3)} = r_{(0)} \otimes r_{(1)}$

- *comultiplication*

$$\begin{aligned}\tilde{\Delta} : T^B \# H^{op} &\longrightarrow T^B \# H^{op} \otimes_{T^B} T^B \# H^{op} \\ r \otimes h &\mapsto r \otimes h_{(1)} \otimes 1 \otimes h_{(2)}\end{aligned}$$

- *counit* $\tilde{\varepsilon} : T^B \# H^{op} \longrightarrow T^B$, $r \otimes h \mapsto r\varepsilon_H(h)$

Proof. Applying the Galois map β to the source and target maps of $((T \otimes_B T)^B)^{op}$ yields for $r \in T^B$

$$\begin{aligned}\tilde{s}(r) &= \beta(s^{op}(r)) = \beta(r \otimes 1) = r \otimes 1 \\ \tilde{t}(r) &= \beta(t^{op}(r)) = \beta(1 \otimes r) = r^{(1)} \otimes r^{(2)} \otimes r^{(3)} = r_{(0)} \otimes r_{(1)}\end{aligned}$$

The images of $\tilde{s}$ and $\tilde{t}$ commute in $T^B \# H^{op}$, since $\tilde{s}(x)\tilde{t}(y) = (x \otimes 1)(y_{(0)} \otimes y_{(1)}) = xy_{(0)} \otimes y_{(1)}$ and $\tilde{t}(y)\tilde{s}(x) = (y_{(0)} \otimes y_{(1)})(x \otimes 1) = y_{(0)}(y_{(1)} \triangleright x) \otimes y_{(2)} = y_{(0)}{y_{(1)}}^{[1]} x {y_{(1)}}^{[2]} \otimes y_{(2)} = xy_{(0)} \otimes y_{(1)}$ by (1.10) for all $x, y \in T^B$.

For the comultiplication $\tilde{\Delta}$, we consider the diagram

$$\begin{array}{ccc} ((T \otimes_B T)^B)^{op} & \xrightarrow{\quad\beta\quad} & T^B \# H^{op} \\ \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle \tilde{\Delta}} \\ ((T \otimes_B T)^B)^{op} \otimes_{T^B} ((T \otimes_B T)^B)^{op} & \xrightarrow[\beta \otimes_{T^B} \beta]{} & T^B \# H^{op} \otimes_{T^B} T^B \# H^{op} \end{array}$$

Note that the map $\beta \otimes_{T^B} \beta$ is well-defined, since the (T^B, T^B)-bimodule structure on $T^B \# H^{op}$ is induced by β. So we obtain for $r \otimes h \in T^B \# H$

$$\begin{aligned}\tilde{\Delta}(r \otimes h) &= (\beta \otimes_{T^B} \beta)\Delta\beta^{-1}(r \otimes h) \\ &= (\beta \otimes_{T^B} \beta)\Delta(rh^{[1]} \otimes h^{[2]}) \\ &= (\beta \otimes_{T^B} \beta)(rh^{[1]} \otimes h^{[2](1)} \otimes h^{[2](2)} \otimes h^{[2](3)}) \\ &= rh^{[1]}h^{[2](1)} \otimes h^{[2](2)} \otimes h^{[2](3)} \otimes h^{[2](4)}h^{[2](5)} \otimes h^{[2](6)} \otimes h^{[2](7)} \\ &= rh^{[1]}h^{[2](1)} \otimes h^{[2](2)} \otimes h^{[2](3)} \otimes 1 \otimes h^{[2](4)} \otimes h^{[2](5)} \\ &= r \otimes h_{(1)} \otimes 1 \otimes h_{(2)} \ .\end{aligned}$$

Finally, the induced counit is given by $\tilde{\varepsilon}(r \otimes h) = \varepsilon\beta^{-1}(r \otimes h) = r\varepsilon\beta^{-1}(1 \otimes h) = r\varepsilon(h^{[1]} \otimes h^{[2]}) = rh^{[1]}h^{[2]} = r\varepsilon_H(h)$. $\square$

Remark 4.2.8 The (T^B, T^B)-bimodule structure on the left bialgebroid $T^B \# H^{op}$ is given by

$$x \cdot (r \otimes h) = \tilde{s}(x)(r \otimes h) = (x \otimes 1)(r \otimes h) = x(1 \triangleright r) \otimes 1 \cdot h = xr \otimes h$$

$$(r \otimes h) \cdot x = \tilde{t}(x)(r \otimes h) = (x_{(0)} \otimes x_{(1)})(r \otimes h) = x_{(0)}(x_{(1)} \triangleright r) \otimes x_{(2)} \cdot h = \\ = rx_{(0)} \otimes hx_{(1)}$$

for $x \in T^B$ and $r \otimes h \in T^B \# H^{op}$. The last equality follows from $x_{(0)}(x_{(1)} \triangleright r) = x_{(0)}{x_{(1)}}^{[1]} r {x_{(1)}}^{[2]} = rx$ by (1.10).

The following theorem from Lu [26] was applied by Kadison in [20] to recover a Hopf algebroid structure on the smash product $A^B \# H$, starting from a finite dimensional Hopf algebra H over a field $\mathbb{K}$ and a right H-Galois extension A of B.

Theorem 4.2.9 *([26]) Let H be a finite dimensional Hopf algebra over a field $\mathbb{K}$ with antipode S, and let $D(H) = H^* \bowtie H$ be its Drinfeld double.*
Let A be a left $D(H)$-module algebra and assume that the R-matrix $R = R^1 \otimes R^2$ of $D(H)$ acts on A such that

$$(R^2 \cdot y)(R^1 \cdot x) = xy$$

for $x, y \in A$. Then there is a Hopf algebroid structure over A on the smash product algebra $A \# H$ and the structure maps are given by

$$\begin{aligned} s(a) &= a \otimes 1 \\ t(a) &= \sum (h_i^* \cdot a) \otimes h_i \\ \Delta(a \otimes h) &= a \otimes h_{(1)} \otimes 1 \otimes h_{(2)} \\ \varepsilon(a \otimes h) &= \varepsilon(h) a \\ \varsigma(a \otimes h) &= \sum (1 \otimes S(h)) t((S^2(h_i) h_i^*) \cdot a) \end{aligned}$$

where (h_i) is a basis of H with dual basis (h_i^*) of H^*, and the R-matrix of $D(H)$ is $R = \sum (1 \otimes h_i) \otimes (h_i^* \otimes 1)$.

As we observe immediately, the condition in Lu's theorem $(R^2 \cdot y)(R^1 \cdot x) = xy$ for $x, y \in A$, means that A is a commutative algebra in the category ${}_{D(H)}\mathcal{M}$ of left $D(H)$-modules, since the category ${}_{D(H)}\mathcal{M}$ is monoidal and braided with the braiding given by

$$\sigma : M \otimes N \longrightarrow N \otimes M \ , \ m \otimes n \mapsto R^2 \cdot n \otimes R^1 \cdot m \ .$$

We know from [28] that in case the Hopf algebra H has finite dimension over a field $\mathbb{K}$, there is a category equivalence

$$_{D(H)}\mathcal{M} \cong {}_H\mathcal{YD}^H \ ,$$

where a left $D(H)$-module M becomes a left H-module by the restricted action of $H \subset D(H)$, and a right H^{op}-comodule via the action of $H^{*cop} \subset D(H)$, since $H^{*cop} \cong H^{op*}$ and $_{H^{op*}}\mathcal{M} \cong \mathcal{M}^{H^{op}}$. So we have the relation $f \cdot m = m_{(0)} f(m_{(1)})$ for the action of $f \in H^*$, if $m_{(0)} \otimes m_{(1)}$ denotes the induced H-comodule structure. The left H-module and right H-comodule structure are such that the compatibility condition for left-right Yetter-Drinfeld modules is satisfied. The commutativity condition translates to

$$\begin{aligned} mn &= (R^2 \cdot n)(R^1 \cdot m) \\ &= \sum (h_i^* \cdot n)(h_i \triangleright m) \\ &= \sum n_{(0)} h_i^*(n_{(1)})(h_i \triangleright m) \\ &= n_{(0)}(n_{(1)} \triangleright m) \end{aligned}$$

for $m, n \in M$.

Let H be a Hopf algebra with invertible antipode. Then the category $_H\mathcal{YD}^H$ of left-right Yetter-Drinfeld modules is monoidal and braided as follows: The tensor product of two left-right Yetter-Drinfeld modules $M, N \in {}_H\mathcal{YD}^H$ becomes a Yetter-Drinfeld module via $h \triangleright (m \otimes n) := h_{(1)} \triangleright m \otimes h_{(2)} \triangleright n$ and $\delta(m \otimes n) := m_{(0)} \otimes n_{(0)} \otimes n_{(1)} m_{(1)}$ for $m \otimes n \in M \otimes N$ and $h \in H$. Note that this comodule structure is the codiagonal structure with respect to the coaction of H^{op}. The braiding in $_H\mathcal{YD}^H$ is induced by the braiding in the category $_{D(H)}\mathcal{M}$ and given by

$$\sigma : M \otimes N \longrightarrow N \otimes M \ , \ m \otimes n \mapsto n_{(0)} \otimes n_{(1)} \triangleright m \ .$$

Now let A be a left $D(H)$-module algebra. Then A is a left-right Yetter-Drinfeld module, such that the two equivalent conditions

$$h_{(1)} \triangleright a_{(0)} \otimes h_{(2)} a_{(1)} = (h_{(2)} \triangleright a)_{(0)} \otimes (h_{(2)} \triangleright a)_{(1)} h_{(1)} \tag{4.1}$$

$$h_{(2)} \triangleright a_{(0)} \otimes h_{(3)} a_{(1)} S^{-1}(h_{(1)}) = (h \triangleright a)_{(0)} \otimes (h \triangleright a)_{(1)} \tag{4.2}$$

are satisfied for all $a, b \in A$. The compatibility of the algebra structure with the $D(H)$-module structure means that, with respect to the induced structures, A is a left H-module algebra and a right H^{op}-comodule algebra, that is

$$h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b) \quad , \quad h \triangleright 1 = \varepsilon(h)1$$

and

$$\delta(ab) = a_{(0)}b_{(0)} \otimes b_{(1)}a_{(1)} \quad , \quad \delta(1) = 1 \otimes 1$$

for all $a, b \in A$ and $h \in H$. Moreover, A is an algebra in the category ${}_H\mathcal{YD}^H$, by the definition of the monoidal structure. Hence, the commutativity condition

$$ab = \nabla_A \circ \sigma(a \otimes b) = b_{(0)}(b_{(1)} \triangleright a) \tag{4.3}$$

for all $a, b \in A$, says that A is a commutative algebra in the category ${}_H\mathcal{YD}^H$.

Since the definition of Yetter-Drinfeld modules does neither require a finite dimensional Hopf algebra H nor to be working over a field, we rather use the language of Yetter-Drinfeld modules than that of modules over the Drinfeld double. This allows us to drop the assumption that H be finite dimensional and formulate a generalized version of Lu's theorem.

We will use the following two lemmas for the proof:

Lemma 4.2.10 *Let H be a Hopf algebra with invertible antipode S and let A be a left H-module algebra with action $\triangleright : H \otimes A \longrightarrow A$. The the opposite smash product algebra $(A\#H)^{op}$ is isomorphic to the smash product algebra $A^{op}\#H^{op}$, where A^{op} becomes an H^{op}-module algebra by the action $h \blacktriangleright a := S^{-1}(h) \triangleright a$ for $a \in A$ and $h \in H$.*

Proof. A direct calculation shows that the action $\blacktriangleright$ defines a left H^{op}-module algebra structure on A. The map

$$\phi : A^{op}\#H^{op} \longrightarrow (A\#H)^{op} \ , \ a \otimes h \mapsto h_{(1)} \triangleright a \otimes h_{(2)}$$

is bijective with inverse $\phi^{-1}(a \otimes h) = h_{(1)} \blacktriangleright a \otimes h_{(2)}$, since

$$\begin{aligned}
\phi^{-1} \circ \phi(a \otimes h) &= \phi^{-1}(h_{(1)} \triangleright a \otimes h_{(2)}) \\
&= h_{(2)} \blacktriangleright (h_{(1)} \triangleright a) \otimes h_{(3)} \\
&= (S^{-1}(h_{(2)})h_{(1)}) \triangleright a \otimes h_{(3)} \\
&= r \otimes h
\end{aligned}$$

and

$$\begin{aligned}
\phi \circ \phi^{-1}(a \otimes h) &= \phi(h_{(1)} \blacktriangleright a \otimes h_{(2)}) \\
&= h_{(2)} \triangleright (S^{-1}(h_{(1)}) \triangleright a) \otimes h_{(3)} \\
&= (h_{(2)}S^{-1}(h_{(1)})) \triangleright a \otimes h_{(3)} \\
&= a \otimes h \ .
\end{aligned}$$

It is an algebra morphism, since for $a\otimes h, b\otimes g \in A^{op}\#H^{op}$

$$\begin{aligned}\phi((a\otimes h)\cdot(b\otimes g)) &= \phi((h_{(1)} \blacktriangleright b)a \otimes gh_{(2)})\\ &= g_{(1)}h_{(2)} \triangleright ((h_{(1)} \blacktriangleright b)a) \otimes g_{(2)}h_{(3)}\\ &= (g_{(1)}h_{(2)}S^{-1}(h_{(1)}) \triangleright b)(g_{(2)}h_{(3)} \triangleright a) \otimes g_{(3)}h_{(4)}\\ &= (g_{(1)} \triangleright b)(g_{(2)}h_{(1)} \triangleright a) \otimes g_{(3)}h_{(2)}\end{aligned}$$

and

$$\begin{aligned}\phi(a\otimes h)\cdot_{op}\phi(b\otimes g) &= (g_{(1)} \triangleright b \otimes g_{(2)})(h_{(1)} \triangleright a \otimes h_{(2)})\\ &= (g_{(1)} \triangleright b)(g_{(2)} \triangleright (h_{(1)} \triangleright a)) \otimes g_{(3)}h_{(2)}\\ &= (g_{(1)} \triangleright b)(g_{(2)}h_{(1)} \triangleright a) \otimes g_{(3)}h_{(2)}\ .\end{aligned}$$

□

Lemma 4.2.11 *Let H be a Hopf algebra with invertible antipode S and let A be a commutative algebra in the category ${}_H\mathcal{YD}^H$ of Yetter-Drinfeld modules. Then the map*

$$\psi : A\#H \longrightarrow A^{op}\#H^{op}\ ,\ a\otimes h \mapsto a_{(0)} \otimes S(S(a_{(1)})h)$$

is an algebra isomorphism with inverse $\psi^{-1} : A^{op}\#H^{op} \longrightarrow A\#H\ ,\ a\otimes h \mapsto a_{(0)} \otimes a_{(1)}S^{-1}(h)$.

Proof. It is quite obvious that the given maps are inverse to each other. Let $a\otimes h, b\otimes g \in A\#H$. Then, using the Yetter-Drinfeld condition (4.2), we have

$$\begin{aligned}\psi((a\otimes h)(b\otimes g)) &= \psi(a(h_{(1)} \triangleright b) \otimes h_{(2)}g)\\ &= (a(h_{(1)} \triangleright b))_{(0)} \otimes S(S((a(h_{(1)} \triangleright b))_{(1)})h_{(2)}g)\\ &= a_{(0)}(h_{(1)} \triangleright b)_{(0)} \otimes S(S((h_{(1)} \triangleright b)_{(1)}a_{(1)})h_{(2)}g)\\ &= a_{(0)}(h_{(2)} \triangleright b_{(0)}) \otimes S(S(h_{(3)}b_{(1)}S^{-1}(h_{(1)})a_{(1)})h_{(4)}g)\\ &= a_{(0)}(h_{(2)} \triangleright b_{(0)}) \otimes S(S(a_{(1)})h_{(1)}S(b_{(1)})g)\ .\end{aligned}$$

The inverse of the braiding in the category ${}_H\mathcal{YD}^H$ is given by

$$\sigma^{-1} : M\otimes N \longrightarrow N\otimes M\ ,\ m\otimes n \mapsto S(m_{(1)}) \triangleright n \otimes m_{(0)}\ ,$$

and we obtain $\nabla_A = \nabla_A\circ\sigma\circ\sigma^{-1} = \nabla_A\circ\sigma^{-1}$ from the commutativity relation

(4.3), that is, $ab = (S(a_{(1)}) \triangleright b)a_{(0)}$ for all $a, b \in A$. Now we have

$$\begin{aligned}
&\psi(a \otimes h) \cdot \psi(b \otimes g) = \\
&\qquad = (a_{(0)} \otimes S(S(a_{(1)})h)) \cdot (b_{(0)} \otimes S(S(b_{(1)})g)) \\
&\qquad = a_{(0)} \cdot (S(S(a_{(1)})h)_{(1)} \blacktriangleright b_{(0)}) \otimes S(S(a_{(1)})h)_{(2)} \cdot S(S(b_{(1)})g) \\
&\qquad = (S(a_{(1)})_{(2)}h_{(2)} \triangleright b_{(0)})a_{(0)} \otimes S(S(b_{(1)})g)S(S(a_{(1)})_{(1)}h_{(1)}) \\
&\qquad = S(a_{(1)}) \triangleright (h_{(2)} \triangleright b_{(0)})a_{(0)} \otimes S(S(a_{(2)})h_{(1)}S(b_{(1)})g) \\
&\qquad = a_{(0)}(h_{(2)} \triangleright b_{(0)}) \otimes S(S(a_{(1)})h_{(1)}S(b_{(1)})g) \ ,
\end{aligned}$$

which shows that ψ is in fact an algebra homomorphism. □

The following generalizes Lu's Theorem 4.2.9 to the case of an arbitrary Hopf algebra:

Theorem 4.2.12 *Let H be a Hopf algebra with invertible antipode S and let A be a commutative algebra in the category ${}_H\mathcal{YD}^H$ of left-right Yetter-Drinfeld modules.*
Then there is a Hopf algebroid structure over A on the smash product algebra $A\#H$ and the structure maps are given by

$$\begin{aligned}
s(a) &= a \otimes 1 \\
t(a) &= a_{(0)} \otimes a_{(1)} \\
\Delta(a \otimes h) &= a \otimes h_{(1)} \otimes 1 \otimes h_{(2)} \\
\varepsilon(a \otimes h) &= \varepsilon(h)a \\
\varsigma(a \otimes h) &= S(S(a_{(1)})h)_{(1)} \triangleright a_{(0)} \otimes S(S(a_{(1)})h)_{(2)}
\end{aligned}$$

Proof. Assuming that H is still finite dimensional as in Lu's theorem, we can determine the structure morphisms of $A\#H$ with respect to the induced H-comodule structure on A. In particular, we get

$$t(a) = \sum (h_i^* \cdot a) \otimes h_i = \sum a_{(0)} h_i^*(a_{(1)}) \otimes h_i = a_{(0)} \otimes a_{(1)} \ ,$$

and determine the antipode as

$$\begin{aligned}
\varsigma(a \otimes h) &= \sum (1 \otimes S(h)) t((S^2(h_i)h_i^*) \cdot a) \\
&= \sum (1 \otimes S(h))\ t(S^2(h_i) \triangleright (a_{(0)} h_i^*(a_{(1)}))) \\
&= (1 \otimes S(h))\ t(S^2(a_{(1)}) \triangleright a_{(0)}) \\
&= (1 \otimes S(h))((S^2(a_{(1)}) \triangleright a_{(0)})_{(0)} \otimes (S^2(a_{(1)}) \triangleright a_{(0)})_{(1)}) \ .
\end{aligned}$$

The inverse of the antipode constructed in the proof of Lu's theorem in [26] is given by

$$\begin{aligned}\varsigma^{-1}(a \otimes h) &= (1 \otimes S^{-1}(h))t(a)\\ &= (1 \otimes S^{-1}(h))(a_{(0)} \otimes a_{(1)})\\ &= S^{-1}(h)_{(1)} \triangleright a_{(0)} \otimes S^{-1}(h)_{(2)} a_{(1)} \ .\end{aligned}$$

We claim that the structure morphisms stated above make $A\#H$ into a Hopf algebroid if A is a commutative algebra in the category ${}_H\mathcal{YD}^H$. So we first check that the axioms in Definition 4.1.5 hold.
Clearly, $s : A \longrightarrow A\#H$ is an algebra homomorphism. The target map t is an algebra homomorphism $t : A^{op} \longrightarrow A\#H$, since

$$\begin{aligned}t(a)t(b) &= (a_{(0)} \otimes a_{(1)})(b_{(0)} \otimes b_{(1)})\\ &= a_{(0)}(a_{(1)} \triangleright b_{(0)}) \otimes a_{(2)} b_{(1)}\\ &= b_{(0)} a_{(0)} \otimes a_{(1)} b_{(1)}\\ &= (ba)_{(0)} \otimes (ba)_{(1)}\\ &= t(ba)\end{aligned}$$

for $a, b \in A$ by the commutativity property (4.3). Similarly, we see that the images of s and t commute:

$$\begin{aligned}t(a)s(b) &= (a_{(0)} \otimes a_{(1)})(b \otimes 1)\\ &= a_{(0)}(a_{(1)} \triangleright b) \otimes a_{(2)}\\ &= b a_{(0)} \otimes a_{(1)}\\ &= (b \otimes 1)(a_{(0)} \otimes a_{(1)})\\ &= s(b)t(a) \ .\end{aligned}$$

Hence, the source and target map induce an A-bimodule structure on $A\#H$, that is given by

$$\begin{aligned}b \cdot (a \otimes h) \cdot b' &= s(b)t(b')(a \otimes h)\\ &= (b b'_{(0)} \otimes b'_{(1)})(a \otimes h)\\ &= b b'_{(0)}(b'_{(1)} \triangleright a) \otimes b'_{(2)} h\\ &= b a b'_{(0)} \otimes b'_{(1)} h \ .\end{aligned}$$

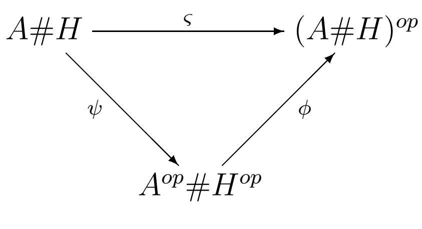
We see immediately that Δ and ε make $A\#H$ into an A-coring, and so it remains to check the identities. Using (4.1), we get

$$\begin{aligned}
\Delta(a\otimes h)(1\otimes 1\otimes s(b)) &= (a\otimes h_{(1)}\otimes 1\otimes h_{(2)})(1\otimes 1\otimes b\otimes 1)\\
&= a\otimes h_{(1)}\otimes h_{(2)}\triangleright b\otimes h_{(3)}\\
&= (a\otimes h_{(1)})\otimes_A (h_{(2)}\triangleright b)\cdot(1\otimes h_{(3)})\\
&= (a\otimes h_{(1)})\cdot(h_{(2)}\triangleright b)\otimes_A (1\otimes h_{(3)})\\
&= a(h_{(2)}\triangleright b)_{(0)}\otimes (h_{(2)}\triangleright b)_{(1)}h_{(1)}\otimes 1\otimes h_{(3)}\\
&= a(h_{(1)}\triangleright b_{(0)})\otimes h_{(2)}b_{(1)}\otimes 1\otimes h_{(3)}\\
&= (a\otimes h_{(1)}\otimes 1\otimes h_{(2)})(b_{(0)}\otimes b_{(1)}\otimes 1\otimes 1)\\
&= \Delta(a\otimes h)(t(b)\otimes 1\otimes 1)\ .
\end{aligned}$$

The remaining identities are straightforward. The formula for the antipode can be simplified further using the Yetter-Drinfeld condition (4.2):

$$\begin{aligned}
\varsigma(a\otimes h) &= (1\otimes S(h))((S^2(a_{(1)})\triangleright a_{(0)})_{(0)}\otimes (S^2(a_{(1)})\triangleright a_{(0)})_{(1)})\\
&= (1\otimes S(h))(S^2(a_{(2)})_{(2)}\triangleright a_{(0)}\otimes S^2(a_{(2)})_{(3)}a_{(1)}S^{-1}(S^2(a_{(2)})_{(1)})\\
&= (1\otimes S(h))(S^2(a_{(3)})\triangleright a_{(0)}\otimes S^2(a_{(4)})a_{(1)}S(a_{(2)}))\\
&= (1\otimes S(h))(S^2(a_{(1)})\triangleright a_{(0)}\otimes S^2(a_{(2)}))\\
&= S(h)_{(1)}\triangleright (S^2(a_{(1)})\triangleright a_{(0)})\otimes S(h)_{(2)}S^2(a_{(2)})\\
&= S(h_{(2)})S^2(a_{(1)})\triangleright a_{(0)}\otimes S(h_{(1)})S^2(a_{(2)})\\
&= S(S(a_{(1)})h)_{(1)}\triangleright a_{(0)}\otimes S(S(a_{(1)})h)_{(2)}\ .
\end{aligned}$$

We have to show that ς satisfies the axioms of an antipode in Definition 4.1.9. In order to verify that $\varsigma : A\#H \longrightarrow (A\#H)^{op}$ is an algebra morphism, we observe that the diagram



commutes, where ϕ is the algebra morphism from Lemma 4.2.10, and ψ is given in Lemma 4.2.11. So since ς is the composition of two algebra morphisms $\phi\circ\psi(a\otimes h) = \phi(a_{(0)}\otimes S(S(a_{(1)})h)) = S(S(a_{(1)})h)_{(1)}\triangleright a_{(0)}\otimes S(S(a_{(1)})h)_{(2)} = \varsigma(a\otimes h)$, it is an algebra homomorphism itself. Since both

ϕ and ψ are isomorphisms, we receive an inverse for ς by

$$\begin{aligned}\varsigma^{-1}(a\otimes h) &= \psi^{-1}\circ\phi^{-1}(a\otimes h)\\ &= \psi^{-1}(S^{-1}(h_{(1)})\triangleright a\otimes h_{(2)})\\ &= (S^{-1}(h_{(1)})\triangleright a)_{(0)}\otimes(S^{-1}(h_{(1)})\triangleright a)_{(1)}S^{-1}(h_{(2)})\\ &= (S^{-1}(h)_{(2)}\triangleright a)_{(0)}\otimes(S^{-1}(h)_{(2)}\triangleright a)_{(1)}S^{-1}(h)_{(1)}\\ &= S^{-1}(h)_{(1)}\triangleright a_{(0)}\otimes S^{-1}(h)_{(2)}a_{(1)}\ ,\end{aligned}$$

where we used (4.1). This is the same formula as in Lu's theorem.

Finally, we check that ς satisfies the properties in Definition 4.1.9. We have for $a\in A$

$$\begin{aligned}\varsigma\circ t(a) &= \varsigma(a_{(0)}\otimes a_{(1)})\\ &= S(S(a_{(1)})a_{(2)})_{(1)}\triangleright a_{(0)}\otimes S(S(a_{(1)})a_{(2)})_{(2)}\\ &= a\otimes 1\\ &= s(a)\ .\end{aligned}$$

By the Yetter-Drinfeld condition (4.2) we compute for $a\otimes h\in A\#H$

$$\begin{aligned}&t\circ\varepsilon\circ\varsigma(a\otimes h) =\\ &\quad= t(S(S(a_{(1)})h)\triangleright a_{(0)})\\ &\quad= (S(S(a_{(1)})h)\triangleright a_{(0)})_{(0)}\otimes(S(S(a_{(1)})h)\triangleright a_{(0)})_{(1)}\\ &\quad= S(S(a_{(2)})h)_{(2)}\triangleright a_{(0)}\otimes S(S(a_{(2)})h)_{(3)}a_{(1)}S^{-1}(S(S(a_{(2)})h)_{(1)})\\ &\quad= S((S(a_{(2)})h)_{(2)})\triangleright a_{(0)}\otimes S((S(a_{(2)})h)_{(1)})a_{(1)}(S(a_{(2)})h)_{(3)}\\ &\quad= S(S(a_{(3)})h_{(2)})\triangleright a_{(0)}\otimes S(S(a_{(4)})h_{(1)})a_{(1)}S(a_{(2)})h_{(3)}\\ &\quad= S(S(a_{(1)})h_{(2)})\triangleright a_{(0)}\otimes S(S(a_{(2)})h_{(1)})h_{(3)}\\ &\quad= S(S(a_{(1)})h_{(1)})_{(1)}\triangleright a_{(0)}\otimes S(S(a_{(1)})h_{(1)})_{(2)}h_{(2)}\\ &\quad= \nabla_{A\#H}(S(S(a_{(1)})h_{(1)})_{(1)}\triangleright a_{(0)}\otimes S(S(a_{(1)})h_{(1)})_{(2)}\otimes 1\otimes h_{(2)})\\ &\quad= \nabla_{A\#H}(\varsigma\otimes\mathrm{id})(a\otimes h_{(1)}\otimes 1\otimes h_{(2)})\\ &\quad= \nabla_{A\#H}\circ(\varsigma\otimes\mathrm{id})\circ\Delta(a\otimes h)\ ,\end{aligned}$$

which is the second property. The map

$$\gamma: A\#H\otimes A\#H\longrightarrow A\#H\otimes A\#H\ ,\ a\otimes h\otimes b\otimes g\mapsto ab_{(0)}\otimes b_{(1)}h\otimes 1\otimes g$$

factors over the quotient $A\#H \otimes_A A\#H$, since for all $x \in A$

$$\begin{aligned}\gamma((a \otimes h) \cdot x \otimes (b \otimes g)) &= \gamma(ax_{(0)} \otimes x_{(1)}h \otimes b \otimes g)\\ &= ax_{(0)}b_{(0)} \otimes b_{(1)}x_{(1)}h \otimes 1 \otimes g\\ &= a(xb)_{(0)} \otimes (xb)_{(1)}h \otimes 1 \otimes g\\ &= \gamma(a \otimes h \otimes xb \otimes g)\\ &= \gamma(a \otimes h \otimes x \cdot (b \otimes g))\ .\end{aligned}$$

The factorization $\bar{\gamma} : A\#H \otimes_A A\#H \longrightarrow A\#H \otimes A\#H$ is a section for the projection $\pi : A\#H \otimes A\#H \longrightarrow A\#H \otimes_A A\#H$, since

$$\begin{aligned}\pi \circ \bar{\gamma}(a \otimes h \otimes b \otimes g) &= \pi(ab_{(0)} \otimes b_{(1)}h \otimes 1 \otimes g)\\ &= \pi((a \otimes h) \cdot b \otimes 1 \otimes g)\\ &= a \otimes h \otimes b \otimes g\ .\end{aligned}$$

Now

$$\begin{aligned}\nabla_{A\#H}(\mathrm{id} \otimes \varsigma)\bar{\gamma}\Delta(a \otimes h) &= \nabla_{A\#H}(\mathrm{id} \otimes \varsigma)(a \otimes h_{(1)} \otimes 1 \otimes h_{(2)})\\ &= \nabla_{A\#H}(a \otimes h_{(1)} \otimes 1 \otimes S(h_{(2)}))\\ &= a \otimes h_{(1)}S(h_{(2)})\\ &= a \otimes \varepsilon(h)1\\ &= s \circ \varepsilon(a \otimes h)\ ,\end{aligned}$$

which finishes the proof. □

We return to the case of a B-torsor T and the left T^B-bialgebroid $T^B\#H^{op}$ arising from it. It turns out that, due to T^B being a commutative Yetter-Drinfeld algebra in $\mathcal{YD}_H^H$, the bialgebroid $T^B\#H^{op}$ is actually a Hopf algebroid.

Theorem 4.2.13 *Let T be a B-torsor and assume that the Hopf algebra H, that coacts on T by Theorem 4.2.2, has an invertible antipode S.*
Then there exists an antipode for the bialgebroid $T^B\#H^{op}$ and it is given by the formula

$$\varsigma(r \otimes h) := S^{-1}(hS^{-1}(r_{(1)}))_{(1)} \triangleright r_{(0)} \otimes S^{-1}(hS^{-1}(r_{(1)}))_{(2)}\ ,$$

where $\triangleright : H^{op} \otimes T^B \longrightarrow T^B$, $h \triangleright r := h^{[1]}rh^{[2]}$ is the Miyashita-Ulbrich action of H^{op} on T^B.

The associated morphism $\vartheta : T^B \longrightarrow T^B$ from Proposition 4.1.11 is given by

$$\vartheta(r) = S^{-2}(r_{(1)}) \triangleright r_{(0)} = S^{-2}(r_{(1)})^{[1]}r_{(0)}S^{-2}(r_{(1)})^{[2]}\ .$$

Proof. We note that S^{-1} is the antipode for the opposite bialgebra H^{op}. We have seen in Section 1.2 that T^B is a commutative algebra in the category $\mathcal{YD}_H^H$ of Yetter-Drinfeld modules, where the right H-comodule structure is given as the restriction of the right H-comodule structure on T and the right H-module structure is the Miyashita-Ulbrich action. It is easy to see that a right-right H-Yetter-Drinfeld module satisfies the left-right H^{op}-Yetter-Drinfeld condition (4.1) with respect to the corresponding H^{op}-module and comodule structure. This implies that T^B is an algebra in the category ${}_{H^{op}}\mathcal{YD}^{H^{op}}$, and is moreover commutative by commutativity of T^B in $\mathcal{YD}_H^H$. Hence, T^B and H^{op} satisfy the conditions of the previous theorem. The resulting bialgebroid structure on $T^B \# H^{op}$ is the same as the one constructed in Proposition 4.2.7.
With respect to the opposite multiplication in H^{op}, the formula for the antipode yields

$$\varsigma(r \otimes h) = S^{-1}(hS^{-1}(r_{(1)}))_{(1)} \triangleleft r_{(0)} \otimes S^{-1}(hS^{-1}(r_{(1)}))_{(2)}$$

for $r \otimes h \in T^B \# H^{op}$. The map ϑ can now be computed via the formula in Proposition 4.1.11 as

$$\begin{aligned}
\vartheta(r) &= \varepsilon \circ \varsigma \circ s(r) \\
&= \varepsilon \circ \varsigma(r \otimes 1) \\
&= \varepsilon(S^{-2}(r_{(1)}) \triangleright r_{(0)} \otimes S^{-2}(r_{(2)})) \\
&= (S^{-2}(r_{(1)}) \triangleright r_{(0)})\varepsilon_H(S^{-2}(r_{(2)})) \\
&= S^{-2}(r_{(1)}) \triangleright r_{(0)} \\
&= S^{-2}(r_{(1)})^{[1]} r_{(0)} S^{-2}(r_{(1)})^{[2]} \ .
\end{aligned}$$

□

Proposition 4.2.14 *The antipode ς of the Hopf algebroid $T^B \# H^{op}$ has the inverse*

$$\varsigma^{-1} : T^B \# H^{op} \longrightarrow T^B \# H^{op} \ , \ r \otimes h \mapsto S(h_{(2)}) \triangleright r_{(0)} \otimes r_{(1)} S(h_{(1)}) \ .$$

The inverse for the underlying morphism $\vartheta : T^B \longrightarrow T^B$ is given by

$$\vartheta^{-1} : T^B \longrightarrow T^B \ , \ r \mapsto S(r_{(1)}) \triangleright r_{(0)} = S(r_{(1)})^{[1]} r_{(0)} S(r_{(1)})^{[2]} \ .$$

Proof. The inverse of ς was computed in the proof of Theorem 4.2.12, and a straightforward computation shows that the inverse of ϑ is given by the second map. □

Remark 4.2.15 In Proposition 4.1.11 the underlying map ϑ of the antipode ς is defined via the property $t \circ \vartheta = \varsigma \circ s$.
Applying this on the source map $s(r) = r \otimes 1$ and target map $t(r) = r_{(0)} \otimes r_{(1)} = \delta(r)$ of $T^B \# H^{op}$ that we determined in Proposition 4.2.7, we obtain the equality

$$\vartheta(r)_{(0)} \otimes \vartheta(r)_{(1)} = \vartheta(r_{(0)}) \otimes S^{-2}(r_{(1)})$$

for all $r \in T^B$.

We note that the formula for ϑ in Theorem 4.2.12 equals the formula (1.22) for the inverse of the Grunspan map θ of a quantum torsor, i.e. in the case $B = k$. Since the Miyashita-Ulbrich action, and therefore also this formula, is really just well-defined on the centralizer T^B (this is due to $h^{[1]} \otimes h^{[2]} \in T \otimes_B T$ for all $h \in H$), we cannot expect to have such a map defined on the whole of T. Nevertheless, we will see that the map $\theta : T^B \longrightarrow T^B$ interacts in a Hopf algebroid structure for the bialgebroid $((T \otimes_B T)^B)^{op}$. This connection can then be used to give a new interpretation for the axioms of the Grunspan map.

We continue recovering some more Hopf algebroid structures:

By Lemma 4.2.10, there exists an algebra isomorphism ϕ between the opposite algebra $(T^B \# H^{op})^{op}$ and the smash product algebra $(T^B)^{op} \# H$, where $(T^B)^{op}$ becomes a left H-module algebra by

$$h \blacktriangleright r := S(h) \triangleright r = S(h)^{[1]} r S(h)^{[2]}$$

(here we have to use that the antipode of H^{op} is given by S^{-1}).
Moreover, defining a right H-comodule structure on $(T^B)^{op}$ by

$$\delta(t) = t_{<0>} \otimes t_{<1>} := t_{(0)} \otimes S^{-2}(t_{(1)}) \ ,$$

we receive a smash product algebra structure of the type considered in Theorem 4.2.12 and hence a Hopf algebroid structure on $(T^B)^{op} \# H$:

Proposition 4.2.16 *The smash product algebra $(T^B)^{op} \# H$ is a left Hopf algebroid over $(T^B)^{op}$ with the structure maps*

$$\begin{aligned} s(r) &= r \otimes 1 \\ t(r) &= r_{(0)} \otimes S^{-2}(r_{(1)}) \\ \Delta(r \otimes h) &= r \otimes h_{(1)} \otimes 1 \otimes h_{(2)} \\ \varepsilon(r \otimes h) &= \varepsilon(h) r \\ \varsigma(r \otimes h) &= S(S(h_{(2)}) r_{(1)}) \triangleright r_{(0)} \otimes S(h_{(1)}) r_{(2)} \end{aligned}$$

Proof. We have to show that $(T^B)^{op}$ is a commutative algebra in the category ${}_H\mathcal{YD}^H$. It satisfies the left-right Yetter-Drinfeld condition

$$h_{(1)} \blacktriangleright r_{<0>} \otimes h_{(2)} r_{<1>} = (h_{(2)} \blacktriangleright r)_{<0>} \otimes (h_{(2)} \blacktriangleright r)_{<1>} h_{(1)}$$

for $h \in H$ and $r \in (T^B)^{op}$, since

$$\begin{aligned}
&(h_{(2)} \blacktriangleright r)_{<0>} \otimes (h_{(2)} \blacktriangleright r)_{<1>} h_{(1)} = \\
&\quad = (S(h_{(2)}) \triangleright r)_{(0)} \otimes S^{-2}((S(h_{(2)}) \triangleright r)_{(1)}) h_{(1)} \\
&\quad = S(h_{(2)})^{[1]}{}_{(0)} r_{(0)} S(h_{(2)})^{[2]}{}_{(0)} \otimes S^{-2}(S(h_{(2)})^{[1]}{}_{(1)} r_{(1)} S(h_{(2)})^{[2]}{}_{(1)}) h_{(1)} \\
&\quad = S(h_{(2)})_{(2)}{}^{[1]} r_{(0)} S(h_{(2)})_{(2)}{}^{[2]} \otimes S^{-2}(S(S(h_{(2)})_{(1)}) r_{(1)} S(h_{(2)})_{(3)}) h_{(1)} \\
&\quad = S(h_{(3)})^{[1]} r_{(0)} S(h_{(3)})^{[2]} \otimes S^{-2}(S^2(h_{(4)}) r_{(1)} S(h_{(2)})) h_{(1)} \\
&\quad = S(h_{(3)}) \triangleright r_{(0)} \otimes h_{(4)} S^{-2}(r_{(1)}) S^{-1}(h_{(2)}) h_{(1)} \\
&\quad = h_{(1)} \blacktriangleright r_{(0)} \otimes h_{(2)} S^{-2}(r_{(1)}) \\
&\quad = h_{(1)} \blacktriangleright r_{<0>} \otimes h_{(2)} r_{<1>}
\end{aligned}$$

by (1.4) and (1.5), and it is obvious that $(T^B)^{op}$ is an algebra in ${}_H\mathcal{YD}^H$. As we have already verified in the proof of Theorem 4.2.13, T^B is a commutative algebra in the category of Yetter-Drinfeld modules, and hence $rs = s_{(0)}(s_{(1)} \triangleright r)$ for all $r, s \in T^B$. This implies commutativity of $(T^B)^{op}$ in ${}_H\mathcal{YD}^H$, since

$$\begin{aligned}
s_{<0>} \cdot (s_{<1>} \blacktriangleright r) &= (S^{-2}(s_{(1)}) \blacktriangleright r) s_{(0)} \\
&= (S^{-1}(s_{(1)}) \triangleright r) s_{(0)} \\
&= s_{(0)}(s_{(1)} \triangleright (S^{-1}(s_{(2)}) \triangleright r)) \\
&= s_{(0)}(S^{-1}(s_{(2)}) s_{(1)} \triangleright r) \\
&= sr \\
&= r \cdot s
\end{aligned}$$

for all $r, s \in (T^B)^{op}$. Now we can apply Theorem 4.2.13 to obtain the structure morphisms for the Hopf algebroid $(T^B)^{op} \# H$ as stated above. In particular, the antipode is given by

$$\begin{aligned}
\varsigma(r \otimes h) &= S(S(r_{<1>})h)_{(1)} \blacktriangleright r_{<0>} \otimes S(S(r_{<1>})h)_{(2)} \\
&= S(S(S^{-2}(r_{(1)}))h)_{(1)} \blacktriangleright r_{(0)} \otimes S(S(S^{-2}(r_{(1)}))h)_{(2)} \\
&= S(S^{-1}(r_{(1)})h)_{(1)} \blacktriangleright r_{(0)} \otimes S(S^{-1}(r_{(1)})h)_{(2)} \\
&= S(S(h_{(2)}) r_{(1)}) \triangleright r_{(0)} \otimes S(h_{(1)}) r_{(2)} \ .
\end{aligned}$$

□

Now we can use the isomorphism

$$\phi : (T^B)^{op}\#H \longrightarrow (T^B\#H^{op})^{op}\ ,\ r\otimes h \mapsto h_{(1)} \triangleright r \otimes h_{(2)}$$

from Lemma 4.2.10 to transfer the left $(T^B)^{op}$-bialgebroid structure on $(T^B)^{op}\#H$ to the opposite algebra $(T^B\#H^{op})^{op}$, making it a left $(T^B)^{op}$-bialgebroid.
This yields a structure which is different from the canonical right T^B-bialgebroid structure on $(T^B\#H^{op})^{op}$ that is imposed by Remark 4.1.7.

Proposition 4.2.17 *The isomorphism ϕ induces a left $(T^B)^{op}$-bialgebroid structure on the opposite algebra $(T^B\#H^{op})^{op}$ with the structure maps*

- *source map* $\tilde{s}(r) = r \otimes 1$
- *target map* $\tilde{t}(r) = S^{-2}(r_{(1)}) \triangleright r_{(0)} \otimes S^{-2}(r_{(2)})$
- *comultiplication* $\tilde{\Delta}(r\otimes h) = r \otimes h_{(1)} \otimes 1 \otimes h_{(2)}$
- *counit* $\tilde{\varepsilon}(r \otimes h) = S(h) \triangleright r$

We are going to denote this bialgebroid by ${}^{op}(T^B\#H^{op})$.

Proof. We apply the isomorphism ϕ on the structure morphisms of $(T^B)^{op}\#H$ that we computed in Proposition 4.2.16, and obtain the source map as

$$\tilde{s}(r) = \phi \circ s(r) = \phi(r \otimes 1) = r \otimes 1\ ,$$

and the target map as

$$\begin{aligned}\tilde{t}(r) = \phi \circ t(r) &= \phi(r_{(0)} \otimes S^{-2}(r_{(1)}))\\ &= S^{-2}(r_{(1)})_{(1)} \triangleright r_{(0)} \otimes S^{-2}(r_{(1)})_{(2)}\\ &= S^{-2}(r_{(1)}) \triangleright r_{(0)} \otimes S^{-2}(r_{(2)})\end{aligned}$$

for $r \in (T^B)^{op}$. As in the proof of Proposition 4.2.7, we get the comonoid structure maps for $r \otimes h \in (T^B\#H^{op})^{op}$ by

$$\begin{aligned}\tilde{\Delta}(r\otimes h) &= (\phi \otimes_{T^B} \phi)\Delta\phi^{-1}(r\otimes h)\\ &= (\phi \otimes_{T^B} \phi)\Delta(h_{(1)} \blacktriangleright r \otimes h_{(2)})\\ &= (\phi \otimes_{T^B} \phi)(h_{(1)} \blacktriangleright r \otimes h_{(2)} \otimes 1 \otimes h_{(3)})\\ &= h_{(2)} \triangleright (S(h_{(1)} \triangleright r) \otimes h_{(2)} \otimes h_{(4)} \triangleright 1 \otimes h_{(5)}\\ &= S(h_{(1)})h_{(2)} \triangleright r \otimes h_{(3)} \otimes 1 \otimes h_{(4)}\\ &= r \otimes h_{(1)} \otimes 1 \otimes h_{(2)}\end{aligned}$$

and

$$\begin{aligned}\varepsilon(r \otimes h) &= \tilde{\varepsilon} \circ \phi^{-1}(r \otimes h) \\ &= \tilde{\varepsilon}(h_{(1)} \blacktriangleright r \otimes h_{(2)}) \\ &= \varepsilon_H(h_{(2)})h_{(1)} \blacktriangleright r \\ &= S(h) \triangleright r \ .\end{aligned}$$

□

Recall from Remark 4.1.3 how one can construct to a $\times_A$-bialgebra L its coopposite $\times_{\bar{A}}$-bialgebra L^{cop}. In the language of bialgebroids this means that given a left A-bialgebroid L, we obtain the coopposite left A^{op}-bialgebroid L^{cop} by interchanging the source and the target map and composing the comultiplication with a twist.
We apply this construction to the left $(T^B)^{op}$-bialgebroid ${}^{op}(T^B \# H^{op})$:

Corollary 4.2.18 *The coopposite left bialgebroid* ${}^{op}(T^B \# H^{op})^{cop}$ *over* T^B *has the structure maps*

- *source map* $s^{opcop}(r) = S^{-2}(r_{(1)}) \triangleright r_{(0)} \otimes S^{-2}(r_{(2)})$
- *target map* $t^{opcop}(r) = r \otimes 1$
- *comultiplication* $\Delta^{opcop}(r \otimes h) = 1 \otimes h_{(2)} \otimes r \otimes h_{(1)}$
- *counit* $\varepsilon^{opcop}(r \otimes h) = S(h) \triangleright r$

Remark 4.2.19 The T^B-bimodule structure on ${}^{op}(T^B \# H^{op})^{cop}$ that is induced by the source resp. the target map is given by

$$\begin{aligned}x \cdot (r \otimes h) = s^{opcop}(x) \cdot (r \otimes h) &= (S^{-2}(x_{(1)}) \triangleright x_{(0)} \otimes S^{-2}(x_{(2)})) \cdot (r \otimes h) \\ &= (r \otimes h)(S^{-2}(x_{(1)}) \triangleright x_{(0)} \otimes S^{-2}(x_{(2)})) \\ &= r(h_{(1)} \triangleright S^{-2}(x_{(1)}) \triangleright x_{(0)}) \otimes S^{-2}(x_{(2)})h_{(2)} \\ &= r(S^{-2}(x_{(1)})h_{(1)} \triangleright x_{(0)}) \otimes S^{-2}(x_{(2)})h_{(2)}\end{aligned}$$

and

$$\begin{aligned}(r \otimes h) \cdot x = t^{opcop}(x) \cdot (r \otimes h) &= (x \otimes 1) \cdot (r \otimes h) \\ &= (r \otimes h)(x \otimes 1) \\ &= r(h_{(1)} \triangleright x) \otimes h_{(2)}\end{aligned}$$

for $x \in T^B$ and $r \otimes h \in T^B \# H^{op}$.

So far we have obtained a left T^B-Hopf algebroid $T^B\#H^{op}$ in Proposition 4.2.13, and a structure of left T^B-bialgebroid on its opposite algebra, namely the bialgebroid ${}^{op}(T^B\#H^{op})^{cop}$ of the previous corollary. Our next result shows that the antipode ς of $T^B\#H^{op}$ has a property that generalizes in a certain sense a well-known fact for Hopf algebras: It is a bialgebroid morphism from $T^B\#H^{op}$ to ${}^{op}(T^B\#H^{op})^{cop}$. This is consistent with the fact that ς is by definition an algebra anti-morphism.

We need the following lemma for the proof:

Lemma 4.2.20 *Let $x \otimes y \in (T \otimes_B T)^B$. Then*

$$S^{-2}(x_{(1)})^{[1]}x_{(0)} \otimes yS^{-2}(r_{(1)})^{[2]} \in B \otimes_B T \subset T \otimes_B T \ .$$

In particular, we have $S^{-2}(r_{(1)})^{[1]}r_{(0)} \otimes S^{-2}(r_{(1)})^{[2]} \in B \otimes_B T$ for $r \in T^B$.

Proof. We note that the expression is well-defined by faithful flatness of $(T \otimes_B T)^B$. Applying the H-comodule structure map to the left tensorand, we obtain by (1.5) that

$$\begin{aligned}
(\delta \otimes_B \mathrm{id})(S^{-2}(x_{(1)})^{[1]}x_{(0)} \otimes_B yS^{-2}(x_{(1)})^{[2]}) = \\
&= S^{-2}(x_{(2)})^{[1]}{}_{(0)}x_{(0)} \otimes S^{-2}(x_{(2)})^{[1]}{}_{(1)}x_{(1)} \otimes_B yS^{-2}(x_{(2)})^{[2]} \\
&= S^{-2}(x_{(3)})^{[1]}x_{(0)} \otimes S^{-1}(x_{(2)})x_{(1)} \otimes_B yS^{-2}(x_{(3)})^{[2]} \\
&= S^{-2}(x_{(1)})^{[1]}x_{(0)} \otimes 1 \otimes_B yS^{-2}(x_{(1)})^{[2]} \\
&\in (T \otimes H) \otimes_B T \subset T \otimes_B T \ .
\end{aligned}$$

Since T is a faithfully flat right B-module and $T^{coH} = B$, we conclude that $S^{-2}(x_{(1)})^{[1]}x_{(0)} \otimes yS^{-2}(x_{(1)})^{[2]} \in B \otimes_B T$. □

Proposition 4.2.21 *The antipode of the Hopf algebroid $T^B\#H^{op}$ is a bialgebroid morphism*

$$\begin{aligned}
\varsigma : T^B\#H^{op} &\longrightarrow {}^{op}(T^B\#H^{op})^{cop} \\
r \otimes h &\mapsto S^{-1}(hS^{-1}(r_{(1)}))_{(1)} \triangleleft r_{(0)} \otimes S^{-1}(hS^{-1}(r_{(1)}))_{(2)} \ .
\end{aligned}$$

Proof. By Definition 4.1.8 we have to show that ς is a T^B-bimodule morphism that is compatible with both comonoid structures. With respect to the bimodule structures described in Remark 4.2.8 resp. 4.2.19, we obtain for $x \in T^B$ and $r \otimes h \in T^B \# H^{op}$

$$\begin{aligned}
&x \cdot \varsigma(r \otimes h) = \\
&\quad = (S^{-1}(hS^{-1}(r_{(1)}))_{(1)} \triangleright r_{(0)})(S^{-2}(x_{(1)})S^{-1}(hS^{-1}(r_{(1)}))_{(2)} \triangleright x_{(0)}) \otimes \\
&\qquad \otimes S^{-2}(x_{(2)})S^{-1}(hS^{-1}(r_{(1)}))_{(3)} = \\
&\quad = S^{-1}(hS^{-1}(r_{(1)}))_{(1)} \triangleright (r_{(0)}(S^{-2}(x_{(1)}) \triangleright x_{(0)})) \otimes \\
&\qquad \otimes S^{-2}(x_{(2)})S^{-1}(hS^{-1}(r_{(1)}))_{(2)} = \\
&\quad = S^{-1}(hS^{-1}(r_{(1)}))_{(1)} \triangleright (r_{(0)}S^{-2}(x_{(1)})^{[1]}x_{(0)}S^{-2}(x_{(1)})^{[2]}) \otimes \\
&\qquad \otimes S^{-2}(x_{(2)})S^{-1}(hS^{-1}(r_{(1)}))_{(2)} = \\
&\quad = S^{-1}(hS^{-1}(r_{(1)}))_{(1)} \triangleright (S^{-2}(x_{(1)})^{[1]}x_{(0)}r_{(0)}S^{-2}(x_{(1)})^{[2]}) \otimes \\
&\qquad \otimes S^{-2}(x_{(2)})S^{-1}(hS^{-1}(r_{(1)}))_{(2)} = \\
&= S^{-2}(x_{(1)})S^{-1}(hS^{-1}(r_{(1)}))_{(1)} \triangleright (x_{(0)}r_{(0)}) \otimes S^{-2}(x_{(2)})S^{-1}(hS^{-1}(r_{(1)}))_{(2)} = \\
&\quad = S^{-1}(hS^{-1}(x_{(1)}r_{(1)}))_{(1)} \triangleright (x_{(0)}r_{(0)}) \otimes S^{-1}(hS^{-1}(x_{(1)}r_{(1)}))_{(2)} = \\
&\quad = S^{-1}(hS^{-1}((xr)_{(1)}))_{(1)} \triangleright (xr)_{(0)} \otimes S^{-1}(hS^{-1}((xr)_{(1)}))_{(2)} = \\
&\qquad = \varsigma(xr \otimes h) = \varsigma(x \cdot (r \otimes h)) \ ,
\end{aligned}$$

using Lemma 4.2.20 for the fourth equality. Letting $x \in T^B$ act on the right, we have

$$\begin{aligned}
&\varsigma((r \otimes h) \cdot x) = S(rx_{(0)} \otimes hx_{(1)})) = \\
&= S^{-1}(hx_{(2)}S^{-1}(r_{(1)}x_{(1)}))_{(1)} \triangleright r_{(0)}x_{(0)} \otimes S^{-1}(hx_{(2)}S^{-1}(r_{(1)}x_{(1)}))_{(2)} \\
&= S^{-1}(hS^{-1}(r_{(1)}))_{(1)} \triangleright r_{(0)}x \otimes S^{-1}(hS^{-1}(r_{(1)}))_{(2)} \\
&= (S^{-1}(hS^{-1}(r_{(1)}))_{(1)} \triangleright r_{(0)})(S^{-1}(hS^{-1}(r_{(1)}))_{(2)} \triangleright x) \otimes S^{-1}(hS^{-1}(r_{(1)}))_{(3)} \\
&= \varsigma(r \otimes h) \cdot x \ .
\end{aligned}$$

This shows that ς is a morphism of T^B-bimodules. Finally, we prove compatibility of ς with the comonoid structures of $T^B \# H^{op}$ and ${}^{op}(T^B \# H^{op})^{cop}$. We have

$$\begin{aligned}
\varepsilon^{opcop} \circ \varsigma(r \otimes h) &= S(S^{-1}(hS^{-1}(r_{(1)}))_{(2)}) \triangleright S^{-1}(hS^{-1}(r_{(1)}))_{(1)} \triangleright r_{(0)} \\
&= (S^{-1}(hS^{-1}(r_{(1)}))_{(1)}S(S^{-1}(hS^{-1}(r_{(1)}))_{(2)})) \triangleright r_{(0)} \\
&= \varepsilon(hS^{-1}(r_{(1)}))r_{(0)} \\
&= \varepsilon(h)r \\
&= \varepsilon(r \otimes h)
\end{aligned}$$

and

$$
\begin{aligned}
(\varsigma \otimes_{T^B} \varsigma)\Delta(r \otimes h) &= (\varsigma \otimes_{T^B} \varsigma)(r \otimes h_{(1)} \otimes 1 \otimes h_{(2)}) = \\
&= \varsigma(r \otimes h_{(1)}) \otimes 1 \otimes S^{-1}(h_{(2)}) = \\
&= S^{-1}(h_{(1)}S^{-1}(r_{(1)}))_{(1)} \triangleright r_{(0)} \otimes S^{-1}(h_{(1)}S^{-1}(r_{(1)}))_{(2)} \otimes 1 \otimes S^{-1}(h_{(2)}) = \\
&= (1 \otimes S^{-1}(h_{(1)}S^{-1}(r_{(1)}))) \cdot r_{(0)} \otimes 1 \otimes S^{-1}(h_{(2)}) = \\
&= 1 \otimes S^{-1}(h_{(1)}S^{-1}(r_{(1)})) \otimes r_{(0)} \cdot (1 \otimes S^{-1}(h_{(2)})) = \\
&= 1 \otimes S^{-1}(h_{(1)}S^{-1}(r_{(3)})) \otimes S^{-2}(r_{(1)})S^{-1}(h_{(2)})_{(1)} \triangleright r_{(0)} \otimes \\
&\qquad \otimes S^{-2}(r_{(2)})S^{-1}(h_{(2)})_{(2)} = \\
&= 1 \otimes S^{-1}(h_{(1)}S^{-1}(r_{(3)})) \otimes S^{-1}(h_{(3)}S^{-1}(r_{(1)})) \triangleright r_{(0)} \otimes S^{-1}(h_{(2)}S^{-1}(r_{(2)})) = \\
&= 1 \otimes S^{-1}(hS^{-1}(r_{(1)}))_{(3)} \otimes S^{-1}(hS^{-1}(r_{(1)}))_{(1)} \triangleright r_{(0)} \otimes S^{-1}(hS^{-1}(r_{(1)}))_{(2)} = \\
&= \Delta^{opcop} \circ \varsigma(r \otimes h) \ .
\end{aligned}
$$

This proves the claim. □

We could now expect the antipode ς to be also an antipode for the bialgebroid ${}^{op}(T^B \# H^{op})^{cop}$. But although it is easy to show that ς, considered as a map $\varsigma : {}^{op}(T^B \# H^{op})^{cop} \longrightarrow {}^{op}(T^B \# H^{op})^{cop}$, satisfies the first two axioms in Definition 4.1.9, there seems to be no good candidate for a section $\gamma : {}^{op}(T^B \# H^{op})^{cop} \otimes_{T^B} {}^{op}(T^B \# H^{op})^{cop} \longrightarrow {}^{op}(T^B \# H^{op})^{cop} \otimes {}^{op}(T^B \# H^{op})^{cop}$ of the canonical projection such that the third property would hold.

Nevertheless, we can now return to the left T^B-bialgebroid $((T \otimes_B T)^B)^{op}$ that is isomorphic to $T^B \# H^{op}$ via the Galois map β. The antipode ς of $T^B \# H^{op}$ can be transfered back to $((T \otimes_B T)^B)^{op}$ as follows:

Proposition 4.2.22 *The left T^B-bialgebroid $((T \otimes_B T)^B)^{op}$ is a Hopf algebroid with the antipode given by the algebra morphism*

$$\bar{\varsigma} : ((T \otimes_B T)^B)^{op} \longrightarrow (T \otimes_B T)^B \ , \ x \otimes y \mapsto S^{-2}(x_{(1)})^{[1]} x_{(0)} y \otimes S^{-2}(x_{(1)})^{[2]} \ .$$

The underlying map of the antipode ς is given by

$$\bar{\vartheta} : T^B \longrightarrow T^B \ , \ r \mapsto S^{-2}(r_{(1)})^{[1]} r_{(0)} S^{-2}(r_{(1)})^{[2]} = \vartheta(r) \ .$$

Proof. Since the bialgebroid structure of $T^B \# H^{op}$ was constructed such that the Galois map β became a bialgebroid morphism, we obtain an antipode $\bar{\varsigma}$ for $((T \otimes_B T)^B)^{op}$ such that the diagram

$$
\begin{array}{ccc}
((T \otimes_B T)^B)^{op} & \xrightarrow{\ \beta\ } & T^B \# H^{op} \\
\Big\downarrow{\scriptstyle \bar{\varsigma}} & & \Big\downarrow{\scriptstyle \varsigma} \\
(T \otimes_B T)^B & \xrightarrow[\ \beta\ ]{} & (T^B \# H^{op})^{op}
\end{array}
$$

commutes. Hence, we can calculate $\bar{\varsigma}$ for $x \otimes y \in ((T \otimes_B T)^B)^{op}$ as

$$\begin{aligned}
&\beta^{-1} \circ \varsigma \circ \beta(x \otimes y) = \beta^{-1} \circ \varsigma(xy_{(0)} \otimes y_{(1)}) = \\
&= \beta^{-1}(S^{-1}(y_{(2)}S^{-1}(x_{(1)}y_{(1)}))_{(1)} \rhd x_{(0)}y_{(0)} \otimes S^{-1}(y_{(2)}S^{-1}(x_{(1)}y_{(1)}))_{(2)}) = \\
&= \beta^{-1}(S^{-2}(x_{(1)})_{(1)} \rhd x_{(0)}y \otimes S^{-2}(x_{(1)})_{(2)}) = \\
&= (S^{-2}(x_{(1)})_{(1)} \rhd x_{(0)}y)S^{-2}(x_{(1)})_{(2)}{}^{[1]} \otimes S^{-2}(x_{(1)})_{(2)}{}^{[2]} = \\
&= S^{-2}(x_{(1)})_{(1)}{}^{[1]}x_{(0)}yS^{-2}(x_{(1)})_{(1)}{}^{[2]}S^{-2}(x_{(1)})_{(2)}{}^{[1]} \otimes S^{-2}(x_{(1)})_{(2)}{}^{[2]} = \\
&= S^{-2}(x_{(1)})^{[1]}x_{(0)}y \otimes S^{-2}(x_{(1)})^{[2]}
\end{aligned}$$

using (1.9). Since β is an isomorphism of bialgebroids, it is clear that $\bar{\varsigma}$ should satisfy the axioms of an antipode. The first two axioms can be easily checked by

$$(\bar{\varsigma} \circ t)(r) = \bar{\varsigma}(1 \otimes r) = r \otimes 1 = s(r)$$

for all $r \in T^B$, and

$$\begin{aligned}
\nabla^{op}(\bar{\varsigma} \otimes \mathrm{id})\Delta(x \otimes y) &= \nabla^{op}(\bar{\varsigma} \otimes \mathrm{id})(x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)}) \\
&= \nabla^{op}(S^{-2}(x_{(1)})^{[1]}x_{(0)}y^{(1)} \otimes S^{-2}(x_{(1)})^{[2]} \otimes y^{(2)} \otimes y^{(3)}) \\
&= S^{-2}(x_{(1)})^{[1]}x_{(0)}y^{(1)}y^{(2)} \otimes y^{(3)}S^{-2}(x_{(1)})^{[2]} \\
&= S^{-2}(x_{(1)})^{[1]}x_{(0)} \otimes yS^{-2}(x_{(1)})^{[2]} \\
&= 1 \otimes S^{-2}(x_{(1)})^{[1]}x_{(0)}yS^{-2}(x_{(1)})^{[2]} \\
&= t(S^{-2}(x_{(1)})^{[1]}x_{(0)}yS^{-2}(x_{(1)})^{[2]}) \\
&= (t \circ \varepsilon \circ \bar{\varsigma})(x \otimes y)
\end{aligned}$$

for all $x \otimes y \in ((T \otimes_B T)^B)^{op}$ by Lemma 4.2.20.
A section for the canonical projection $((T \otimes_B T)^B)^{op} \otimes ((T \otimes_B T)^B)^{op} \longrightarrow ((T \otimes_B T)^B)^{op} \otimes_{T^B} ((T \otimes_B T)^B)^{op}$ is given by $\gamma(x \otimes y \otimes v \otimes w) := x \otimes yvw^{(1)} \otimes x^{(2)} \otimes w^{(3)}$, and so we have

$$\begin{aligned}
&\nabla^{op}(\mathrm{id} \otimes \bar{\varsigma})\gamma\Delta(x \otimes y) = \nabla^{op}(\mathrm{id} \otimes \bar{\varsigma})\gamma(x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)}) \\
&= \nabla^{op}(\mathrm{id} \otimes \bar{\varsigma})(x \otimes y^{(1)}y^{(2)}y^{(3)(1)} \otimes y^{(3)(2)} \otimes y^{(3)(3)}) \\
&= \nabla^{op}(\mathrm{id} \otimes \bar{\varsigma})(x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)}) \\
&= \nabla^{op}(\mathrm{id} \otimes \bar{\varsigma})(x \otimes y_{(0)} \otimes y_{(1)}{}^{[1]} \otimes y_{(1)}{}^{[2]}) \\
&= \nabla^{op}(x \otimes y_{(0)} \otimes S^{-2}(y_{(1)}{}^{[1]}{}_{(1)})^{[1]}y_{(1)}{}^{[1]}{}_{(0)}y_{(1)}{}^{[2]} \otimes S^{-2}(y_{(1)}{}^{[1]}{}_{(1)})^{[2]}) \\
&= xS^{-1}(y_{(1)})^{[1]}y_{(2)}{}^{[1]}y_{(2)}{}^{[2]} \otimes S^{-1}(y_{(1)})^{[2]}y_{(0)} \\
&= xS^{-1}(y_{(1)})^{[1]} \otimes S^{-1}(y_{(1)})^{[2]}y_{(0)} \\
&= xy \otimes 1 \\
&= s \circ \varepsilon(x \otimes y)
\end{aligned}$$

by (1.5) and (1.11). So indeed, $((T \otimes_B T)^B)^{op}$ is a left T^B-Hopf algebroid. □

This result shows that the coaction of a Hopf algebra on each B-torsor leads to an antipode for the bialgebroid $((T \otimes_B T)^B)^{op}$.
We observe that the formula for the antipode $\bar{\varsigma}$ of $((T \otimes_B T)^B)^{op}$ is such, that applied to $r \otimes s \in (T \otimes_B T)^B$ with $r, s \in T^B$, we have

$$\begin{aligned}
\bar{\varsigma}(r \otimes s) &= S^{-2}(r_{(1)})^{[1]} r_{(0)} s \otimes S^{-2}(r_{(1)})^{[2]} \\
&= s S^{-2}(r_{(1)})^{[1]} r_{(0)} \otimes S^{-2}(r_{(1)})^{[2]} \\
&= s \otimes S^{-2}(r_{(1)})^{[1]} r_{(0)} S^{-2}(r_{(1)})^{[2]} \\
&= s \otimes \vartheta(r)
\end{aligned}$$

by Lemma 4.2.20. So in this special case the expression (1.22) appears as a part of the antipode $\bar{\varsigma}$. Moreover, this formula resembles very much the form of the antipode for the Hopf algebra $H_r(T)$ of a quantum torsor T (in the case $B = k$) in Theorem 1.4.5. We are going to investigate this relationship in the following section.

4.3 Grunspan Axioms for B-Torsors

We keep the assumptions of the previous section and let T be a B-torsor that is right faithfully flat over B and assume that $(T \otimes_B T)^B$ is faithfully flat over k.
In the previous section we have seen that the underlying map of both antipodes ς for $T^B \# H^{op}$ and $\bar{\varsigma}$ for $((T \otimes_B T)^B)^{op}$ is given by $\vartheta : T^B \longrightarrow T^B$. This map has the same formula (1.22) as the inverse of the Grunspan map θ for a quantum torsor. The reason why we obtained a formula for the inverse, rather than of the Grunspan map (1.21) itself, was that we had to consider the coaction of the opposite Hopf algebra H^{op}.
But it turns out that the formula (1.21) plays a role in a Hopf algebroid structure on the right T^B-bialgebroid $(T \otimes_B T)^B$ that we originally constructed in Proposition 4.2.3.

Proposition 4.3.1 *The right T^B-bialgebroid $(T \otimes_B T)^B$ that is associated with each B-torsor by Proposition 4.2.3 is a right Hopf algebroid. Its antipode is given by the algebra morphism*

$$\Theta : (T \otimes_B T)^B \longrightarrow ((T \otimes_B T)^B)^{op} \ , \ x \otimes y \mapsto S(y_{(1)})^{[1]} \otimes x y_{(0)} S(y_{(1)})^{[2]} \ ,$$

where S is the antipode of the Hopf algebra $H \subset (T \otimes_B T)^B$ from Theorem 4.2.2. The underlying map of the antipode Θ is

$$\theta : T^B \longrightarrow T^B \ , \ r \mapsto S(r_{(1)})^{[1]} r_{(0)} S(r_{(1)})^{[2]} \ .$$

Proof. By the axioms in Definition 4.1.9 and the analogy between left and right bialgebroids, it is clear that we should call $(T \otimes_B T)^B$ a right Hopf algebroid, if there exists an algebra anti-morphism $\Theta : (T \otimes_B T)^B \longrightarrow (T \otimes_B T)^B$ such that $\Theta \circ t = s$, $\nabla(\mathrm{id} \otimes \Theta)\Delta = t \circ \varepsilon \circ \Theta$ and if there exists a section $\gamma : (T \otimes_B T)^B \otimes_{T^B} (T \otimes_B T)^B \longrightarrow (T \otimes_B T)^B \otimes (T \otimes_B T)^B$ for the canonical projection such that $\nabla(\Theta \otimes \mathrm{id})\gamma\Delta = s \circ \varepsilon$ holds.
We note that the expression $S(y_{(1)})^{[1]} \otimes x y_{(0)} S(y_{(1)})^{[2]}$ is well-defined. The first property is obviously satisfied, and we obtain the second one as

$$\begin{aligned}
\nabla(\mathrm{id} \otimes \Theta)\Delta(x \otimes y) &= \nabla(\mathrm{id} \otimes \Theta)(x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(2)}) \\
&= \nabla(x \otimes y^{(1)} \otimes S({y^{(3)}}_{(1)})^{[1]} \otimes y^{(2)} {y^{(3)}}_{(0)} S({y^{(3)}}_{(1)})^{[2]}) \\
&= S({y^{(3)}}_{(1)})^{[1]} x \otimes y^{(1)} y^{(2)} {y^{(3)}}_{(0)} S({y^{(3)}}_{(1)})^{[2]} \\
&= S(y_{(1)})^{[1]} x \otimes y_{(0)} S(y_{(1)})^{[2]} \\
&= S(y_{(1)})^{[1]} x y_{(0)} S(y_{(1)})^{[2]} \otimes 1 \\
&= (t \circ \varepsilon \circ \theta)(x \otimes y) \ ,
\end{aligned}$$

using that $S(y_{(1)})^{[1]} x \otimes y_{(0)} S(y_{(1)})^{[2]} \in T \otimes_B B \subset T \otimes_B T$ by a proof similar to the one in Lemma 4.2.20.
A section $\gamma : (T \otimes_B T)^B \otimes_{T^B} (T \otimes_B T)^B \longrightarrow (T \otimes_B T)^B \otimes (T \otimes_B T)^B$ for the canonical projection is given by $\gamma(x \otimes y \otimes v \otimes w) := x \otimes yvw^{(1)} \otimes w^{(2)} \otimes w^{(3)}$, and we have

$$\begin{aligned}
\nabla(\Theta \otimes \mathrm{id})\gamma\Delta(x \otimes y) &= \nabla(\Theta \otimes \mathrm{id})(x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)}) \\
&= \nabla(S({y^{(1)}}_{(1)})^{[1]} \otimes x {y^{(1)}}_{(0)} S({y^{(1)}}_{(1)})^{[2]} \otimes y^{(2)} \otimes y^{(3)}) \\
&= y^{(2)} S({y^{(1)}}_{(1)})^{[1]} \otimes x {y^{(1)}}_{(0)} S({y^{(1)}}_{(1)})^{[2]} y^{(3)} \\
&= {y_{(2)}}^{[1]} S(y_{(1)})^{[1]} \otimes x y_{(0)} S(y_{(1)})^{[2]} {y_{(2)}}^{[2]} \\
&= (S(y_{(1)}) y_{(2)})^{[1]} \otimes x y_{(0)} (S(y_{(1)}) y_{(2)})^{[2]} \\
&= 1 \otimes xy \\
&= s \circ \varepsilon(x \otimes y) \ ,
\end{aligned}$$

which proves the claim. □

Remark 4.3.2 The map $\bar{\varsigma} : ((T \otimes_B T)^B)^{op} \longrightarrow (T \otimes_B T)^B$ from Proposition 4.2.22 is inverse to the algebra morphism Θ. This follows from a straightforward calculation and is in accordance to the fact that the underlying maps ϑ and θ are each other's inverses. So we rename $\bar{\varsigma} =: \Theta^{-1}$ and $\vartheta =: \theta^{-1}$ for the rest of this section.

In the special case $B = k$, the above proposition says that $T^{op} \otimes T$ is a Hopf algebroid with the antipode

$$\Theta(x \otimes y) = S(y_{(1)})^{[1]} \otimes x y_{(0)} S(y_{(1)})^{[2]} = S(y_{(1)})^{[1]} y_{(0)} S(y_{(1)})^{[2]} \otimes x = \theta(y) \otimes x \ ,$$

for $x, y \in T$, using that $S(y_{(1)})^{[1]} \otimes y_{(0)} S(y_{(1)})^{[2]} \in T \otimes k \subset T \otimes T$ by [40]. The Grunspan map $\theta : T \longrightarrow T$ can be isolated from the antipode as $\theta(x) = \varepsilon \circ \Theta(1 \otimes x)$.
We note that this yields a structure of Hopf algebroid on the canonical bialgebroid $T \otimes T^{op}$ that is different from the trivial Hopf algebroid structure on $T \otimes T^{op}$ we would obtain by Example 4.1.10.

We can now show that the properties of the Grunspan map in Definition 1.4.1 can be derived from the way the expression (1.22) interacts in the Hopf algebroid structures that we recovered in the previous section.

Let T be a quantum torsor with torsor structure map $\mu : T \longrightarrow T \otimes T^{op} \otimes T$. We recall that the Grunspan map $\theta : T \longrightarrow T$ is an algebra map that satisfies the two properties

$$(\mathrm{id} \otimes \mathrm{id} \otimes \theta \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \mu)\mu = (\mathrm{id} \otimes \mu^{op} \otimes \mathrm{id})\mu \tag{4.4}$$

$$(\theta \otimes \theta \otimes \theta)\mu = \mu \circ \theta \ . \tag{4.5}$$

The first equation can be understood better by expressing in in terms of the inverse θ^{-1}. We then have the conditions

$$t^{(1)} \otimes t^{(2)} \otimes t^{(3)(1)} \otimes t^{(3)(2)} \otimes t^{(3)(3)} = t^{(1)} \otimes t^{(2)(3)} \otimes \theta^{-1}(t^{(2)(2)}) \otimes t^{(2)(1)} \otimes t^{(3)} \tag{4.6}$$

$$\theta^{-1}(t^{(1)}) \otimes \theta^{-1}(t^{(2)}) \otimes \theta^{-1}(t^{(3)}) = \theta^{-1}(t)^{(1)} \otimes \theta^{-1}(t)^{(2)} \otimes \theta^{-1}(t)^{(3)} \tag{4.7}$$

for all $t \in T$.

We claim that for a B-torsor T, the map $\Theta : (T \otimes_B T)^B \longrightarrow (T \otimes_B T)^B$ resp. its inverse Θ^{-1} satisfies the equation

$$t^{(1)} \otimes t^{(2)} \otimes t^{(3)(1)} \otimes t^{(3)(2)} \otimes t^{(3)(3)} = t^{(1)} \otimes \Theta^{-1}(t^{(2)(2)} \otimes t^{(2)(3)}) \otimes t^{(2)(1)} \otimes t^{(3)}$$

in $T \otimes (T \otimes_B T)^B \otimes (T \otimes_B T)^B$ for all $t \in T$, and that

$$\theta^{-1}(r^{(1)}) \otimes \Theta^{-2}(r^{(2)} \otimes r^{(3)}) = \theta^{-1}(r)^{(1)} \otimes \theta^{-1}(r)^{(2)} \otimes \theta^{-1}(r)^{(3)}$$

in $T^B \otimes (T \otimes_B T)^B$ for all $r \in T^B$. These are obviously generalizations of the properties above, and indicate how they really arise in this context.

We gather some consequences of results in the previous section:

Remark 4.3.3 Commutativity of the diagram in the proof of Proposition 4.2.22 means that $\Theta^{-1} \circ \beta^{-1} = \beta^{-1} \circ \varsigma$, and so we can derive the equality

$$\begin{aligned}\Theta^{-1}(h^{[1]} \otimes h^{[2]}) &= S^{-2}(h^{[1]}{}_{(1)})^{[1]} h^{[1]}{}_{(0)} h^{[2]} \otimes S^{-2}(h^{[1]}{}_{(1)})^{[2]} \\ &= S^{-1}(h)^{[1]} \otimes S^{-1}(h)^{[2]}\end{aligned}$$

in $(T \otimes_B T)^B$ for all $h \in H^{op} \subset ((T \otimes_B T)^B)^{op}$.

We moreover have the following implicit formula for the antipode of H:

Proposition 4.3.4 *The antipode $\bar{\varsigma} = \Theta^{-1} : ((T \otimes_B T)^B)^{op} \longrightarrow (T \otimes_B T)^B$ of the Hopf algebroid $((T \otimes_B T)^B)^{op}$ restricts to the antipode of the Hopf algebra $H^{op} \subset ((T \otimes_B T)^B)^{op}$. That is*

$$S^{-1}(x \otimes y) = \Theta^{-1}(x \otimes y) = S^{-2}(h^{[1]}{}_{(1)})^{[1]} h^{[1]}{}_{(0)} h^{[2]} \otimes S^{-2}(h^{[1]}{}_{(1)})^{[2]}$$

for all $x \otimes y \in H^{op}$.

Proof. We use the fact that the restricted Galois map

$$\beta : ((T \otimes_B T)^B)^{op} \xrightarrow{\cong} T^B \# H^{op} \ , \ x \otimes y \mapsto xy^{(1)} \otimes y^{(2)} \otimes y^{(3)}$$

maps elements $x \otimes y \in H^{op} \subset ((T \otimes_B T)^B)^{op}$ to $1 \otimes x \otimes y \in T^B \otimes H^{op}$, as follows from the definition of $H = \{x \otimes y \in T \otimes_B T \mid xy^{(1)} \otimes y^{(2)} \otimes y^{(3)} = 1 \otimes x \otimes y\}$ in Theorem 4.2.2. Then the commutative diagram in the proof of the previous proposition yields for $x \otimes y \in H$

$$\begin{aligned}S^{-1}(x \otimes y) &= \beta^{-1}(1 \otimes S^{-1}(x \otimes y)) \\ &= \beta^{-1} \circ \varsigma(1 \otimes x \otimes y) \\ &= \beta^{-1} \circ \varsigma(xy^{(1)} \otimes y^{(2)} \otimes y^{(3)}) \\ &= \beta^{-1} \circ \varsigma \circ \beta(x \otimes y) \\ &= S^{-2}(x_{(1)})^{[1]} x_{(0)} y \otimes S^{-2}(x_{(1)})^{[2]} \ .\end{aligned}$$

Now the claimed formula for the antipode of H is obvious. □

Now we arrive at a proof of the first of the above equations. It can be seen as an analogue of the property (4.6) for the general case of B-torsors:

Proposition 4.3.5 *Let T be a B-torsor with the torsor structure map $\mu : T \longrightarrow T \otimes (T \otimes_B T)^B$, $t \mapsto t^{(1)} \otimes t^{(2)} \otimes t^{(3)}$. Then the algebra morphism*

$$\Theta^{-1} : (T \otimes_B T)^B \longrightarrow ((T \otimes_B T)^B)^{op} \ , \ x \otimes y \mapsto S^{-2}(x_{(1)})^{[1]} x_{(0)} y \otimes S^{-2}(x_{(1)})^{[2]}$$

satisfies the equation

$$t^{(1)} \otimes t^{(2)} \otimes t^{(3)(1)} \otimes t^{(3)(2)} \otimes t^{(3)(3)} = t^{(1)} \otimes \Theta^{-1}(t^{(2)(2)} \otimes t^{(2)(3)}) \otimes t^{(2)(1)} \otimes t^{(3)}$$

in $T \otimes (T \otimes_B T)^B \otimes (T \otimes_B T)^B$ for all $t \in T$.

Proof. We have to show that the expression on the right hand side of the equation is well-defined. Consider the map

$$\alpha : T \otimes T \longrightarrow (T \otimes_B T)^B \otimes T \otimes_B T \ , \ x \otimes y \mapsto x^{(2)} \otimes x^{(3)} \otimes x^{(1)} \otimes y \ .$$

Since $\alpha(xb \otimes y) = x^{(2)} \otimes x^{(3)} \otimes x^{(1)} b \otimes y = x^{(2)} \otimes x^{(3)} \otimes x^{(1)} \otimes by = \alpha(x \otimes by)$ for all $b \in B$, it factors over the quotient $T \otimes_B T$ as $\bar{\alpha}$. Now let $x \otimes y \in (T \otimes_B T)^B$. Then $\bar{\alpha}(bx \otimes y) = \bar{\alpha}(x \otimes yb)$, and hence $x^{(2)} \otimes x^{(3)} \otimes bx^{(1)} \otimes y = x^{(2)} \otimes x^{(3)} \otimes x^{(1)} \otimes yb$ which implies that the image of $\bar{\alpha}$ lies in $(T \otimes_B T)^B \otimes (T \otimes_B T)^B$ by faithful flatness of $(T \otimes_B T)^B$ over k. Applying $\Theta^{-1} \otimes \mathrm{id}$ yields

$$\begin{aligned}(\Theta^{-1} \otimes \mathrm{id}_{(T \otimes_B T)^B})\bar{\alpha}(x \otimes y) &= (\Theta^{-1} \otimes \mathrm{id})(x^{(2)} \otimes x^{(3)} \otimes x^{(1)} \otimes y) \\ &= \Theta^{-1}(x^{(2)} \otimes x^{(3)}) \otimes x^{(1)} \otimes y \ .\end{aligned}$$

So the right hand side of the equation can be expressed as the composition of well-defined maps

$$\begin{aligned}&(\mathrm{id} \otimes \Theta^{-1} \otimes \mathrm{id}_{(T \otimes_B T)^B})(\mathrm{id} \otimes \bar{\alpha})\mu(t) = \\ &\qquad = (\mathrm{id} \otimes \Theta^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \bar{\alpha})(t^{(1)} \otimes t^{(2)} \otimes t^{(3)}) \\ &\qquad = t^{(1)} \otimes \Theta^{-1}(t^{(2)(2)} \otimes t^{(2)(3)}) \otimes t^{(2)(1)} \otimes t^{(3)} \ ,\end{aligned}$$

for all $\in T$, and is therefore well-defined. Using that the torsor structure map is given by $\mu(t) = t^{(1)} \otimes t^{(2)} \otimes t^{(3)} = t_{(0)} \otimes {t_{(1)}}^{[1]} \otimes {t_{(1)}}^{[2]}$ with respect to the induced H-comodule structure on T, we obtain for $t \in T$

$$\begin{aligned}\Theta^{-1}(t^{(2)} \otimes t^{(3)}) \otimes t^{(1)} &= \Theta^{-1}({t_{(1)}}^{[1]} \otimes {t_{(1)}}^{[2]}) \otimes t_{(0)} \\ &= S^{-1}(t_{(1)})^{[1]} \otimes S^{-1}(t_{(1)})^{[2]} \otimes t_{(0)}\end{aligned}$$

by the formula in Remark 4.3.3. Hence we have by (1.5) and (1.4)

$$\begin{aligned}
&t^{(1)} \otimes \Theta^{-1}(t^{(2)(2)} \otimes t^{(2)(3)}) \otimes t^{(2)(1)} \otimes t^{(3)} = \\
&\quad = t_{(0)} \otimes \Theta^{-1}({t_{(1)}{}^{[1]}{}_{(1)}}^{[1]} \otimes {t_{(1)}{}^{[1]}{}_{(1)}}^{[2]}) \otimes t_{(1)}{}^{[1]}{}_{(0)} \otimes t_{(1)}{}^{[2]} \\
&\quad = t_{(0)} \otimes S^{-1}(t_{(1)}{}^{[1]}{}_{(1)})^{[1]} \otimes S^{-1}(t_{(1)}{}^{[1]}{}_{(1)})^{[2]} \otimes t_{(1)}{}^{[1]}{}_{(0)} \otimes t_{(1)}{}^{[2]} \\
&\quad = t_{(0)} \otimes S^{-1}(S(t_{(1)}))^{[1]} \otimes S^{-1}(S(t_{(1)}))^{[2]} \otimes t_{(2)}{}^{[1]} \otimes t_{(2)}{}^{[2]} \\
&\quad = t_{(0)} \otimes t_{(1)}{}^{[1]} \otimes t_{(1)}{}^{[2]} \otimes t_{(2)}{}^{[1]} \otimes t_{(2)}{}^{[2]} \\
&\quad = t_{(0)} \otimes t_{(1)}{}^{[1]} \otimes t_{(1)}{}^{[2]}{}_{(0)} \otimes {t_{(1)}{}^{[2]}{}_{(1)}}^{[1]} \otimes {t_{(1)}{}^{[2]}{}_{(1)}}^{[2]} \\
&\quad = t^{(1)} \otimes t^{(2)} \otimes t^{(3)(1)} \otimes t^{(3)(2)} \otimes t^{(3)(3)} ,
\end{aligned}$$

which is the claimed property. □

Finally, we can also deduce the analogue to (4.7), which says that the Grunspan map θ is a morphism of quantum torsors. Recall that we have shown in Remark 4.2.15 that $\delta \circ \theta^{-1}(r) = \theta^{-1}(r_{(0)}) \otimes S^{-2}(r_{(1)})$ for all $r \in T^B$. By the result in Proposition 4.3.4, we can express the antipode S^{-1} of H^{op} in terms of Θ^{-1} and obtain $S^{-2}(x \otimes y) = \Theta^{-2}(x \otimes y)$. This implies the following:

Corollary 4.3.6 *The morphism $\theta^{-1} : T^B \longrightarrow T^B$ satisfies the equation*

$$(\theta^{-1} \otimes \Theta^{-2}) \circ \mu = \mu \circ \theta^{-1} ,$$

that is

$$\theta^{-1}(r^{(1)}) \otimes \Theta^{-2}(r^{(2)} \otimes r^{(3)}) = \theta^{-1}(r)^{(1)} \otimes \theta^{-1}(r)^{(2)} \otimes \theta^{-1}(r)^{(3)}$$

in $T^B \otimes (T \otimes_B T)^B$ for all $r \in T^B$.

Altogether we can interpret the two properties (4.6) and (4.7) of a Grunspan map as follows:
The first property is due the particular way the Hopf algebra H coacts on T. It is essentially determined by the inner structure of H, which is such that $\Theta^{-1}(x^{(2)} \otimes x^{(3)}) \otimes x^{(1)} \otimes y = x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)}$ for all $x \otimes y \in H \subseteq (T \otimes_B T)^B$. The second property follows directly from the connection between the antipode of the T^B-Hopf algebroid $(T \otimes_B T)^B$ and its underlying morphism.

Even though we are not able to recover a Grunspan map for B-torsors, we still have an endomorphism $\theta : T^B \longrightarrow T^B$ of the centralizer and an implicit formula for the antipode of the Hopf algebra H. They both give rise to generalized Grunspan axioms as derived above.

Chapter 5

A-B-Torsors and Depth Two Extensions

As we have seen so far, faithfully flat Hopf-Galois extensions can be completely described within the concept of quantum torsor and its more general version of B-torsor. This is done such that the coaction of the Hopf algebra is encoded in the respective torsor structure map. In this chapter, we show that one can also go in the opposite direction: We uncover torsor structures on extensions of depth two. Then our previous results imply that there are actions of Hopf algebras that lead to Hopf-Galois extensions.
Inspired by this, we introduce the notion of A-B-torsor as the most general concept of a noncommutative principal homogeneous space.

5.1 B-Torsors and Irreducible Depth Two Extensions

The notion of depth two comes from the theory of invariants and subfactors. Finite depth is a property of the standard invariant of the Jones tower for a subfactor [19]. Jones towers can be used to investigate inclusions of semisimple algebras and Frobenius extensions [21]. We use the following definition of depth two for ring extensions as was introduced by Kadison and Szlachányi in [22]:

Definition 5.1.1 A ring extension $N \subset M$ is called *left depth two* or *left D2*, if the canonical (N, M)-bimodule ${}_N M \otimes_N M_M$ is isomorphic to a direct summand of the free module $\bigoplus^m {}_N M_M$ for some positive integer m.

A ring extension $N \subset M$ is called *right depth two* or *right D2* if the canonical (M,N)-bimodule ${}_M M \otimes_N M_N$ is isomorphic to a direct summand of the free module $\bigoplus^n {}_M M_N$ for some positive integer n.

An extension $N \subset M$ is called *depth two* or *D2* if it is both left depth two and right depth two.

Depth two extensions are characterized by a quasibasis property. In the sequel, we shall concentrate on right depth two extensions. The following proof is outlined for left D2 in [22]:

Lemma 5.1.2 *A ring extension $N \subset M$ is right depth two if and only if there exist $c_i = c_i^1 \otimes c_i^2 \in (M \otimes_N M)^N$ and $\gamma_i \in \mathrm{End}\,{}_N M_N$, called a right D2 quasibasis, such that*

$$\sum_i \gamma_i(m) c_i^1 \otimes c_i^2 = 1 \otimes m \tag{5.1}$$

in $M \otimes_N M$ for all $m \in M$.

Proof. Let $N \subset M$ be a right D2 extension. Then, by definition, there exists a split epimorphism $\pi : \bigoplus^n {}_M M_N \longrightarrow {}_M M \otimes_N M_N$ with a section $\sigma : {}_M M \otimes_N M_N \longrightarrow \bigoplus^n {}_M M_N$ such that $\pi \circ \sigma = \mathrm{id}_{M \otimes_N M}$. Let $\{e_i\}_{i=1}^n$ be the standard basis of the free module $\bigoplus^n {}_M M$, and let $p_i : \bigoplus^n {}_M M_N \longrightarrow {}_M M_N$ denote the standard projections.
We define $c_i := \pi(e_i)$ for all $i = 1, \dots n$, and obtain $c_i \in (M \otimes_N M)^N$ since $\pi \in \mathrm{End}\,{}_M M_N$. Now we consider the map $\iota : {}_N M_N \longrightarrow {}_N M \otimes_N M_N$, $m \mapsto 1 \otimes m$, and let $\gamma_i := p_i \circ \sigma \circ \iota \in \mathrm{End}\,{}_N M_N$. Then (c_i, γ_i) is a right D2 quasibasis, since

$$\begin{aligned} 1 \otimes m = \iota(m) = \pi \circ \sigma \circ \iota(m) = \pi(\sum_{i=1}^n p_i \circ \sigma \circ \iota(m) e_i) = \pi(\sum_{i=1}^n \gamma_i(m) e_i) \\ = \sum_{i=1}^n \gamma_i(m) \pi(e_i) = \sum_{i=1}^n \gamma_i(m) c_i^1 \otimes c_i^2 \ . \end{aligned}$$

Assume now that there exists a right D2 quasibasis $(c_i, \gamma_i)_{i=1}^n$ for the ring extension $N \subset M$. As above, let $\{e_i\}_{i=1}^n$ be the standard basis of the free module $\bigoplus^n {}_M M$, and define two (M,N)-bimodule maps $\pi : \bigoplus^n {}_M M_N \longrightarrow {}_M M \otimes_N M_N$ by $\pi(e_i) := c_i$ for all $i = 1, \dots n$, and $\sigma : {}_M M \otimes_N M_N \longrightarrow \bigoplus^n {}_M M_N$, $x \otimes y \mapsto \sum_i x \gamma_i(y) e_i$.

Then we have

$$\pi \circ \sigma(x \otimes y) = \pi(\sum_i x\gamma_i(y)e_i) = \sum_i x\gamma_i(y)\pi(e_i)$$
$$= \sum_i x\gamma_i(y)c_i = \sum_i x\gamma_i(y)c_i^1 \otimes c_i^2 = x \otimes y \ ,$$

which shows that ${}_M M \otimes_N M_N$ is isomorphic to a direct summand of the free module $\bigoplus^n {}_M M_N$. Hence, $N \subset M$ is right D2. □

Various examples of depth two extensions, such as centrally projective ring extensions and H-separable extensions, are discussed in [22]. It is shown in [20] that each Hopf-Galois extension $B \subset A$ with respect to a finite-dimensional $\mathbb{K}$-Hopf algebra H is depth two. The finiteness condition on H allows to explicitly construct a quasibasis. However, this does not work for arbitrary Hopf algebras.

We observe that the quasibasis property of Lemma 5.1.2 looks very much like the first axiom for an N-torsor M in Definition 4.2.1. So we are going to investigate whether there exists a connection between depth two extensions and torsors.

For the rest of this section, we let $A := \operatorname{End}{}_N M_N$ and $B := (M \otimes_N M)^N$. As in the previous chapter, the algebra structure on the centralizer B is given by $(x \otimes y)(x' \otimes y') = x'x \otimes yy'$ for $x \otimes y, x' \otimes y' \in (M \otimes_N M)^N$.
We will sometimes abuse the notation, and treat elements in subsets of tensor products as if they were decomposable tensors.

We state a main theorem for depth two extensions from [22], which shows that a left D2 extension always comes along with a Morita context.

Theorem 5.1.3 *([22]) If $N \subset M$ is a left D2 extension, then $R := M^N$ and $C := \operatorname{End}_N(M \otimes_N M)_M$ are Morita equivalent rings with invertible bimodules ${}_C B_R$ and ${}_R A_C$. In particular, B_R and ${}_R A$ are finitely generated projective generators with the following isomorphisms:*

- $B \otimes_R A \xrightarrow{\cong} C \ , \ b \otimes \alpha \mapsto (m \otimes m' \mapsto b\alpha(m)m')$
- $B_R \xrightarrow{\cong} \operatorname{Hom}({}_R A, {}_R R)_R \ , \ b = b^1 \otimes b^2 \mapsto (\alpha \mapsto \alpha(b^1)b^2)$
- $B \otimes_R M \xrightarrow{\cong} M \otimes_N M \ , \ b \otimes m \mapsto bm$
- $M \otimes_N M \xrightarrow{\cong} \operatorname{Hom}({}_R A, {}_R M) \ , \ m \otimes m' \mapsto (\alpha \mapsto \alpha(m)m')$
- $C \cong \operatorname{End} B_R$ *via* $c \mapsto (b \mapsto c(b))$

- $C \cong \mathrm{End}_R A$ *via* $c \mapsto (\alpha \mapsto \nabla_M(\alpha \otimes \mathrm{id}_M) c\iota_1)$*, where* $\iota_1 : M \longrightarrow M \otimes M$, $m \mapsto m \otimes 1$.

Of course, a corresponding version of the above theorem holds also for right D2 extensions. This implies, in particular, the following statement:

Lemma 5.1.4 *Let* $N \subset M$ *be a right D2 extension with the quasibasis* (c_i, γ_i)*. There is an isomorphism of* (M, N)*-bimodules*

$$\rho : M \otimes_R (M \otimes_N M)^N \xrightarrow{\cong} M \otimes_N M$$

given by $\rho(m \otimes \sum_j x_j \otimes y_j) := \sum_j m x_j \otimes y_j$ *with inverse* $\rho^{-1}(x \otimes y) := \sum_i x\gamma_i(y) \otimes c_i^1 \otimes c_i^2$.

Proof. Obviously, both ρ and ρ^{-1} are (M, N)-bimodule maps. Let $x \otimes y \in (M \otimes_N M)^N$. Then we have $nx \otimes y = x \otimes yn$ for all $n \in N$, and therefore $nx\gamma_i(y) = x\gamma_i(yn) = x\gamma_i(y)n$ for all i, showing that $x\gamma_i(y) \in M^N = R$. Now we get

$$\begin{aligned} \rho^{-1} \circ \rho(m \otimes x \otimes y) &= \sum_i mx\gamma_i(y) \otimes c_i^1 \otimes c_i^2 \\ &= \sum_i m \otimes x\gamma_i(y) c_i^1 \otimes c_i^2 \\ &= m \otimes x \otimes y \end{aligned}$$

and

$$\rho \circ \rho^{-1}(x \otimes y) = \sum_i x\gamma_i(y) c_i^1 \otimes c_i^2 = x \otimes y \ ,$$

which proves the claim. □

Since the B-torsor axioms in Definition 4.2.1 are somewhat symmetrical, we will have to work with right depth two extensions that have an additional property. It turns out that the following restriction is sufficient:

Definition 5.1.5 A k-algebra extension $N \subset M$ is called *irreducible* if the centralizer $R := M^N$ of N in M is trivial, i.e. $R \cong k$.

Lemma 5.1.4 now allows us to establish a connection between depth two extensions and torsors:

Proposition 5.1.6 *Let* $N \subset M$ *be an irreducible right D2 algebra extension with* M *faithfully flat over* k*. Then* M *is a right* N*-torsor.*

Proof. Let (c_i, γ_i) be a right D2 quasibasis for the extension $N \subset M$. We claim that the map

$$\mu : M \longrightarrow M \otimes (M \otimes_N M)^N \ , \ m \mapsto \sum \gamma_i(m) \otimes c_i^1 \otimes c_i^2$$

defines an N-torsor structure on M. Since $R \cong k$, we have the isomorphism

$$\rho : M \otimes (M \otimes_N M)^N \xrightarrow{\cong} M \otimes_N M \ , \ m \otimes \sum x_j \otimes y_j \mapsto \sum_j m x_j \otimes y_j$$

from Lemma 5.1.4. Let $m, m' \in M$. Then we have

$$\mu(mm') = \sum_i \gamma_i(mm') \otimes c_i^1 \otimes c_i^2 \ ,$$

$$\begin{aligned} \mu(m)\mu(m') &= (\sum_i \gamma_i(m) \otimes c_i^1 \otimes c_i^2)(\sum_j \gamma_j(m') \otimes c_j^1 \otimes c_j^2) \\ &= \sum_{i,j} \gamma_i(m)\gamma_j(m') \otimes c_j^1 c_i^1 \otimes c_i^2 c_j^2 \ . \end{aligned}$$

We apply the the isomorphism ρ to both expressions and obtain

$$\rho(\sum_i \gamma_i(mm') \otimes c_i^1 \otimes c_i^2) = \sum_i \gamma_i(mm') c_i^1 \otimes c_i^2 = 1 \otimes mm' \ ,$$

$$\rho(\sum_{i,j} \gamma_i(m)\gamma_j(m') \otimes c_j^1 c_i^1 \otimes c_i^2 c_j^2) = \sum_i \gamma_i(m) c_i^1 \otimes c_i^2 m' = 1 \otimes mm' \ .$$

Hence, we can conclude that $\mu(mm') = \mu(m)\mu(m')$. Applying the quasibasis property to the unit $1 \in M$ yields $1 \otimes 1 = \sum_i c_i^1 \otimes c_i^2$ and therefore $\mu(1) = 1 \otimes \sum_i c_i^1 \otimes c_i^2 = 1 \otimes 1 \otimes 1$. So μ is an algebra morphism. We prove that μ satisfies the properties of an N-torsor map in Definition 4.2.1:

1) The first N-torsor axiom $m^{(1)} m^{(2)} \otimes m^{(3)} = 1 \otimes m$ is exactly the right D2 quasibasis property.

2) The quasibasis property implies $\sum_i \gamma_i(m) c_i^1 c_i^2 = m$ for all $m \in M$. Since $c_i^1 c_i^2 \in R \cong k$ for all i, it follows that

$$m^{(1)} \otimes m^{(2)} m^{(3)} = \sum_i \gamma_i(m) \otimes c_i^1 c_i^2 = \sum_i \gamma_i(m) c_i^1 c_i^2 \otimes 1 = m \otimes 1$$

for all $m \in M$.

3) Let $n \in N$. Then $\mu(n) = \sum_i \gamma_i(n) \otimes c_i^1 \otimes c_i^2 = \sum_i n\gamma_i(1) \otimes c_i^1 \otimes c_i^2 = n \otimes 1 \otimes 1$.

4) Let $f \in \mathrm{End}_N M_N$. Then we have $1 \otimes f(m) = \sum_i \gamma_i \circ f(m) c_i^1 \otimes c_i^2$ for all $m \in M$ by the right quasibasis property. On the other hand, by applying $\mathrm{id} \otimes f$ to the quasibasis equation, we get $1 \otimes f(m) = \sum_i \gamma_i(m) c_i^1 \otimes f(c_i^2)$. Under the isomorphism ρ this equality implies

$$\sum_i \gamma_i \circ f(m) \otimes c_i^1 \otimes c_i^2 = \sum_i \gamma_i(m) \otimes c_i^1 \otimes f(c_i^2) \ ,$$

and hence

$$\begin{aligned} \mu(m^{(1)}) \otimes m^{(2)} \otimes m^{(3)} &= \sum_{i,j} \gamma_j \circ \gamma_i(m) \otimes c_j^1 \otimes c_j^2 \otimes c_i^1 \otimes c_i^2 \\ &= \sum_{i,j} \gamma_j(m) \otimes c_j^1 \otimes \gamma_i(c_j^2) \otimes c_i^1 \otimes c_i^2 \\ &= m^{(1)} \otimes m^{(2)} \otimes \mu(m^{(3)}) \ . \end{aligned}$$

□

The assumption that M be faithfully flat over k was not needed for the proof, but really just included to meet the requirements of Definition 4.2.1. It is clear that, as in the original definition of a quantum torsor, one could formulate a definition of B-torsor without the assumption of faithful flatness.

But on the other hand, we rather keep this assumption to see that the relation between irreducible depth two extensions and torsors has the following immediate consequence:

Corollary 5.1.7 *Let $N \subset M$ be an irreducible right depth two algebra extension and assume that M is both faithfully flat over k and right faithfully flat over N.*
Then M is a right Hopf-Galois extension of N, where the Hopf algebra that coacts on M is $H := (M \otimes_N M)^N = B$.

Proof. Since M is an N-torsor by Proposition 5.1.6 and faithfully flat as a right N-module, we can apply Theorem 4.2.2. It says that M is an H-Galois extension of N with the Hopf algebra H given by

$$H = \{x \otimes y \in M \otimes_N M \mid xy^{(1)} \otimes y^{(2)} \otimes y^{(3)} = 1 \otimes x \otimes y\} \ .$$

With respect to the torsor structure defined in Proposition 5.1.6, we get

$$xy^{(1)} \otimes y^{(2)} \otimes y^{(3)} = \sum_i x\gamma_i(y) \otimes c_i^1 \otimes c_i^2$$

and

$$1 \otimes x \otimes y = \sum_i \gamma_i(x) c_i^1 \otimes c_i^2 \otimes y$$

for all $x \otimes y \in M \otimes_N M$. The isomorphism ρ from Lemma 5.1.4 yields with $R \cong k$ that $\sum_i x\gamma_i(y)c_i^1 \otimes c_i^2 = x \otimes y = \sum_i \gamma_i(x)c_i^1 c_i^2 \otimes y$. Hence, the defining condition for elements $x \otimes y \in H$ holds for all elements of $(M \otimes_N M)^N$. This implies that $H = (M \otimes_N M)^N = B$. □

Remark 5.1.8 We note that in the above situation the Galois map is of course given by the map $\rho^{-1} : M \otimes_N M \xrightarrow{\cong} M \otimes (M \otimes_N M)^N$, $x \otimes y \mapsto \sum_i x\gamma_i(y) \otimes c_i^1 \otimes c_i^2 = x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)}$ from Lemma 5.1.4.

We have just seen that irreducible depth two extensions $N \subset M$ can be considered as a special case of N-torsors. It turns out that also the converse holds true if we impose a finiteness condition on M:

Proposition 5.1.9 *Let $N \subset M$ be an irreducible algebra extension such that M is finitely generated and projective as a left N-module. If there exists an N-torsor structure $\mu : M \longrightarrow M \otimes (M \otimes_N M)^N$ on M, then $N \subset M$ is a right depth two extension.*

Proof. We know by the dual basis lemma that ${}_N M$ being finitely generated and projective is equivalent to the existence of a dual basis $f_i \in \mathrm{Hom}_N(M, N)$, $p_i \in M$ such that $m = \sum_i f_i(m)p_i$ for all $m \in M$.
Now the N-torsor axiom 1) yields $1 \otimes m = m^{(1)}m^{(2)} \otimes m^{(3)} \in M \otimes_N M$ for all $m \in M$, and we obtain on the other hand that $1 \otimes m = 1 \otimes \sum_i f_i(m)p_i = \sum_i f_i(m) \otimes p_i = \sum_i f_i(m){p_i}^{(1)}{p_i}^{(2)} \otimes {p_i}^{(3)}$. For all i there exist elements $x_{ij}, y_{ij}, z_{ij} \in M$ such that $\mu(p_i) = \sum_j x_{ij} \otimes y_{ij} \otimes z_{ij} \in M \otimes (M \otimes_N M)^N$.

We claim that a right quasibasis for $N \subset M$ is given by $(c_{ij}, \gamma_{ij})_{i,j}$ with $c_{ij}^1 \otimes c_{ij}^2 := y_{ij} \otimes z_{ij} \in (M \otimes_N M)^N$ and $\gamma_{ij}(m) := f_i(m)x_{ij}$ for all $m \in M$. We have

$$\begin{aligned}
\sum_{i,j} \gamma_{ij}(m) c_{ij}^1 \otimes c_{ij}^2 &= \sum_i \sum_j f_i(m) x_{ij} y_{ij} \otimes z_{ij} \\
&= \sum_i f_i(m) \sum_j x_{ij} y_{ij} \otimes z_{ij} \\
&= \sum_i f_i(m) {p_i}^{(1)} {p_i}^{(2)} \otimes {p_i}^{(3)} \\
&= \sum_i f_i(m) \otimes p_i = 1 \otimes \sum_i f_i(m) p_i \\
&= 1 \otimes m \in M \otimes_N M
\end{aligned}$$

by what we have shown above.
It remains to be proved that $\gamma_{ij} \in \operatorname{End}_N M_N$. It is clear that γ_{ij} is left N-linear, since f_i is. For right N-linearity of γ_{ij}, we show that for all $m \in M$, $n \in N$ and i, j we have $f_i(mn)x_{ij} \otimes y_{ij} \otimes z_{ij} = f_i(m)x_{ij}ny_{ij} \otimes z_{ij}$. This follows via the isomorphism $\rho : M \otimes (M \otimes_N M)^N \longrightarrow M \otimes_N M$ from Lemma 5.1.4, since $\rho(\sum f_i(mn)x_{ij} \otimes y_{ij} \otimes z_{ij}) = \sum f_i(mn)x_{ij}y_{ij} \otimes z_{ij} = 1 \otimes mn$ and $\rho(\sum f_i(m)x_{ij}n \otimes y_{ij} \otimes z_{ij}) = \sum f_i(m)x_{ij}ny_{ij} \otimes z_{ij} = f_i(m)x_{ij}y_{ij} \otimes z_{ij}n = 1 \otimes mn$. □

We are now going to show that the theory of N-torsors can be used to prove and generalize results from [22] and [21] about Frobenius towers for depth two extensions.

Let $N \subset M$ be an algebra extension. We recall the following definitions, see for instance [22]:

Definition 5.1.10 A ring extension $N \subset M$ is called a *Frobenius extension* if there exist (a so-called *Frobenius homomorphism*) $E \in \operatorname{Hom}_{N,N}(M, N)$ and (so-called *dual bases*) $x_i, y_i \in M$ such that

$$\sum_i \lambda(x_i)E\lambda(y_i) = \mathrm{id}_M = \sum_i \rho(y_i)E\rho(x_i) \ ,$$

where λ and ρ denote the left resp. right multiplication on M.

Definition 5.1.11 Let $N \subset M$ be an algebra extension. Then M becomes naturally a left $\operatorname{End} M_N$-module by $f \cdot m := f(m)$ for $m \in M, f \in \operatorname{End} M_N$. The right N-module M_N is called *balanced* if the left endomorphism ring of ${}_{\operatorname{End} M_N}M$ is naturally anti-isomorphic to N : $\operatorname{End}_{\operatorname{End} M_N} M \cong N$.

In [22] Kadison and Szlachányi prove that an irreducible Frobenius extension $N \subset M$ of depth two is a $B = (M \otimes_N M)^N$-Galois extension, provided M_N is balanced. This result is a corollary of a number of propositions that involve quite a lot of work. Replacing Frobenius and balanced by a faithful flatness assumption, we arrived at a similar statement in Corollary 5.1.7 above, using just the theory of N-torsors.

Let $N \subset M$ be a Frobenius extension with dual bases $x_i, y_i \in M$ and a Frobenius homomorphism $E \in \operatorname{Hom}_{N,N}(M, N)$. We recall the construction of a generalized Jones tower for Frobenius extensions from [22]. It is based on the fundamental construction described in [15].

Proposition 5.1.12 *([22]) Let $N \subset M$ be a Frobenius extension. Then there is an isomorphism* $\operatorname{End} M_N \cong M \otimes_N M$, $f \mapsto \sum_i f(x_i) \otimes y_i$ *with the* *inverse given by* $m \otimes m' \mapsto \lambda(m)E\lambda(m')$*. Moreover,* $\operatorname{End} M_N$ *is a Frobenius extension of* M*.*

We denote by $M_1 := M \otimes_N M$ the ring with the multiplication induced by the isomorphism $\operatorname{End} M_N \cong M \otimes_N M$. Then M_1 is a Frobenius extension of M.

The centralizer ${M_1}^N$ of N in M_1 is isomorphic as an algebra to $A = \operatorname{End}{}_N M_N$ via $\alpha \mapsto \sum_i \alpha(x_i) \otimes y_i$. Note that the multiplication on ${M_1}^N = (M \otimes_N M)^N$ differs from the algebra structure on $B = (M \otimes_N M)^N$ in Proposition 5.1.7.

Now that M_1 is a Frobenius extension of M, this construction can be iterated to obtain a tower

$$N \subset M \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow ... \tag{5.2}$$

where also $M_2 = M_1 \otimes_M M_1$ is a Frobenius extension of M_1, and so on.

We use the following results from [22] in the sequel:

Proposition 5.1.13 *([22]) If* $N \subset M$ *is a left D2 Frobenius extension, then it is also right D2.*

Proposition 5.1.14 *([22]) If* $N \subset M$ *is a left D2 Frobenius extension, then* M_1 *is a right D2 Frobenius extension of* M*.*

We can now show that for the higher components $M_1, M_2, \ldots$ in the generalized Jones tower (5.2), the assumption on faithful flatness which we needed in Proposition 5.1.7 can be dropped.

Proposition 5.1.15 *Let* $N \subset M$ *be an irreducible left depth two Frobenius extension with* M *faithfully flat over* k*.*
Then M_1 *is a right Hopf-Galois extension of* M*. The Hopf algebra that coacts on* M_1 *is* $H_1 := (M_1 \otimes_M M_1)^M$*.*

Proof. We first compute the centralizer of M in M_1. Recall that we have an isomorphism $\operatorname{End} M_N \cong M_1$ by Proposition 5.1.12. The endomorphism ring $\operatorname{End} M_N$ is a natural M-bimodule with $(m \cdot f)(x) = mf(x)$ and $(f \cdot m)(x) = f(mx)$. Hence, the centralizer of M in $\operatorname{End} M_N$ consists of the right M-linear endomorphisms $f \in \operatorname{End} M_N$, i.e. $(\operatorname{End} M_N)^M = \operatorname{End}_M M_N$. Since $\operatorname{End}{}_M M$ becomes an N-bimodule by $(n \cdot f)(x) = f(xn)$ and $(f \cdot n)(x) = f(x)n$, the same reasoning shows that $\operatorname{End}{}_M M_N = (\operatorname{End}{}_M M)^N$, and we obtain

$${M_1}^M \cong (\operatorname{End} M_N)^M = \operatorname{End}_M M_N = (\operatorname{End}_M M)^N \cong M^N = R\ .$$

Thus, ${M_1}^M \cong R \cong k$, since we assumed $N \subset M$ to be irreducible.
Now $N \subset M$ is a depth two extension and so, by definition, $(M \otimes_N M)_M$

is isomorphic to a direct summand of the free M-module $\bigoplus^n M_M$ for some positive integer n. It is therefore flat over M.
In order to show that $M \otimes_N M$ is right faithfully flat over M, we let $0 \neq X$ be a left M-module. We have to show that $M \otimes_N M \otimes_M X \neq 0$. By Theorem 5.1.3 there is an isomorphism of (N, M)-bimodules $B \otimes M \cong M \otimes_N M$, and $B = (M \otimes_N M)^N$ is a finitely generated projective generator in the category $\mathcal{M}_k$ of k-modules (since $R \cong k$). So B is a faithfully flat k-module, and we have

$$M \otimes_N M \otimes_M X \cong B \otimes M \otimes_M X \cong B \otimes X \neq 0 \ ,$$

which proves that $M_1 = M \otimes_N M$ is right faithfully flat over M. We can also deduce from the above isomorphism that M_1 is a faithfully flat k-module, since both B and M are faithfully flat over k. Altogether we arrive at the situation that $M \subset M_1$ is a k-faithfully flat irreducible right D2 Frobenius extension with M_1 faithfully flat as a right M-module. By Corollary 5.1.7 then $H_1 = (M_1 \otimes_M M_1)^M$ is a Hopf algebra, and M_1 is a right H_1-Galois extension of M. □

Remark 5.1.16 Above we have proved that M_1 is a faithfully flat M-module using a property of B. In fact, M is isomorphic to a direct summand of M_1, since the maps $\iota : M \longrightarrow M_1 \ , \ m \mapsto m \otimes 1$ and $p : M_1 \longrightarrow M \ , \ m \otimes m' \mapsto mm'$ satisfy $p \circ \iota = \mathrm{id}_M$.

We can now successively apply the Proposition 5.1.15 to each of the higher components $M_2 = M_1 \otimes_M M_1$, $M_3 = M_2 \otimes_{M_1} M_2$, etc. in the Jones tower (5.2). This leads to the following result:

Theorem 5.1.17 *Let $N \subset M$ be an irreducible right depth two Frobenius extension with M a faithfully flat k-module.*
Then in the generalized Jones tower for Frobenius extensions

$$N \subset M := M_0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow \dots \ ,$$

each component M_i is an H_i-Galois extension of M_{i-1}, and the Hopf algebra H_i that coacts on M_i is given by $H_i := (M_i \otimes_{M_{i-1}} M_i)^{M_{i-1}}$.

We remark again that the algebra structure on H_i in the above theorem is not equal to the algebra structure on ${M_{i+1}}^{M_{i-1}}$ which is induced by the isomorphism $M_{i+1} \cong \mathrm{End}\, {M_i}_{M_{i-1}}$.

It was proved in [22] that ${M_2}^M \cong B$ as algebras via the composition of isomorphisms

$${M_2}^M \cong \mathrm{End}_M\, {M_1}_M \cong (M \otimes_N M)^N \ ,$$

using that $N \subset M$ is a Frobenius extension. So we can apply the same isomorphism to the Frobenius extension $M \subset M_1$ and get

$$M_3{}^{M_1} \cong (M_1 \otimes_M M_1)^M = H_1 \ ,$$

as well as

$$M_{i+1}{}^{M_{i-1}} \cong (M_{i-1} \otimes_{M_{i-2}} M_{i-1})^{M_{i-2}} = H_{i-1}$$

for the higher components of the Jones tower (5.2)

We summarize a few more results from [22]: The authors show that for an arbitrary depth two extension $N \subset M$, the algebra $A = \mathrm{End}\,{}_N M_N$ has the structure of a left R-bialgebroid and the algebra $B = (M \otimes_N M)^N$ is a right R-bialgebroid. This bialgebroid structure on B is actually the same as the one we constructed in Proposition 4.2.3 on the centralizer $(M \otimes M)^N$ for an N-torsor M.
One main result from [22] says that A and B possess dual left and right R-bialgebroid structures. This refers to a definition of left and right duals for bialgebroids, that generalizes the notion of left and right dual for finitely generated projective k-bialgebras. So B is isomorphic as R-bialgebroids to both the left and right bialgebroid dual of A. In case $R \cong k$, that is for an irreducible D2 Frobenius extension $N \subset M$, we can conclude that the bialgebra A is the dual of B, i.e. $A^* \cong B$. Note that A and B both possess duals since they are finitely generated projective k-modules by Theorem 5.1.3.
Using that $A \cong M_1{}^N$ and $B \cong M_2{}^M$, we obtain by iteration

$$A \cong M_1{}^N \cong (M_2{}^M)^* \cong M_3{}^{M_1} \cong \dots$$

and

$$B \cong M_2{}^M \cong (M_3{}^{M_1})^* \cong M_4{}^{M_2} \cong \dots$$

as bialgebras. Hence, we conclude that the Hopf algebras H_i in Theorem 5.1.17 are given by

$$A \cong H_1 \cong H_3 \cong \dots \quad \text{and} \quad B \cong H_2 \cong H_4 \cong \dots$$

In the paper [21] of Kadison and Nikshych, which is a predecessor of [22], only the first three components of the Jones tower are considered under more restricted conditions than ours. One of them assumes k to be a field, and another one requires M_N to be balanced. The authors prove that M/N and M_2/M_1 are B-Galois extensions and that M_1/M is an A-Galois extension. We replace the condition that M_N be balanced by a faithfully flatness assumption on M_N as before, and get a result as follows:

Theorem 5.1.18 *Let $N \subset M$ be an irreducible depth two Frobenius extension with M faithfully flat over k. Then in the generalized Jones tower for Frobenius extensions*

$$N \subset M := M_0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow ...$$

the odd numbered components M_{2i+1} are A-Galois extensions of M_{2i} and the even numbered components M_{2i+2} are B-Galois extensions of M_{2i+1} for all $i \geq 0$.
Moreover, if M is a faithfully flat right N-module, then M is a B-Galois extension of N.

This theorem is of course a direct consequence of what we have shown above. Nevertheless, we can give another proof of the last statement, which indicates where exactly the faithful flatness assumption is needed. We use the following theorem that is stated in [28] (Theorem 8.3.3) in case k is a field, but holds also for a commutative ring k, as in the original proofs of [25] and [53].

Theorem 5.1.19 *Let H be a finitely generated projective Hopf algebra over k and let M be a left H-module algebra.*
Then M is a right H^*-Galois extension of $M^H = \{m \in M \mid h \cdot m = \varepsilon(h)m \ \forall \ h \in H\}$ iff M is a finitely generated projective right M^H-module and the map

$$\pi : M\#H \longrightarrow \mathrm{End}(M_{M^H}) \ , \ m \otimes h \mapsto (m' \mapsto m(h \cdot m'))$$

is an algebra isomorphism.

Here is the alternative proof of Theorem 5.1.18:

Proof. It remains to be shown that M is a B-Galois extension of N. Now B is finitely-generated by Theorem 5.1.3, and M becomes a left A-module algebra via evaluation. The set of invariants $M^A = \{x \in M \mid \alpha(x) = \alpha(1)x \ \forall \ \alpha \in \mathrm{End}\,_N M_N = A\}$ clearly contains N. We also observe that $\alpha(1) \in M^N \cong k$ for all $\alpha \in A$, and hence $\alpha(1)m = m\alpha(1)$ for all $m \in M$. This implies together with the quasibasis property (5.1) that we have for $x \in M^A$

$$1 \otimes x = \sum \gamma_i(x)c_i^1 \otimes c_i^2 = \sum \gamma_i(1)xc_i^1 \otimes c_i^2 = \sum x\gamma_i(1)c_i^1 \otimes c_i^2 = x \otimes 1$$

in $M \otimes_N M$, and hence $x \in N$ by faithful flatness of M over N. So we have $M^A = N$, and M is a finitely generated projective right N-module, since $N \subset M$ is a Frobenius extension.

By [22] Corollary 8.2., M_1 is isomorphic to a smash product algebra $M\#A \cong M_1$, $m \otimes \alpha \mapsto \sum_i m\alpha(x_i) \otimes y_i$. Composing this map with the isomorphism $M_1 \cong \operatorname{End} M_N$, $m \otimes m' \mapsto \lambda(m)E\lambda(m')$ from Proposition 5.1.12, we obtain $m \otimes \alpha \mapsto \sum_i \lambda(m\alpha(x_i))E\lambda(y_i)$ and $\sum_i \lambda(m\alpha(x_i))E\lambda(y_i)(m') = \sum_i m\alpha(x_i)E(y_i m') = \sum_i m\alpha(x_i E(y_i m')) = m\alpha(m')$. This shows that there is an isomorphism

$$M\#A \xrightarrow{\cong} \operatorname{End} M_N \ , \ m \otimes \alpha \mapsto (m' \mapsto m\alpha(m')) \ .$$

Since $B \cong A^*$, it follows by Theorem 5.1.19 that M is a B-Galois extension of N. □

Remark 5.1.20 The condition that M_N be balanced in [22] was really just needed to show that $M^A = N$. We used faithful flatness for this part. So it might be interesting at this point to compare these two conditions that do not seem to be linked at first sight.
We let H be a finitely generated projective Hopf algebra over k and $N \subset M$ an H^*-Galois extension with M finitely generated and projective as a right N-module. Then we obtain by Theorem 5.1.19 an isomorphism $M\#H \cong \operatorname{End} M_N$. This means that the canonical left $\operatorname{End} M_N$-module structure on M is determined through the action of $M\#H$ on M. We have $\operatorname{End}_M M \cong M$, and so the endomorphisms of M that are simultaneously left M-linear and left H-linear can be identified with the centralizer M^H. So we see that for this extension $\operatorname{End}_{\operatorname{End} M_N} M \cong \operatorname{End}_{M\#H} M \cong M^H = N$, which means that M_N is balanced according to Definition 5.1.11. But of course, M does not necessarily have to be right faithfully flat over N.
So our faithful flatness condition is really somewhat stronger than that of M_N being balanced. Nevertheless, faithful flatness allows us to use the theory of N-torsors which provided the above results with significantly less effort than the methods used in [22] and [21].

5.2 A-B-Torsors and $\times_A$-Bialgebras

We have seen in the previous section that the notion of B-torsor is only applicable to irreducible depth two extensions. In order to cover depth two extensions without this restriction, we introduce the more general notion of A-B-torsor. It generalizes the notion of quantum torsor and that of a B-torsor. Moreover, it has the advantage of having a torsor structure map that is symmetric with respect to the roles of A and B.

Recall from Section 4.1 that given two k-algebras A and B, an $A\otimes B$-ring T is a k-algebra T together with an algebra map $i_T : A\otimes B \longrightarrow T$. This induces an $A\otimes B$-bimodule structure on T via multiplication with the image of i_T. We are going to denote the image $i_T(a\otimes 1)\in T$ of $a\in A$ simply as $a\in T$, and similarly, $b = i_T(1\otimes b)\in T$ for $b\in B$.

Definition 5.2.1 Let A and B be k-algebras, and let T be an $A\otimes B$-ring. An *A-B-torsor structure* on T is a map

$$\mu : T \longrightarrow T\otimes_A T\otimes_B T\ ,\ x\mapsto x^{(1)}\otimes x^{(2)}\otimes x^{(3)}\ ,$$

that satisfies the following properties:

1) $ax^{(1)}\otimes x^{(2)}\otimes x^{(3)} = x^{(1)}\otimes x^{(2)}a\otimes x^{(3)}$ for all $a\in A$

2) $x^{(1)}\otimes bx^{(2)}\otimes x^{(3)} = x^{(1)}\otimes x^{(2)}\otimes x^{(3)}b$ for all $b\in B$

3) $\mu(xy) = x^{(1)}y^{(1)}\otimes y^{(2)}x^{(2)}\otimes x^{(3)}y^{(3)}$ for all $x,y\in T$

4) $x^{(1)}x^{(2)}\otimes x^{(3)} = 1\otimes x \ \in T\otimes_B T$

5) $x^{(1)}\otimes x^{(2)}x^{(3)} = x\otimes 1 \ \in T\otimes_A T$

6) $\mu(a) = 1\otimes 1\otimes a$ for all $a\in A$

7) $\mu(b) = b\otimes 1\otimes 1$ for all $b\in B$

8) $\mu(x^{(1)})\otimes x^{(2)}\otimes x^{(3)} = x^{(1)}\otimes x^{(2)}\otimes \mu(x^{(3)}) \ \in T\otimes_A T\otimes_B T\otimes_A T\otimes_B T$

We note that the composition of maps in axiom 8) is well-defined because μ is a (B,A)-bimodule map by the axioms 6) and 7).

Remark 5.2.2 The fact that T is an $A\otimes B$-ring in the definition implies that the images of $a\in A$ and $b\in B$ in T have the property $ab = ba$.
If we consider A and B as subalgebras of T via $i_T : A\otimes B\longrightarrow T$, then the centralizer $(T\otimes_A T)^A$, which becomes an algebra with the structure induced from $T\otimes T^{op}$, possesses a natural B-bimodule structure. Similarly, $(T\otimes_B T)^B$ with multiplication induced from $T^{op}\otimes T$ becomes an A-bimodule.
So the conditions 1) and 2) in the above definition imply that we have simultaneously $\mathrm{Im}(\mu)\subset (T\otimes_A T)^A\otimes_B T$ and $\mathrm{Im}(\mu)\subset T\otimes_A (T\otimes_B T)^B$. Hence, axiom 3) could be also expressed in requiring that the map μ be an algebra morphism with respect to the corresponding algebra structure on $\mathrm{Im}(\mu)$, which is then induced by $T\otimes T^{op}\otimes T$.

We have seen in Example 1.4.2 that each k-Hopf algebra carries the structure of a quantum torsor. It turns out that this relation holds in an analogous way for $\times_A$-Hopf algebras and $\bar{A}$-A-torsors. This provides an example for our definition of A-B-torsor. We first recall some facts about $\times_A$-Hopf algebras from [43]:

Let L be a $\times_A$-Hopf algebra. Then, by definition, the map

$$\tilde{\beta} : L \otimes_{\bar{A}} L \longrightarrow L \diamond L \ , \ \ell \otimes m \mapsto \ell_{(1)} \otimes \ell_{(2)} m$$

is bijective and we denote

$$\ell_+ \otimes \ell_- := \tilde{\beta}^{-1}(\ell \otimes 1) \in L \otimes_{\bar{A}} L$$

for each $\ell \in L$. This expression has the following properties that were deduced in [43].

Proposition 5.2.3 *Let L be a $\times_a$-Hopf algebra. Then we have*

$$\ell_{+(1)} \otimes \ell_{+(2)} \ell_- = \ell \otimes 1 \in L \diamond L \tag{5.3}$$

$$\ell_{(1)+} \otimes \ell_{(1)-} \ell_{(2)} = \ell \otimes 1 \in L \otimes_{\bar{A}} L \tag{5.4}$$

$$\ell_+ \otimes \ell_- \in \int^b \int_a {}_{\bar{b}}L_{\bar{a}} \otimes {}_{\bar{a}}L_{\bar{b}} \tag{5.5}$$

$$(\ell m)_+ \otimes (\ell m)_- = \ell_+ m_+ \otimes m_- \ell_- \in L \otimes_{\bar{A}} L \tag{5.6}$$

$$1_+ \otimes 1_- = 1 \otimes 1 \tag{5.7}$$

$$\ell_{+(1)} \otimes \ell_{+(2)} \otimes \ell_- = \ell_{(1)} \otimes \ell_{(2)+} \otimes \ell_{(2)-} \in \int^{cd} \int_{ab} {}_{\bar{a}}L_{\bar{c}} \otimes {}_{a\bar{d}}L_{c\bar{b}} \otimes {}_{\bar{b}}L_{\bar{d}} \tag{5.8}$$

$$\ell_+ \otimes \ell_{-(1)} \otimes \ell_{-(2)} = \ell_{++} \otimes \ell_- \otimes \ell_{+-} \in \int^{cd} \int_{ab} {}_{\bar{c}}L_{\bar{a}} \otimes {}_{\bar{b}}L_{\bar{d}} \otimes {}_{b\bar{a}}L_{d\bar{c}} \tag{5.9}$$

$$\ell_+ \overline{\varepsilon(\ell_-)(1)} = \ell \tag{5.10}$$

$$\ell_+ \ell_- = \varepsilon(\ell)(1) \tag{5.11}$$

With some of these properties we can now show:

Proposition 5.2.4 *Let L be a $\times_A$-Hopf algebra. Then L is an $\bar{A}$-A-torsor with the torsor structure map*

$$\mu : L \longrightarrow L \otimes_{\bar{A}} L \otimes_A L \ , \ \ell \mapsto \ell_{(1)+} \otimes \ell_{(1)-} \otimes \ell_{(2)} \ .$$

Proof. L is an $A^e = A \otimes \bar{A}$-ring by definition, and it can obviously also be considered as an $\bar{A} \otimes A$-ring.

In order to show that the map μ is well-defined, we have to verify that the composition of the comultiplication $\Delta : L \longrightarrow L \times_A L$, $\ell \mapsto \ell_{(1)} \otimes \ell_{(2)}$ with $\tilde{\beta}^{-1}(- \otimes 1) \otimes \mathrm{id} : L \otimes L \longrightarrow L \otimes_{\bar{A}} L \otimes L$ followed by the canonical projection onto $L \otimes_{\bar{A}} L \otimes_A L$ is well-defined. The latter maps $\ell \otimes m \in L \otimes L$ to $\ell_+ \otimes \ell_- \otimes m \in L \otimes_{\bar{A}} L \otimes_A L$, and factors over $L \diamond L$. This is because $\bar{a}_+ \otimes \bar{a}_- = \tilde{\beta}^{-1}(\bar{a} \otimes 1) = \tilde{\beta}^{-1}(1 \otimes a) = 1 \otimes a$ for $a \in A$, and so we have

$$(\bar{a}\ell)_+ \otimes (\bar{a}\ell)_- \otimes m = \bar{a}_+\ell_+ \otimes \ell_-\bar{a}_- \otimes m = \ell_+ \otimes \ell_- a \otimes m = \ell_+ \otimes \ell_- \otimes am$$

for $\ell \otimes m \in L \diamond L$ by the property (5.6) above. Hence, μ is well-defined. We check that μ satisfies the axioms of Definition 5.2.1:

1) We have $\bar{a}\ell_{(1)+} \otimes \ell_{(1)-} \otimes \ell_{(2)} = \ell_{(1)+} \otimes \ell_{(1)-}\bar{a} \otimes \ell_{(2)}$ for all $\bar{a} \in \bar{A}$ by the property (5.5).

2) Since $\Delta(L) \subset L \times_A L$ implies $\ell_{(1)} \otimes \ell_{(2)}a = \ell_{(1)}\bar{a} \otimes \ell_{(2)}$, we get

$$\begin{aligned} \ell_{(1)+} \otimes \ell_{(1)-} \otimes \ell_{(2)}a &= (\ell_{(1)}\bar{a})_+ \otimes (\ell_{(1)}\bar{a})_- \otimes \ell_{(2)} \\ &= \ell_{(1)+}\bar{a}_+ \otimes \bar{a}_-\ell_{(1)-} \otimes \ell_{(2)} \\ &= \ell_{(1)+} \otimes a\ell_{(1)-} \otimes \ell_{(2)} \ , \end{aligned}$$

using again that $\bar{a}_+ \otimes \bar{a}_- = 1 \otimes a$ for all $a \in A$.

3) For the multiplicativity property we obtain

$$\begin{aligned} \mu(\ell m) &= (\ell m)_{(1)+} \otimes (\ell m)_{(1)-} \otimes (\ell m)_{(2)} \\ &= (\ell_{(1)}m_{(1)})_+ \otimes (\ell_{(1)}m_{(1)})_- \otimes \ell_{(2)}m_{(2)} \\ &= \ell_{(1)+}m_{(1)+} \otimes m_{(1)-}\ell_{(1)-} \otimes \ell_{(2)}m_{(2)} \end{aligned}$$

for all $\ell, m \in L$ by (5.6).

4) By the $\times_A$-coalgebra axioms, we have $\vartheta'(\varepsilon \times_A L)\Delta(\ell) = \varepsilon(\ell_{(1)})(1)\ell_{(2)} = \ell$, and hence

$$\begin{aligned} \ell_{(1)+}\ell_{(1)-} \otimes \ell_{(2)} &= \varepsilon(\ell_{(1)})(1) \otimes \ell_{(2)} \\ &= 1 \otimes \varepsilon(\ell_{(1)})(1)\ell_{(2)} \\ &= 1 \otimes \ell \in L \otimes_A L \end{aligned}$$

using property (5.11).

5) The equation $\ell_{(1)+} \otimes \ell_{(1)-}\ell_{(2)} = \ell \otimes 1 \in L \otimes_{\bar{A}} L$ is exactly (5.4).

6) We have $\mu(\bar{a}) = \mu(\bar{a}1) = 1_+ \otimes 1_- \otimes \bar{a} = 1 \otimes 1 \otimes \bar{a}$ by (5.7) and the fact that Δ is an A^e-ring map.

7) Because of $\tilde{\beta}(a \otimes 1) = a \otimes 1$, we have $\mu(a) = (a1)_+ \otimes (a1)_- \otimes 1 = a_+ \otimes a_- \otimes 1 = a \otimes 1 \otimes 1$ for all $a \in A$.

8) By (5.8) we get

$$\begin{aligned}
\mu(\ell_{(1)+}) \otimes \ell_{(1)-} \otimes \ell_{(2)} &= \ell_{(1)+(1)+} \otimes \ell_{(1)+(1)-} \otimes \ell_{(1)+(2)} \otimes \ell_{(1)-} \otimes \ell_{(2)} \\
&= \ell_{(1)(1)+} \otimes \ell_{(1)(1)-} \otimes \ell_{(1)(2)+} \otimes \ell_{(1)(2)-} \otimes \ell_{(2)} \\
&= \ell_{(1)+} \otimes \ell_{(1)-} \otimes \ell_{(2)+} \otimes \ell_{(2)-} \otimes \ell_{(3)} \\
&= \ell_{(1)+} \otimes \ell_{(1)-} \otimes \ell_{(2)(1)+} \otimes \ell_{(2)(1)-} \otimes \ell_{(2)(2)} \\
&= \ell_{(1)+} \otimes \ell_{(1)-} \otimes \mu(\ell_{(2)}) \ .
\end{aligned}$$

Here, we applied coassociativity of Δ. It reads $\ell_{(1)(1)} \otimes \ell_{(1)(2)} \otimes \ell_{(2)} = \ell_{(1)} \otimes \ell_{(2)(1)} \otimes \ell_{(2)(2)}$ as elements of $L \diamond L \diamond L$.

□

The definition of A-B-torsor is symmetric with respect to the roles of A and B. We compare this situation to that of a B-torsor T in Section 4.2. In Proposition 4.2.3 we recovered a bialgebroid structure on the centralizer $(T \otimes_B T)^B$. So now we can expect to obtain bialgebroid structures on both $(T \otimes_A T)^A$ and $(T \otimes_B T)^B$ for an A-B-torsor T.

Proposition 5.2.5 *Let T be an A-B-torsor and assume that $(T \otimes_A T)^A$ is right faithfully flat over B with respect to the natural B-bimodule structure. Then there is a left T^A-bialgebroid structure on $(T \otimes_A T)^A$ with the structure maps*

- *source map* $s : T^A \longrightarrow (T \otimes_A T)^A \ , \ r \mapsto r \otimes 1$
- *target map* $t : T^A \longrightarrow (T \otimes_A T)^A \ , \ r \mapsto 1 \otimes r$
- *comultiplication*

$$\begin{aligned}
\Delta : (T \otimes_A T)^A &\longrightarrow (T \otimes_A T)^A \otimes_{T^A} (T \otimes_A T)^A \\
x \otimes y &\mapsto x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes y
\end{aligned}$$

- *counit* $\varepsilon : (T \otimes_A T)^A \longrightarrow T^A \ , \ x \otimes y \mapsto xy$

Proof. We note that, since T is an $A \otimes B$-ring, we have $ab = ba$ in T for the images of $a \in A$ and $b \in B$. This implies $B \subset T^A$ and $A \subset T^B$. We show that the bialgebroid axioms in Definition 4.1.5 are satisfied.

We define a multiplication on $(T \otimes_A T)^A$ by $(a \otimes b)(x \otimes y) := ax \otimes yb$ for $a \otimes b, x \otimes y \in (T \otimes_A T)^A$. With this algebra structure on $(T \otimes_A T)^A$, we see that s is an algebra morphism and t is an algebra anti-morphism satisfying $s(r)t(r')t(r')s(r)$ for all $r, r' \in T^A$. The resulting T^A-bimodule structure on $(T \otimes_A T)^A$ is given by

$$r \cdot (x \otimes y) \cdot r' = s(r)t(r')(x \otimes y) = (r \otimes 1)(1 \otimes r')(x \otimes y) = rx \otimes yr' \ .$$

It corresponds to the natural T^A-bimodule structure on $(T \otimes_A T)^A$.

In order to show that Δ is well-defined, we first consider the map $\Delta_0 : T \otimes_A T \longrightarrow T \otimes_A T \otimes_B T \otimes_A T \ , \ u \otimes v \mapsto u^{(1)} \otimes u^{(2)} \otimes u^{(3)} \otimes v$, that is well-defined by the torsor axiom 6). Now we can show that $\Delta_0((T \otimes_A T)^A) \subseteq (T \otimes_A T)^A \otimes_B (T \otimes_A T)^A$. This expression makes sense, since $(T \otimes_A T)^A$ possesses a natural B-bimodule structure induced by $B \subset T^A$. Let $x \otimes y \in (T \otimes_A T)^A$ and let $a \in A$. Then $\Delta_0(ax \otimes y) = \Delta_0(x \otimes ya)$ and hence using the axiom 6)

$$\begin{aligned} x^{(1)} \otimes x^{(2)} \otimes ax^{(3)} \otimes y &= (ax)^{(1)} \otimes (ax)^{(2)} \otimes (ax)^{(3)} \otimes y \\ &= x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes ya \ . \end{aligned}$$

This proves the claim via faithful flatness of $(T \otimes_A T)^A$ as a right B-module and axiom 1). The map Δ is then obtained by applying Δ_0 followed by the canonical residue class morphism $(T \otimes_A T)^A \otimes_B (T \otimes_A T)^A \longrightarrow (T \otimes_A T)^A \otimes_{T^A} (T \otimes_A T)^A$.

The counit ε is well-defined, since for $x \otimes y \in (T \otimes_A T)^A$ and $a \in A$ we have $axy = xya$ and thus $\varepsilon((T \otimes_A T)^A) \subseteq T^A$.

We check that Δ and ε are T^A-bimodule morphisms. Let $x \otimes y \in (T \otimes_A T)^A$ and $r \in T^A$. Since $ra = ar$ for all $a \in A$, and thus $r^{(1)} \otimes r^{(2)} \otimes ar^{(3)} = r^{(1)} \otimes r^{(2)} \otimes r^{(3)}a$, it follows by right faithful flatness of $(T \otimes_A T)^A$ over B that $\mu(r) \in T \otimes_A T \otimes_B T^A$. This leads to

$$\begin{aligned} \Delta(r \cdot (x \otimes y)) &= \Delta(rx \otimes y) \\ &= (rx)^{(1)} \otimes (rx)^{(2)} \otimes (rx)^{(3)} \otimes y \\ &= r^{(1)}x^{(1)} \otimes x^{(2)}r^{(2)} \otimes r^{(3)}x^{(3)} \otimes y \\ &= r^{(1)}x^{(1)} \otimes x^{(2)}r^{(2)}r^{(3)} \otimes x^{(3)} \otimes y \\ &= rx^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes y \\ &= r \cdot \Delta(x \otimes y) \ . \end{aligned}$$

Multiplying on the other side gives $\Delta((x \otimes y) \cdot r) = \Delta(x \otimes yr) = x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes yr = \Delta(x \otimes y) \cdot r$. For the counit we get $\varepsilon(rx \otimes y) = rxy = r \cdot \varepsilon(x \otimes y)$

and $\varepsilon(x \otimes yr) = xyr = \varepsilon(x \otimes y) \cdot r$.
Coassociativity of Δ follows from coassociativity of μ. The counit axioms are $(\varepsilon \otimes \mathrm{id})\Delta(x \otimes y) = x^{(1)}x^{(2)} \otimes x^{(3)} \otimes y = 1 \otimes x \otimes y \cong x \otimes y$ and $(\mathrm{id} \otimes \varepsilon)\Delta(x \otimes y) = x^{(1)} \otimes x^{(2)} \otimes x^{(3)}y = x^{(1)} \otimes x^{(2)}x^{(3)}y \otimes 1 \cong x \otimes y$ since $x^{(1)} \otimes x^{(2)} \otimes x^{(3)}y \in (T \otimes_A T)^A \otimes_{T^A} T^A$.

The proof of the first identity 1) in Definition 4.1.5 requires the following lemma:

Lemma 5.2.6 *The map*

$$\psi : (T \otimes_A T)^A \otimes_{T^A} (T \otimes_A T)^A \longrightarrow (T \otimes_A T \otimes_A T)^A$$
$$x \otimes y \otimes v \otimes w \mapsto x \otimes yv \otimes w$$

is bijective.

Proof. We claim that the inverse of ψ is given by

$$\phi : (T \otimes_A T \otimes_A T)^A \longrightarrow (T \otimes_A T)^A \otimes_{T^A} (T \otimes_A T)^A$$
$$x \otimes y \otimes z \mapsto x^{(1)} \otimes x^{(2)} \otimes x^{(3)}y \otimes z\ .$$

It is easy to see that both ψ and ϕ are well-defined. They are inverse to each other since $\phi \circ \psi(x \otimes y \otimes v \otimes w) = \phi(x \otimes yv \otimes w) = x^{(1)} \otimes x^{(2)} \otimes x^{(3)}yv \otimes w = x^{(1)} \otimes x^{(2)}x^{(3)}y \otimes v \otimes w = x \otimes y \otimes v \otimes w$, where we used that $x^{(1)} \otimes x^{(2)} \otimes x^{(3)}y \in (T \otimes_A T)^A \otimes_B T^A$, and $\psi \circ \phi(x \otimes y \otimes z) = \psi(x^{(1)} \otimes x^{(2)} \otimes x^{(3)}y \otimes z) = x^{(1)} \otimes x^{(2)}x^{(3)} \otimes z = x \otimes y \otimes z$. □

We have $\Delta(x \otimes y)(1 \otimes s(r)) = (x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes y)(1 \otimes 1 \otimes r \otimes 1) = x^{(1)} \otimes x^{(2)} \otimes x^{(3)}r \otimes y$ and $\Delta(x \otimes y)(t(r) \otimes 1) = (x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes y)(1 \otimes r \otimes 1 \otimes 1) = x^{(1)} \otimes rx^{(2)} \otimes x^{(3)} \otimes y$ for $x \otimes y \in (T \otimes_A T)^A$.
Applying the isomorphism ψ to both expressions yields $\psi(x^{(1)} \otimes x^{(2)} \otimes x^{(3)}r \otimes y) = x^{(1)} \otimes x^{(2)}x^{(3)}r \otimes y = x \otimes r \otimes y$ and $\psi(x^{(1)} \otimes rx^{(2)} \otimes x^{(3)} \otimes y) = x^{(1)} \otimes rx^{(2)}x^{(3)} \otimes y = x \otimes r \otimes y$. This proves that the identity 1) holds. For identity 2) we compute $\Delta((x \otimes y)(v \otimes w)) = \Delta(xv \otimes wy) = (xv)^{(1)} \otimes (xv)^{(2)} \otimes (xv)^{(3)} \otimes wy = x^{(1)}v^{(1)} \otimes v^{(2)}x^{(2)} \otimes x^{(3)}v^{(3)} \otimes wy = (x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes y)(v^{(1)} \otimes v^{(2)} \otimes v^{(3)} \otimes w) = \Delta(x \otimes y)\Delta(v \otimes w)$. We clearly have $\Delta(1 \otimes 1) = 1 \otimes 1 \otimes 1 \otimes 1$, since μ is an algebra morphism. Finally, we obtain 4) and 5) since $\varepsilon((x \otimes y)s\varepsilon(v \otimes w)) = \varepsilon((x \otimes y)(vw \otimes 1)) = \varepsilon(xvw \otimes y) = xvwy$, $\varepsilon((x \otimes y)(v \otimes w)) = \varepsilon(xv \otimes wy) = xvwy$ and $\varepsilon((x \otimes y)t\varepsilon(v \otimes w)) = \varepsilon((x \otimes y)(1 \otimes vw)) = \varepsilon(x \otimes vwy) = xvwy$, and clearly $\varepsilon(1 \otimes 1) = 1$. □

It is obvious that we also have a symmetric version of the above proposition. So we obtain a right T^B-bialgebroid structure on $(T \otimes_B T)^B$, provided it is faithfully flat as a left A-module. Altogether we have:

Corollary 5.2.7 *Let T be an A-B-torsor and assume that $(T \otimes_A T)^A$ is right faithfully flat over B and that $(T \otimes_B T)^B$ is left faithfully flat over A. Then $(T \otimes_A T)^A$ is a left T^A bialgebroid and $(T \otimes_B T)^B$ is a right T^B-bialgebroid.*

Example 5.2.8 Let T be a B-torsor with $(T \otimes_B T)^B$ faithfully flat over k. Then $(T \otimes_B T)^B$ is a right T^B-bialgebroid, as we have seen in Proposition 4.2.3. Considering the left hand side, we recover a left $T^k = T$-bialgebroid structure on $T \otimes T$, which is equal to Lu's trivial bialgebroid $T \otimes T^{op}$ that we described in Example 4.1.10. □

Using the theory of faithfully flat descent as in [42], we can recover two more bialgebroids from the structure map of an A-B-torsor. They are in fact $\times_B$ resp. $\times^A$-bialgebras, and in general different from the bialgebroids constructed above. Moreover, we obtain that they are $\times_B$-resp. $\times^A$-Hopf algebras inducing left and right Hopf-Galois extensions.
We start by observing that each A-B-torsor gives rise to two descent data. This generalizes a fact shown for B-torsors in [42]. A brief explanation of faithfully flat descent can be found in the appendix.

Lemma 5.2.9 *Let T be an A-B-torsor. Then a left descent data from T to A on $T \otimes_B T$ is given by*

$$D_l : T \otimes_B T \longrightarrow T \otimes_A T \otimes_B T \ , \ x \otimes y \mapsto xy^{(1)} \otimes y^{(2)} \otimes y^{(3)} \ .$$

It satisfies $(T \otimes_A D_l)\mu(x) = x^{(1)} \otimes 1 \otimes x^{(2)} \otimes x^{(3)}$.

Proof. The map D_l is well-defined by the A-B-torsor axiom 7) and obviously left T-linear. The composition $(T \otimes_A D_l)\mu$ is well-defined and we obtain for $x \in T$

$$\begin{aligned}
(T \otimes D_l)\mu(x) &= x^{(1)} \otimes D_l(x^{(2)} \otimes x^{(3)}) \\
&= x^{(1)} \otimes x^{(2)} x^{(3)(1)} \otimes x^{(3)(2)} \otimes x^{(3)(3)} \\
&= x^{(1)(1)} \otimes x^{(1)(2)} x^{(1)(3)} \otimes x^{(2)} \otimes x^{(3)} \\
&= x^{(1)} \otimes 1 \otimes x^{(2)} \otimes x^{(3)}
\end{aligned}$$

by the axioms 8) and 5). Now we have

$$\begin{aligned}
(T \otimes_A D_l)D_l(x \otimes y) &= xy^{(1)} \otimes D_l(y^{(2)} \otimes y^{(3)}) \\
&= xy^{(1)} \otimes 1 \otimes y^{(2)} \otimes y^{(3)} \\
&= (T \otimes_A \eta \otimes_A T \otimes_B T)D_l(x \otimes y)
\end{aligned}$$

and

$$(\nabla \otimes_B T)D_l(x \otimes y) = xy^{(1)}y^{(2)} \otimes y^{(3)} = x \otimes y$$

by axiom 4), which proves the claim. □

We note that $D_l(T \otimes_B T) \subset T \otimes_A (T \otimes_B T)^B$ by axiom 2). If we assume that T is right faithfully flat over A, then this implies ${}^{D_l}(T \otimes_B T) \subset (T \otimes_B T)^B$, where ${}^{D_l}(T \otimes_B T) = \{x \otimes y \mid D_l(x \otimes y) = 1 \otimes x \otimes y\}$, as defined in the appendix.

It is obvious that the map

$$D_r : T \otimes_A T \longrightarrow T \otimes_A T \otimes_B T \ , \ x \otimes y \mapsto x^{(1)} \otimes x^{(2)} \otimes x^{(3)}y$$

defines a right descent data from T to B on $T \otimes_A T$ with properties analogous to those described in the previous Lemma.

Now we can generalize Schauenburg's proof of Theorem 4.2.2:

Theorem 5.2.10 *Let T be an A-B-torsor. Assume that T is both left and right faithfully flat over A and B. Then the following hold:*

1) *The $B \otimes \bar{B}$-bimodule*

$$G := (T \otimes_A T)^{D_r} = \{x \otimes y \in T \otimes_A T \mid x^{(1)} \otimes x^{(2)} \otimes x^{(3)}y = x \otimes y \otimes 1\}$$

is faithfully flat over B, and has the structure of a $\times_B$-Hopf algebra.

The algebra T is a left G-Galois extension of A under the G-coaction $\delta_G : T \longrightarrow G \times_B T$ given by $\delta_G(x) = \mu(x)$.

2) *The $\bar{A} \otimes A$-bimodule*

$$H := {}^{D_l}(T \otimes_B T) = \{x \otimes y \in T \otimes_B T \mid xy^{(1)} \otimes y^{(2)} \otimes y^{(3)} = 1 \otimes x \otimes y\}$$

is faithfully flat over A, and has the structure of $\times^A$-Hopf algebra.

The algebra T is a right H-Galois extension of B under the H-coaction $\delta_H : T \longrightarrow T \times^A H$ given by $\delta_H(x) = \mu(x)$.

Proof. We recall from Section 4.1 that we defined $\times^B$-bialgebras to be right versions of $\times_B$-bialgebras. Thus it is clear that part 2) of the theorem can be proved analogously to part 1).

We first observe that G is indeed a subalgebra of $(T \otimes_A T)^A$. We have for $x \otimes y, v \otimes w \in G$

$$\begin{aligned} D_r((x \otimes y)(v \otimes w)) &= D_r(xv \otimes wy) \\ &= (xv)^{(1)} \otimes (xv)^{(2)} \otimes (xv)^{(3)} wy \\ &= x^{(1)} v^{(1)} \otimes v^{(2)} x^{(2)} \otimes x^{(3)} v^{(3)} wy \\ &= x^{(1)} v \otimes w x^{(2)} \otimes x^{(3)} y \\ &= xv \otimes wy \otimes 1 \\ &= (x \otimes y)(v \otimes w) \otimes 1 \ , \end{aligned}$$

and thus $(x \otimes y)(v \otimes w) \in G$. The natural $B^e = B \otimes \bar{B}$-ring structure on G is given by $i : B^e \ni b \otimes \bar{c} \mapsto b \otimes c \in G$ (note that this is well defined because of axiom 7)). It induces the B^e-bimodule structure

$$(b \otimes \bar{c}) \triangleright (x \otimes y) := bx \otimes yc \quad , \quad (x \otimes y) \triangleleft (b \otimes \bar{c}) := xb \otimes cy$$

on G. With this structure we have

$$\begin{aligned} G \diamond G = \int_b {}_{\bar{b}}G \otimes {}_bG = \\ &= (G \otimes G)/ < \bar{b}g \otimes h - g \otimes bh \mid g, h \in G, b \in B > \\ &= (G \otimes G)/ < g^1 \otimes g^2 b \otimes h^1 \otimes h^2 - g^1 \otimes g^2 \otimes bh^1 \otimes h^2 > \ , \end{aligned}$$

and so we can identify $G \diamond G$ with $G \otimes_B G$, the "standard" tensor product over B. Then we obtain

$$\begin{aligned} G \times_B G = \int^c \int_b {}_{\bar{b}}G_{\bar{c}} \otimes {}_bG_c \\ &= \{g \otimes h \in G \otimes_B G \mid g\bar{b} \otimes h = g \otimes hb \ \forall \ b \in B\} \\ &= \{g \otimes h \in G \otimes_B G \mid g^1 \otimes bg^2 \otimes h^1 \otimes h^2 = g^1 \otimes g^2 \otimes h^1 b \otimes h^2\} \ . \end{aligned}$$

The algebra T becomes naturally a B-bimodule via its $A \otimes B$-ring structure. This means that

$$\begin{aligned} G \diamond T = \int_b {}_{\bar{b}}G \otimes {}_bT \\ &= (G \otimes T)/ < g^1 \otimes g^2 b \otimes t - g^1 \otimes g^2 \otimes bt \mid g \in G, b \in B > \\ &= G \otimes_B T \ , \end{aligned}$$

and

$$G \times_B T = \{g \otimes t \in G \otimes_B T \mid g^1 \otimes bg^2 \otimes t = g^1 \otimes g^2 \otimes tb\} \ .$$

We check that the coaction of G on T is well-defined. By left faithful flatness of T over B, $G \otimes_B T$ is given as the equalizer

$$T \otimes_A T \otimes_B T \underset{T\otimes_A T\otimes_B \eta\otimes_B T}{\overset{D_r\otimes_B T}{\rightrightarrows}} T \otimes_A T \otimes_B T \otimes_B T \ .$$

Note that we have $(D_r \otimes_B T)\mu(x) = (T \otimes_A T \otimes_B \eta \otimes_B T)\mu(x)$ by the right symmetric version of Lemma 5.2.9. Moreover, we have $x^{(1)} \otimes bx^{(2)} \otimes x^{(3)} = x^{(1)} \otimes x^{(2)} \otimes x^{(3)}b$ for all $b \in B$ and $x \in T$ by axiom 2). Hence, we see that $\mu(T) \subset G \times_B T$. Then δ_G is an algebra map since μ is, and it is right A-linear by axiom 6).

It follows by faithfully flat descent that the map

$$D_r : T \otimes_A T \longrightarrow G \otimes_B T \ , \ x \otimes y \mapsto x^{(1)} \otimes x^{(2)} \otimes x^{(3)} y$$

is an isomorphism. Of course, D_r is also the potential Galois map. Faithful flatness of T over both A and B implies that G is faithfully flat over B.

We claim that G becomes a $\times_B$-coalgebra with the comultiplication

$$\Delta : G \longrightarrow G \times_B G \ , \ x \otimes y \mapsto x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes y$$

and the counit

$$\varepsilon : G \longrightarrow \mathrm{End}(B) \ , \ x \otimes y \mapsto (b \mapsto xby) \ .$$

The map $\Delta_0 : T \otimes_A T \longrightarrow G \otimes_B T \otimes_A T \ , \ x \otimes y \mapsto x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes y$ is well-defined since μ is right A-linear. To prove that Δ is well-defined, we first show that $\Delta_0(G)$ is contained in $G \diamond G$, which is by faithful flatness of G over B the equalizer

$$G \otimes_B T \otimes_A T \underset{G\otimes_B T\otimes_A T\otimes_B \eta}{\overset{G\otimes_B D_r}{\rightrightarrows}} G \otimes_B T \otimes_A T \otimes_B T \ .$$

But this is true, since we have for $x \otimes y \in G$

$$\begin{aligned}
(G \otimes_B D_r)\Delta_0(x \otimes y) &= (G \otimes_B D_r)(x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes y) \\
&= x^{(1)} \otimes x^{(2)} \otimes x^{(3)(1)} \otimes x^{(3)(2)} \otimes x^{(3)(3)} y \\
&= \mu(x^{(1)}) \otimes x^{(2)} \otimes x^{(3)} y \\
&= (\mu \otimes_A T \otimes_B T) D_r(x \otimes y) \\
&= (\mu \otimes_A T \otimes_B T)(x \otimes y \otimes 1) \\
&= (\Delta_0 \otimes_B \eta)(x \otimes y) \\
&= (G \otimes_B T \otimes_A T \otimes_B \eta)\Delta_0(x \otimes y) \ .
\end{aligned}$$

By axiom 2) the image of Δ has moreover the property

$$x^{(1)} \otimes bx^{(2)} \otimes x^{(3)} \otimes y = x^{(1)} \otimes x^{(2)} \otimes x^{(3)}b \otimes y$$

for all $b \in B$, and so we obtain $\Delta(G) \subset G \times_B G$. Coassociativity of Δ follows from axiom 7) for the A-B-torsor T. It is straightforward to see that Δ is a B^e-bimodule map.

We observe that for $x \otimes y \in G$ and $b \in B$ we have $x \otimes by \in G$, since $x^{(1)} \otimes x^{(2)} \otimes x^{(3)}by = x^{(1)} \otimes bx^{(2)} \otimes x^{(3)}y = x \otimes by \otimes 1$ by axiom 2). This implies that the counit map is well-defined, since $1 \otimes xby = x^{(1)}x^{(2)} \otimes x^{(3)}by = xby \otimes 1$ in $T \otimes_B T$, and hence $xby \in B$ by faithful flatness of T over B. The map ε is clearly a B^e-bimodule map.

Compatibility of Δ and ε, as required in the definition of a $\times_B$-coalgebra, follows from

$$\begin{aligned}\vartheta(G \times_B \varepsilon)\Delta(x \otimes y) &= \overline{(x^{(3)}y)}(x^{(1)} \otimes x^{(2)}) \\ &= x^{(1)} \otimes x^{(2)}x^{(3)}y \\ &= x \otimes y\end{aligned}$$

and

$$\begin{aligned}\vartheta'(\varepsilon \times_B G)\Delta(x \otimes y) &= (x^{(1)}x^{(2)})(x^{(3)} \otimes y) \\ &= x^{(1)}x^{(2)}x^{(3)} \otimes y \\ &= x \otimes y \ .\end{aligned}$$

Moreover, Δ an algebra map since μ satisfies axiom 3), and it is a map of B^e-rings since it commutes with the respective maps from B^e:

$$\begin{aligned}\Delta \circ i_G(b \otimes \bar{c}) &= \Delta(b \otimes c) \\ &= b^{(1)} \otimes b^{(2)} \otimes b^{(3)} \otimes c \\ &= b \otimes 1 \otimes 1 \otimes c \\ &= i_{G \times_B G}(b \otimes \bar{c}) \ .\end{aligned}$$

The same holds for the counit ε, since

$$\varepsilon((x \otimes y)(v \otimes w))(b) = \varepsilon(xv \otimes wy)(b) = xvbwy = \varepsilon(x \otimes y) \circ \varepsilon(v \otimes w)(b)$$

for $b \in B$, and

$$\varepsilon \circ i(b \otimes \bar{c})(b') = bb'c = i_{\mathrm{End}(B)}(b \otimes \bar{c})(b') \ .$$

Altogether we see that the given maps define a $\times_B$-bialgebra structure on G. It is obvious that T becomes a left G-comodule algebra.

The set of coinvariants ${}^{coG}T := \{t \in T \mid \delta(t) = 1 \otimes 1 \otimes t\}$ contains A by axiom 6). Each $t \in {}^{coG}T$ has the property $1 \otimes t = t^{(1)} \otimes t^{(2)}t^{(3)} = t \otimes 1 \in T \otimes_A T$. This implies $t \in A$ by faithful flatness of T over A. Hence, we have ${}^{coG}T = A$.

So the Galois map for the left G-comodule algebra T is given by

$$\beta : T \otimes_A T \longrightarrow G \diamond T \ , \ x \otimes y \mapsto x^{(1)} \otimes x^{(2)} \otimes x^{(3)} y \ ,$$

and it is an isomorphism by faithfully flat descent.

We have $T \otimes_A T^B = \{\sum x_i \otimes y_i \in T \otimes_A T \mid \sum x_i \otimes b y_i = \sum x_i \otimes y_i b \ \forall \ b \in B\}$ by faithful flatness of T over A. Applying the Galois map β yields

$$\beta(x \otimes by) = x^{(1)} \otimes x^{(2)} \otimes x^{(3)} by = x^{(1)} \otimes bx^{(2)} \otimes x^{(3)} y$$

and

$$\beta(x \otimes yb) = x^{(1)} \otimes x^{(2)} \otimes x^{(3)} yb \ .$$

This means that $\beta(T \otimes_A T^B) = G \times_B T$.

Hence, T is a left G-Galois extension of A by Definition 4.1.20. Eventually, Lemma 4.1.21 implies that G is a $\times_B$-Hopf algebra.

Although it is clear that part 2) of the theorem follows in a similar way using the left descent data $D_l : T \otimes_B T \longrightarrow T \otimes_A T \otimes_B T$, we mention a few facts from the proof:

The algebra structure on H is given by $(x \otimes y)(v \otimes w) = (vx \otimes yw)$ for $x \otimes y, v \otimes w \in H$ and makes H into a subalgebra of $(T \otimes_B T)^B$.
An A_e-ring structure on $H = {}^{D_l}(T \otimes_B T)$ is then given by $i_H : \bar{A} \otimes A \longrightarrow H \ , \ \bar{a} \otimes b \mapsto a \otimes b$, and it induces the $\bar{A} \otimes A$-bimodule structure

$$(\bar{a} \otimes b) \triangleright (x \otimes y) = xa \otimes by \quad , \quad (x \otimes y) \triangleleft (\bar{a} \otimes b) = ax \otimes yb$$

for $\bar{a} \otimes b \in \bar{A} \otimes A$. This implies in particular that the map $\varepsilon : H \longrightarrow \mathrm{End}(A)$ given by $\varepsilon(x \otimes y)(a) := xay$, is an A_e-ring map $H \longrightarrow \mathrm{End}(A)^{op}$, which proves one of the axioms for a $\times^A$-bialgebra. The comultiplication on H is given by

$$\Delta : H \longrightarrow H \times^A H \ , \ x \otimes y \mapsto x \otimes y^{(1)} \otimes y^{(2)} \otimes y^{(3)} \ ,$$

and T is a right H-Galois extension of B with the Galois map

$$T \otimes_B T \longrightarrow T \vartriangle H \ , \ x \otimes y \mapsto xy^{(1)} \otimes y^{(2)} \otimes y^{(3)} \ .$$

By what we have shown above, it is clear that H is a $\times^A$-Hopf algebra. □

This result generalizes both the results of Grunspan for quantum torsors and Schauenburg's theorem for B-torsors. If $A = k$ or $B = A = k$, then the corresponding $\times^A$-Hopf algebra is a k-Hopf algebra and T is a regular Hopf-Galois extension.

Our result above says, that we can associate to each A-B-torsor a $\times_B$-Hopf algebra and a $\times^A$-Hopf algebra. We apply this construction method to the canonical $\bar{A}$-A-torsor structure on a $\times_A$-Hopf algebra L that we constructed in Proposition 5.2.4.

Proposition 5.2.11 *Let L be a $\times_A$-Hopf algebra and assume that it is faithfully flat over A. Then L is isomorphic to the $\times_A$-Hopf algebra*

$$G = \{x \otimes y \in L \otimes_{\bar{A}} L \mid x^{(1)} \otimes x^{(2)} \otimes x^{(3)} y = x \otimes y \otimes 1\}$$

that arises by Theorem 5.2.10 from the $\bar{A}$-A-torsor structure

$$\mu : L \longrightarrow L \otimes_{\bar{A}} L \otimes_A L \ , \ \ell \mapsto \ell_{(1)+} \otimes \ell_{(1)-} \otimes \ell_{(2)}$$

on L.

Proof. We consider the map $\kappa : L \longrightarrow G \ , \ \ell \mapsto \ell_+ \otimes \ell_-$. It is well-defined since

$$\begin{aligned} {\ell_+}^{(1)} \otimes {\ell_+}^{(2)} \otimes {\ell_+}^{(3)} \ell_- &= \ell_{+(1)+} \otimes \ell_{+(1)-} \otimes \ell_{+(2)} \ell_- \\ &= \ell_+ \otimes \ell_- \otimes 1 \end{aligned}$$

for all $\ell \in L$ by property (5.3) in Proposition 5.2.3.
We use $\bar{a}_+ \otimes \bar{a}_- = 1 \otimes a$ for all $a \in A$, to obtain $\kappa(\bar{a}\ell) = \bar{a}_+ \ell_+ \otimes \ell_- \bar{a}_- = \ell_+ \otimes \ell_- a = \bar{a} \cdot (\ell_+ \otimes \ell_-) = \bar{a}\kappa(\ell)$ by definition of the A^e-bimodule structure on G. This shows that κ is left $\bar{A}$-linear. Now G can be considered as a right A-module via the left $\bar{A}$-action, and L becomes an (A, A)-bimodule via the given left $A \otimes \bar{A}$-module structure. Then the map

$$\kappa \otimes_A L : L \otimes_A L \longrightarrow G \otimes_A L \ , \ \ell \otimes m \mapsto \ell_+ \otimes \ell_- \otimes m$$

is well-defined, and has the inverse

$$G \otimes_A L \longrightarrow L \otimes_A L \ , \ x \otimes y \otimes m \mapsto x_{(1)} \otimes x_{(2)} y m \ .$$

This follows from $\ell_{+(1)} \otimes \ell_{+(2)} \ell_- m = \ell \otimes m$ by (5.3), and $x_{(1)+} \otimes x_{(1)-} \otimes x_{(2)} y m = x \otimes y \otimes m$ by definition of G. So we can conclude by faithful flatness of L over A that the map κ is bijective.

It remains to be shown that κ is a morphism of $\times_A$-bialgebras. It is clearly a map of $A \otimes \bar{A}$-rings, and we get by (5.8) for all $\ell \in L$

$$\begin{aligned}\Delta_G \circ \kappa(\ell) &= \Delta_G(\ell_+ \otimes \ell_-)\\ &= \ell_{+(1)+} \otimes \ell_{+(1)-} \otimes \ell_{+(2)} \otimes \ell_-\\ &= \ell_{(1)+} \otimes \ell_{(1)-} \otimes \ell_{(2)+} \otimes \ell_{(2)-}\\ &= (\kappa \times_A \kappa)\Delta_L(\ell)\ .\end{aligned}$$

Finally we have

$$(\varepsilon_G \circ \kappa)(\ell)(1) = \varepsilon_G(\ell_+ \otimes \ell_-)(1) = \ell_+\ell_- = \varepsilon_L(\ell)(1)$$

by (5.11), showing that $\varepsilon_G \circ \kappa = \varepsilon_L$ as A^e-ring maps $L \longrightarrow \mathrm{End}(A)$. □

The previous proposition is consistent with what we already know for k-Hopf algebras: Each $\times_A$-Hopf algebra L that is faithfully flat over A is a left L-Galois extension of A.
By what we have seen in Theorem 5.2.10, the $\bar{A}$-A-torsor structure on a $\times_A$-Hopf algebra L also induces a $\times^{\bar{A}}$-Hopf algebra $H \subseteq (T \otimes_{\bar{A}} T)^{\bar{A}}$. In the sequel, we explain how it is linked to L.

Given a $\times_A$-bialgebra L with the A^e-ring structure $i_L : A^e = A \otimes \bar{A} \longrightarrow L$, we know from Remark 4.1.7 that the opposite algebra L^{op} is a $\times^A$-bialgebra (this expresses exactly our construction of $\times^A$-bialgebra in Section 4.1). Its A_e-ring structure is given by $i_{L^{op}} : \bar{A} \otimes A \longrightarrow L^{op}$, $i_{L^{op}}(\bar{a} \otimes b) := i_L(a \otimes \bar{b})$, and consequently the right action of A resp. $\bar{A}$ on L^{op} comes from the left action of $\bar{A}$ resp. A on L. The left actions are induced analogously by the respective right actions. This implies that, as k-modules, $L^{op} \vartriangle L^{op} = \int_a {L^{op}}_a \otimes {L^{op}}_{\bar{a}}$ equals $\int_a {}_{\bar{a}}L \otimes {}_aL = L \diamond L$, and $L^{op} \otimes_{\bar{A}} L^{op}$ equals $\int_a {}_aL \otimes L_a$.

We have seen in Remark 4.1.3 that endowing L with the coopposite comultiplication

$$\Delta^{cop} : L \xrightarrow{\Delta} L \times_A L \xrightarrow{\sigma} \int^b \int_a {}_aL_b \otimes {}_{\bar{a}}L_{\bar{b}}\ ,\ \ell \mapsto \ell_{(2)} \otimes \ell_{(1)}$$

leads to a $\times_{\bar{A}}$-bialgebra L^{cop} with the $\bar{A}^e$-ring structure $i_{L^{cop}} : \bar{A} \otimes A \longrightarrow L^{cop}$, $i_{L^{cop}}(\bar{a} \otimes b) := i_L(b \otimes \bar{a})$. Here, the actions of A resp. $\bar{A}$ stay the same.

Applying both these constructions to L yields a $\times^{\bar{A}}$-bialgebra L^{opcop} with $i_{L^{opcop}} : \bar{A}_e = A \otimes \bar{A} \longrightarrow L^{opcop}$, $i_{L^{opcop}}(a \otimes \bar{b}) = i_{L^{cop}}(\bar{a} \otimes b) = i_L(b \otimes \bar{a})$. The A resp. $\bar{A}$-bimodule structure on L^{opcop} is such that, as k-modules, $\int_a {L^{opcop}}_{\bar{a}} \otimes {L^{opcop}}_a = \int_a {}_aL \otimes {}_{\bar{a}}L$, and $L^{opcop} \otimes_A L^{opcop} = \int_a {}_{\bar{a}}L \otimes L_{\bar{a}}$.

We assume now that L is a $\times_A$-Hopf algebra, i.e.

$$\beta : L \otimes_{\bar{A}} L \longrightarrow L \diamond L = \int_a {}_{\bar{a}}L \otimes {}_aL \ , \ \ell \otimes m \mapsto \ell_{(1)} \otimes \ell_{(2)} m$$

is a bijective map. The symmetry in $\mathcal{M}_k$ induces the isomorphisms

$$\rho_1 : \int_a {}_{\bar{a}}L \otimes L_{\bar{a}} \longrightarrow \int_a L_{\bar{a}} \otimes {}_{\bar{a}}L = L \otimes_{\bar{A}} L \ , \ \ell \otimes m \mapsto m \otimes \ell$$

and

$$\rho_2 : \int_a {}_{\bar{a}}L \otimes {}_aL \longrightarrow \int_a {}_aL \otimes {}_{\bar{a}}L \ , \ \ell \otimes m \mapsto m \otimes \ell \ .$$

The $\times^{\bar{A}}$-bialgebra L^{opcop} is a $\times^{\bar{A}}$-Hopf algebra by Definition 4.1.17, if the map

$$\beta_{opcop} : L^{opcop} \otimes_A L^{opcop} \longrightarrow \int_a {L^{opcop}}_{\bar{a}} \otimes {L^{opcop}}_a \ , \ \ell \otimes m \mapsto m_{(2)}\ell \otimes m_{(1)}$$

is bijective. We obtain for $\ell \otimes m \in \int_a {}_{\bar{a}}L \otimes L_{\bar{a}}$

$$\begin{aligned} \rho_2 \circ \beta \circ \rho_1(\ell \otimes m) &= \rho_2 \circ \beta(m \otimes \ell) = \rho_2(m_{(1)} \otimes m_{(2)}\ell) \\ &= m_{(2)}\ell \otimes m_{(1)} = \beta_{opcop}(\ell \otimes m) \ . \end{aligned}$$

With the identifications we made above, it follows that bijectivity of β implies that β_{opcop} is bijective, and so L^{opcop} is a $\times^{\bar{A}}$-Hopf algebra.

We can now prove that this opposite coopposite $\times^{\bar{A}}$-Hopf algebra L^{opcop} arises in Theorem 5.2.10, when we apply it to the $\bar{A}$-A-torsor structure on the $\times_A$-Hopf algebra L.

Theorem 5.2.12 *Let L be a $\times_A$-Hopf algebra that is faithfully flat over both A and $\bar{A}$. Assume that L^{cop} is a $\times_{\bar{A}}$-Hopf algebra.*
Then the $\times^{\bar{A}}$-Hopf algebra

$$H = \{x \otimes y \in L \otimes_A L \mid xy_{(1)+} \otimes y_{(1)-} \otimes y_{(2)} = 1 \otimes x \otimes y\}$$

that arises from the $\bar{A}$-A-torsor structure on L is isomorphic to L^{opcop}.

Proof. By definition, L^{cop} is a $\times_{\bar{A}}$-Hopf algebra iff the map

$$\beta_{cop} : L \otimes_A L \longrightarrow \int_a {}_aL \otimes {}_{\bar{a}}L \ , \ \ell \otimes m \mapsto \ell_{(2)} \otimes \ell_{(1)} m$$

is bijective. We are going to denote $\ell^+ \otimes \ell^- := {\beta_{cop}}^{-1}(\ell \otimes 1) \in L \otimes_A L$ for $\ell \in L$. Then, as can be seen by modifying the results of Proposition 5.2.3, the following hold:

$$\ell^+{}_{(2)} \otimes \ell^+{}_{(1)}\ell^- = \ell \otimes 1 \in \int_a {}_aL \otimes {}_{\bar{a}}L \tag{5.12}$$

$$\ell_{(2)}{}^+ \otimes \ell_{(2)}{}^- \ell_{(1)} = \ell \otimes 1 \in L \otimes_A L \tag{5.13}$$

$$(\ell m)^+ \otimes (\ell m)^- = \ell^+ m^+ \otimes m^- \ell^- \in L \otimes_A L \tag{5.14}$$

$$\ell^+{}_{(2)} \otimes \ell^+{}_{(1)} \otimes \ell^- = \ell_{(2)} \otimes \ell_{(1)}{}^+ \otimes \ell_{(1)}{}^- \in \int_{ab} {}_aL \otimes {}_{\bar{a}}L_b \otimes {}_bL \tag{5.15}$$

$$\ell^+ \otimes \ell^-{}_{(2)} \otimes \ell^-{}_{(1)} = \ell^{++} \otimes \ell^- \otimes \ell^{+-} \in \int_{ab} L_a \otimes {}_bL \otimes {}_{a\bar{b}}L \tag{5.16}$$

$$\ell^+ \ell^- = \varepsilon(\ell)(1) \tag{5.17}$$

We also note that $a^+ \otimes a^- = 1 \otimes \bar{a}$ and $\bar{a}^+ \otimes \bar{a}^- = \bar{a} \otimes 1$ for all $a \in A$.

By Proposition 5.2.4, L^{cop} becomes an A-$\bar{A}$-torsor with the torsor structure map

$$\mu_{L^{cop}} : L \longrightarrow L \otimes_A L \otimes_{\bar{A}} L \ , \ \ell \mapsto \ell_{(2)}{}^+ \otimes \ell_{(2)}{}^- \otimes \ell_{(1)} \ .$$

Then L^{cop} is isomorphic as $\times_{\bar{A}}$-Hopf algebras to $\tilde{G} := \{x \otimes y \in L \otimes_A L \mid x_{(2)}{}^+ \otimes x_{(2)}{}^- \otimes x_{(1)}y = x \otimes y \otimes 1\}$ by Theorem 5.2.11. The isomorphism $L^{cop} \longrightarrow \tilde{G}$ is of course given by $\ell \mapsto \ell^+ \otimes \ell^-$.

We show that the sets $\tilde{G}$ and H are equal. Let $x \otimes y \in H$, that is $xy_{(1)+} \otimes y_{(1)-} \otimes y_{(2)} = 1 \otimes x \otimes y$ in $L \otimes_{\bar{A}} L \otimes_A L$. We apply the map

$$\lambda : L \otimes_{\bar{A}} L \otimes_A L \longrightarrow L \otimes_A L \otimes_{\bar{A}} L \ , \ \ell \otimes m \otimes n \mapsto \ell m_{(2)}{}^+ \otimes m_{(2)}{}^- \otimes m_{(1)}n$$

to both sides of the equation. Note that λ well-defined, since we have $\ell(\bar{a}m_{(2)})^+ \otimes (\bar{a}m_{(2)})^- \otimes m_{(1)}n = \ell\bar{a}m_{(2)}{}^+ \otimes m_{(2)}{}^- \otimes m_{(1)}n$ for $\bar{a} \in \bar{A}$ using that Δ is an $\bar{A}^e$-module map. We obtain for $x \otimes y \in H$

$$\lambda(1 \otimes x \otimes y) = x_{(2)}{}^+ \otimes x_{(2)}{}^- \otimes x_{(1)}y$$

and

$$\begin{aligned}\lambda(xy_{(1)+} \otimes y_{(1)-} \otimes y_{(2)}) &= xy_{(1)+}y_{(1)-(2)}{}^+ \otimes y_{(1)-(2)}{}^- \otimes y_{(1)-(1)}y_{(2)} \\ &= xy_{(1)++}y_{(1)+-}{}^+ \otimes y_{(1)+-}{}^- \otimes y_{(1)-}y_{(2)} \\ &= xy_+y_-{}^+ \otimes y_-{}^- \otimes 1 \ .\end{aligned}$$

Here, we use for the second equality that applying (5.9) to $y_{(1)} \otimes y_{(2)} \in L \diamond L$ is well-defined, and derive the third equation from (5.4).

We note that the expression ${y_+y_-}^+ \otimes {y_-}^- \in L \otimes_A L$ is well-defined, and claim that ${y_+y_-}^+ \otimes {y_-}^- = 1 \otimes y \in L \otimes_A L$. This can be verified by applying the isomorphism $\beta_{cop} : L \otimes_A L \longrightarrow \int_a {}_aL \otimes {}_{\bar{a}}L$ to both expressions. We have $\beta_{cop}(1 \otimes y) = 1 \otimes y$, and

$$\begin{aligned}\beta_{cop}({y_+y_-}^+ \otimes {y_-}^-) &= y_{+(2)}{y_-^+}_{(2)} \otimes y_{+(1)}{y_-^+}_{(1)}{y_-}^- \\ &= y_{+(2)}y_- \otimes y_{+(1)} \\ &= 1 \otimes y \ ,\end{aligned}$$

where the second equation follows from (5.12), and the third one from (5.3). We have to verify that the second expression is indeed well-defined: We have $y_+ \otimes y_- \in L \otimes_{\bar{A}} L$ and get for $\ell \otimes m \in L \otimes L$ and $a \in A$ that $(\ell\bar{a})_{(2)}{m^+}_{(2)} \otimes (\ell\bar{a})_{(1)}{m^+}_{(1)}m^- = \ell_{(2)}\bar{a}{m^+}_{(2)} \otimes \ell_{(1)}{m^+}_{(1)}m^- = \ell_{(2)}{(\bar{a}m)^+}_{(2)} \otimes \ell_{(1)}{(\bar{a}m)^+}_{(1)}(\bar{a}m)^-$.

Returning to our previous calculation, we now have

$$\begin{aligned}\lambda(xy_{(1)+} \otimes y_{(1)-} \otimes y_{(2)}) &= x{y_+y_-}^+ \otimes {y_-}^- \otimes 1 \\ &= x \otimes y \otimes 1 \ .\end{aligned}$$

So it follows that $x \otimes y \in H$ satisfies the equation $x_{(2)+} \otimes x_{(2)-} \otimes x_{(1)}y = x \otimes y \otimes 1 \in L \otimes_A L \otimes_{\bar{A}} L$, which means that $x \otimes y \in \tilde{G}$.

It is clear that we can show $\tilde{G} \subset H$ by similar reasoning, such that we have altogether $H = \tilde{G}$. This implies by the result in Proposition 5.2.11 that there is a bijection $L^{cop} \longrightarrow H$, $\ell \mapsto \ell^+ \otimes \ell^-$, and it remains to be shown that it induces an isomorphism of $\times^{\bar{A}}$-bialgebras $\tilde{\kappa} : L^{opcop} \longrightarrow H$.

Now $\tilde{\kappa}$ is an algebra morphism, since $\tilde{\kappa}(\ell \cdot m) = \tilde{\kappa}(m\ell) = (m\ell)^+ \otimes (m\ell)^- = m^+\ell^- \otimes \ell^- m^- = (\ell^+ \otimes \ell^-)(m^+ \otimes m^-) = \tilde{\kappa}(\ell)\tilde{\kappa}(m)$ for $\ell, m \in L^{opcop}$. Hence, $\tilde{\kappa}$ is a $\bar{A}_e$-ring map. We have

$$\begin{aligned}\Delta_H \circ \tilde{\kappa}(\ell) &= \Delta_H(\ell^+ \otimes \ell^-) \\ &= \ell^+ \otimes {\ell^-}_{(1)+} \otimes {\ell^-}_{(1)-} \otimes {\ell^-}_{(2)} \\ &= \ell^{++} \otimes {\ell^{+-}}_+ \otimes {\ell^{+-}}_- \otimes \ell^-\end{aligned}$$

and

$$\begin{aligned}(\tilde{\kappa} \times^{\bar{A}} \tilde{\kappa})\Delta_{L^{opcop}}(\ell) &= (\tilde{\kappa} \times^{\bar{A}} \tilde{\kappa})(\ell^+ \otimes \ell^-) \\ &= {\ell_{(2)}}^+ \otimes {\ell_{(2)}}^- \otimes {\ell_{(1)}}^+ \otimes {\ell_{(1)}}^- \\ &= {{\ell^+}_{(2)}}^+ \otimes {{\ell^+}_{(2)}}^- \otimes {\ell^+}_{(1)} \otimes \ell^- \ .\end{aligned}$$

In order to show that both expressions are equal in $\int_{\bar{a}} H_{\bar{a}} \otimes H_a$, which can be identified with $L \otimes_A L \otimes_{\bar{A}} L \otimes_A L$, we apply the isomorphism $L \otimes_A \beta \otimes_A L$ to them. This is well-defined, since $\beta : L \otimes_{\bar{A}} L \longrightarrow L \diamond L$, $\ell \otimes m \mapsto \ell_{(1)} \otimes \ell_{(2)} m$ is left and right A-linear. We obtain

$$\begin{aligned}(L \otimes_A \beta \otimes_A L)({\ell^+}_{(2)}{}^+ \otimes {\ell^+}_{(2)}{}^- \otimes {\ell^+}_{(1)} \otimes \ell^-) &= \\ = {\ell^+}_{(2)}{}^+ \otimes {{\ell^+}_{(2)}{}^-}_{(1)} \otimes {{\ell^+}_{(2)}{}^-}_{(2)} {\ell^+}_{(1)} \otimes \ell^- &= \\ = {\ell^+}_{(2)}{}^{++} \otimes {\ell^+}_{(2)}{}^{+-} \otimes {\ell^+}_{(2)}{}^- {\ell^+}_{(1)} \otimes \ell^- &= \\ = \ell^{++} \otimes \ell^{+-} \otimes 1 \otimes \ell^- ,\end{aligned}$$

where the second equality follows from (5.16). This operation is well-defined, since ${\ell^+}_{(2)} \otimes {\ell^+}_{(1)} \otimes \ell^- \in \int_{ab} {}_a L \otimes_{\bar{a}} L_b \otimes_b L$ and $(a\ell)^+ \otimes {(a\ell)^-}_{(1)} \otimes {(a\ell)^-}_{(2)} m \otimes n = \ell^+ \otimes (\ell^- \bar{a})_{(1)} \otimes (\ell^- \bar{a})_{(2)} m \otimes n = \ell^+ \otimes {\ell^-}_{(1)} \otimes {\ell^-}_{(2)} \bar{a} m \otimes n$ for all $\ell, m, n \in L$ and $a \in A$. The last equality is a consequence of (5.13) applied to $\ell^+ \otimes \ell^-$. The other expression is mapped to

$$\begin{aligned}(L \otimes_A \beta \otimes_A L)(\ell^{++} \otimes {\ell^{+-}}_+ \otimes {\ell^{+-}}_- \otimes \ell^-) &= \\ = \ell^{++} \otimes {\ell^{+-}}_{+(1)} \otimes {\ell^{+-}}_{+(2)} {\ell^{+-}}_- \otimes \ell^- &= \\ = \ell^{++} \otimes \ell^{+-} \otimes 1 \otimes \ell^-,\end{aligned}$$

which can be deduced from (5.3), verifying that the expression in fact well-defined: We have $\ell^{++} \otimes \ell^{+-} \otimes \ell^- \in \int_{ab} L_a \otimes_{a\bar{b}} L \otimes_b L$, and obtain for $\ell \otimes m \otimes n \in L \otimes L \otimes L$ that $\ell \otimes (am)_{+(1)} \otimes (am)_{+(2)} (am)_- \otimes n = \ell \otimes a m_{+(1)} \otimes m_{+(2)} m_- \otimes n = \ell a \otimes m_{+(1)} \otimes m_{+(2)} m_- \otimes n$ in $L \otimes_A L \diamond L \otimes_A L$, and $\ell \otimes (\bar{a}m)_{+(1)} \otimes (\bar{a}m)_{+(2)} (\bar{a}m)_- \otimes n = \ell \otimes m_{+(1)} \otimes m_{+(2)} m_- a \otimes n = \ell \otimes m_{+(1)} \otimes m_{+(2)} m_- \otimes an$ in $L \otimes_A L \diamond L \otimes_A L$.

So altogether we have proved that $\tilde{\kappa} : L^{opcop} \longrightarrow H$ is compatible with the respective comultiplications. Finally, we have

$$(\varepsilon_H \circ \tilde{\kappa})(\ell)(1) = \varepsilon_H(\ell^+ \otimes \ell^-)(1) = \ell^+ \ell^- = \varepsilon_{L^{opcop}}(\ell)(1) .$$

This shows that $\tilde{\kappa} : L^{opcop} \longrightarrow H$ is an isomorphism of $\times^{\bar{A}}$-bialgebras and therefore an isomorphism of $\times^{\bar{A}}$-Hopf algebras. □

5.3 A Jones Tower with Coactions of $\times^R$-Hopf Algebras

We apply our results for A-B-torsors that we obtained in the previous section to the theory of depth two extensions. The notion of A-B-torsor allows a torsor structure map with tensor products over two nontrivial rings. Therefore, we do no longer have to restrict our considerations to irreducible depth two extensions, but have the following more general result:

Proposition 5.3.1 *Let $N \subset M$ be a right depth two algebra extension and set $R := M^N$. Then M is an R-N-torsor.*

Proof. Both $R = M^N$ and N subalgebras of M with the property $rn = nr$ for all $r \in R$ and $n \in N$. So there exits an $R \otimes N$-ring structure on M, given by $i_M : R \otimes N \longrightarrow M$, $i_M(r \otimes n) := rn$.
Let (c_i, γ_i) be a right D2 quasibasis for the extension $N \subset M$ according to Lemma 5.1.2. We claim that the map

$$\mu : M \longrightarrow M \otimes_R M \otimes_N M \ , \ m \mapsto \sum \gamma_i(m) \otimes c_i^1 \otimes c_i^2$$

makes M into an R-N-torsor. This map obviously satisfies axiom 2) in Definition 5.2.1. Since $(M \otimes_N M)^N$ is a natural R-bimodule, we can treat μ as a map $\mu : M \longrightarrow M \otimes_R (M \otimes_N M)^N$. Now, using the isomorphism

$$\rho : M \otimes_R (M \otimes_N M)^N \xrightarrow{\cong} M \otimes_N M \ , \ m \otimes \sum x_j \otimes y_j \mapsto \sum m x_j \otimes y_j$$

from Lemma 5.1.4, the axioms of an R-N-torsor can be proved just as in Proposition 5.1.6 for the special case $R = k$. □

We have shown in Proposition 5.2.5 that each A-B-torsor T gives rise to a left T^A-bialgebroid structure on $(T \otimes_A T)^A$ and a right T^B-bialgebroid structure on $(T \otimes_B T)^B$. For a right D2 extension $N \subset M$, this recovers the right $R = M^N$-bialgebroid $(M \otimes_N M)^N$ from [22]:

Proposition 5.3.2 *Let $N \subset M$ be a right D2 extension and set $R := M^N$. Then there is a right R-bialgebroid structure on $B = (M \otimes_N M)^N$.*
The structure maps are given by

- *source* $s : R \longrightarrow B$, $r \mapsto 1 \otimes r$
- *target* $t : R^{op} \longrightarrow B$, $r \mapsto r \otimes 1$
- $\Delta : B \longrightarrow B \otimes_R B$, $\sum_j x_j \otimes y_j \mapsto \sum_{i,j} x_j \otimes \gamma_i(y_j) \otimes c_i^1 \otimes c_i^2$
- $\varepsilon : B \longrightarrow R$, $\sum_j x_j \otimes y_j \mapsto \sum_j x_j y_j$

where (γ_i, c_i) is a right quasibasis for $N \subset M$.

Proof. The algebra M is an R-N-torsor by Proposition 5.3.1. By the right symmetric version of Theorem 5.1.3, ${}_R B$ is a finitely generated projective generator and therefore left faithfully flat over R. So the conditions of Proposition 5.2.5 are satisfied and we obtain the above structure maps for the right R-bialgebroid $B = (M \otimes_N M)^N$. □

Remark 5.3.3 Let $N \subset M$ be a depth two extension. Then M is both an N-R-torsor and an R-N-torsor. The resulting structure of left R-bialgebroid on $(M \otimes_N M)^N$ equals the left R-bialgebroid structure on B^{op} that arises naturally from the right R-bialgebroid structure on B in the previous proposition.

Let $N \subset M$ be a right depth two extension such that M is faithfully flat over both R and N. Then we can apply Theorem 5.2.10 and obtain by faithfully flat descent a $\times_N$-Hopf algebra that coacts on M on the left and a $\times^R$-Hopf algebra that coacts on the right. As in the irreducible case, it turns out that the latter is the same as the R-bialgebroid from Proposition 5.3.2:

Proposition 5.3.4 *Let $N \subset M$ be a right depth two extension such that M is faithfully flat over both R and N.*
Then the right R-bialgebroid $(M \otimes_N M)^N$ from Proposition 5.3.2 is isomorphic as R-bialgebroids (resp. $\times^R$-bialgebras) to the $\times^R$-Hopf algebra

$$H := {}^{D_l}(M \otimes_N M) = \{x \otimes y \in M \otimes_N M \mid \sum x\gamma_i(y) \otimes c_i^1 \otimes c_i^2 = 1 \otimes x \otimes y\}$$

from Theorem 5.2.10.

Proof. It follows by right faithful flatness of M over R that H is a subset of $(M \otimes_N M)^N$. Recall from Lemma 5.1.4 that we have the isomorphism $\rho : M \otimes_R (M \otimes_N M)^N \longrightarrow M \otimes_N M$, $m \otimes \sum_j x_j \otimes y_j \mapsto \sum m x_j \otimes y_j$. We obtain for $x \otimes y \in (M \otimes_N M)^N$

$$\begin{aligned} \rho(\sum x\gamma_i(y) \otimes c_i^1 \otimes c_i^2) &= \sum x\gamma_i(y) c_i^1 \otimes c_i^2 \\ &= x \otimes y \\ &= \rho(1 \otimes x \otimes y) \end{aligned}$$

by the quasibasis property. Thus, each element of $(M \otimes_N M)^N$ is contained in H, and hence we have $H = (M \otimes_N M)^N$. We have explicitly shown in Proposition 4.1.14 that the right R-bialgebroid structure on $(M \otimes_N M)^N$ corresponds to a structure of $\times^R$-bialgebra. It is obvious that this is the same structure as the $\times^R$-bialgebra structure on H from Theorem 5.2.10. □

Combining Proposition 5.3.1 and Theorem 5.2.10 yields the following result:

Corollary 5.3.5 *Let $N \subset M$ be a right depth two extension with M faithfully flat over both R and N.*
Then $B = (M \otimes_N M)^N$ is a $\times^R$-Hopf algebra, and M is a B-Galois extension of N.

Finally, we apply our results to the generalized Jones tower for Frobenius extensions that we considered in Section 5.1 for the irreducible case.
Let $N \subset M$ be a right depth two Frobenius extension. We consider the Jones tower for Frobenius extensions (5.2)

$$N \subset M \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow ...$$

where $M_1 = M \otimes_N M$, $M_2 = M_1 \otimes_M M_1$, and so on. We have shown in Proposition 5.1.15 that ${M_1}^M \cong R$. If we assume in addition that M is faithfully flat over both R and N, then it follows that M_1 is faithfully flat over both R and M. So M_1 is an R-M-torsor and we obtain as above that $(M_1 \otimes_M M_1)^M$ is a $\times^R$-Hopf algebra and that M is a $(M_1 \otimes_M M_1)^M$-Galois extension of M.
This reasoning can be successively applied to each of the other components of the Jones tower. We obtain ${M_i}^{M_{i-1}} \cong R$ for all $i \in \mathbb{N}$, and M_i is faithfully flat over both R and M_{i-1}. Thus, M_i is an R-M_{i-1}-torsor with a coaction of the $\times^R$-Hopf algebra $(M_i \otimes_{M_{i-1}} M_i)^{M_{i-1}}$. This leads to our final result:

Theorem 5.3.6 *Let $N \subset M$ be a depth two Frobenius extension. Assume that M is faithfully flat over both R and N. Then the Jones tower*

$$N := M_{-1} \subset M := M_0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow ...$$

consists of $\times^R$-Hopf-Galois extensions.
More precisely, each M_i is an H_i-Galois extension of M_{i-1}. The $\times^R$-Hopf algebra that coacts on M_i is given by $H_i := (M_i \otimes_{M_{i-1}} M_i)^{M_{i-1}}$.

This Theorem extends the results of [22], where only bialgebroid coactions on the components M, M_1 and M_2 were established. It turns out that Schauenburg's concept of a $\times^R$-Hopf algebra admits a natural generalization of the results in Section 5.1. The Hopf algebra coactions in the irreducible case are special cases of $\times^R$-Hopf algebra coactions in the case of arbitrary depth two Frobenius extensions. The corresponding Hopf-Galois extensions are particular cases of $\times^R$-Hopf-Galois extensions in the Jones tower.

So it turned out that the intrinsic description of noncommutative principal homogeneous spaces in terms of torsor structures is indeed of practical use. It allowed us to recover $\times^R$-Hopf algebras and $\times^R$-Hopf-Galois extensions from arbitrary depth two extensions.

While principal homogeneous spaces inherit their structure from the groups that act on them, we can draw conclusions in the other direction: A noncommutative torsor provides information about the Hopf algebras that coact on it.

Appendix

We give a short exposition on some tools and notions that are used throughout the main part of the text.

A Monoidal Categories

The theory of monoidal categories, or tensor categories (as they are sometimes called), was developed in the 1960's. References are for instance [27] and [23].

Definition A.1 A *monoidal category* $(\mathcal{C}, \otimes, \alpha, I, \lambda, \rho)$ consists of a category $\mathcal{C}$ together with a bifunctor

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C} \ , \ (A, B) \mapsto A \otimes B \ ,$$

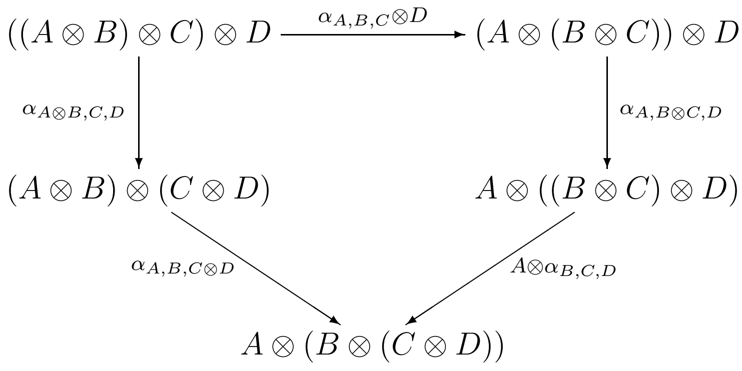
a natural isomorphism

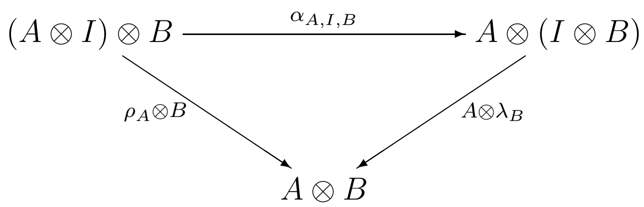
$$\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \ ,$$

a unit object $I \in \mathcal{C}$ and natural isomorphisms

$$\lambda_A : I \otimes A \longrightarrow A$$
$$\rho_A : A \otimes I \longrightarrow A$$

such that the following *coherence diagrams* commute:





A monoidal category is called *strict*, if the coherence morphisms α, λ and ρ are identities.

The standard example of a monoidal category is the category $\mathcal{M}_k$ of modules over a commutative ring k with the tensor product $\otimes_k$ over k. The unit object is of course k, and the tensor product is clearly associative.

Another example is the category ${}_R\mathcal{M}_R$ of bimodules over an arbitrary ring R together with the tensor product over R.

Let B be a bialgebra over k. Then the category ${}_B\mathcal{M}$ of left B-modules is monoidal with the tensor product $\otimes = \otimes_k$ of the underlying category $\mathcal{M}_k$. The tensor product $U \otimes V$ of two left B-modules $U, V \in {}_B\mathcal{M}$ becomes a B-module via the diagonal module structure $b \cdot (u \otimes v) = b_{(1)} \cdot u \otimes b_{(2)} \cdot v$ for $b \in B, u \in U, v \in V$. The left B-module structure on the unit object k is given as the trivial action of B via the counit ε.
Similarly, the category $\mathcal{M}^B$ of right B-comodules is monoidal with the tensor product over k. For $X, Y \in \mathcal{M}^B$, the B-comodule structure on $X \otimes Y$ is the codiagonal structure, given by $\delta(x \otimes y) = x_{(0)} \otimes y_{(0)} \otimes x_{(1)} y_{(1)}$ for $x \in X, y \in Y$. The unit object k has the trivial B-comodule structure $\delta(1) = 1 \otimes 1_B$, that is induced by the unit of B.

In all of the above examples, the coherence morphisms for the respective monoidal category are trivial. In fact, as was proved for instance in [23], every monoidal category is monoidally equivalent to a strict one. Another result of [45] shows that each monoidal category, whose objects are sets with an algebraic structure, can be endowed with an isomorphic tensor product. This tensor product is such that it makes the category into a strict monoidal category. Therefore, one can usually omit denoting the coherence morphisms at all, and treat all monoidal categories as if they were strict.

Definition A.2 Let $(\mathcal{C}, \otimes, \alpha, I, \lambda, \rho)$ and $(\mathcal{D}, \boxtimes, \tilde{\alpha}, J, \tilde{\lambda}, \tilde{\rho})$ be monoidal categories. A *monoidal functor* $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ consists of a natural isomorphism

$$\xi_{A,B} : \mathcal{F}(A) \boxtimes \mathcal{F}(B) \xrightarrow{\cong} \mathcal{F}(A \otimes B)$$

and an isomorphism

$$\xi_0 : J \xrightarrow{\cong} \mathcal{F}(I)$$

such that the following diagrams commute:

$$\begin{array}{ccccc}
(\mathcal{F}(A) \boxtimes \mathcal{F}(B)) \boxtimes \mathcal{F}(C) & \xrightarrow{\xi \boxtimes 1} & \mathcal{F}(A \otimes B) \boxtimes \mathcal{F}(C) & \xrightarrow{\xi} & \mathcal{F}((A \otimes B) \otimes C) \\
\downarrow{\scriptstyle \tilde{\alpha}} & & & & \downarrow{\scriptstyle \mathcal{F}(\alpha)} \\
\mathcal{F}(A) \boxtimes (\mathcal{F}(B) \boxtimes \mathcal{F}(C)) & \xrightarrow[1 \boxtimes \xi]{} & \mathcal{F}(A) \boxtimes \mathcal{F}(B \otimes C) & \xrightarrow[\xi]{} & \mathcal{F}(A \otimes (B \otimes C))
\end{array}$$

$$\begin{array}{ccccc}
J \boxtimes \mathcal{F}(A) & \xrightarrow{\xi_0 \boxtimes 1} & \mathcal{F}(I) \boxtimes \mathcal{F}(A) & \xrightarrow{\xi} & \mathcal{F}(I \otimes A) \\
 & {\scriptstyle \tilde{\lambda}} \searrow & & \swarrow {\scriptstyle \mathcal{F}(\lambda)} & \\
 & & \mathcal{F}(A) & &
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{F}(A) \boxtimes J & \xrightarrow{1 \boxtimes \xi_0} & \mathcal{F}(A) \boxtimes \mathcal{F}(I) & \xrightarrow{\xi} & \mathcal{F}(A \otimes I) \\
 & {\scriptstyle \tilde{\rho}} \searrow & & \swarrow {\scriptstyle \mathcal{F}(\rho)} & \\
 & & \mathcal{F}(A) & &
\end{array}$$

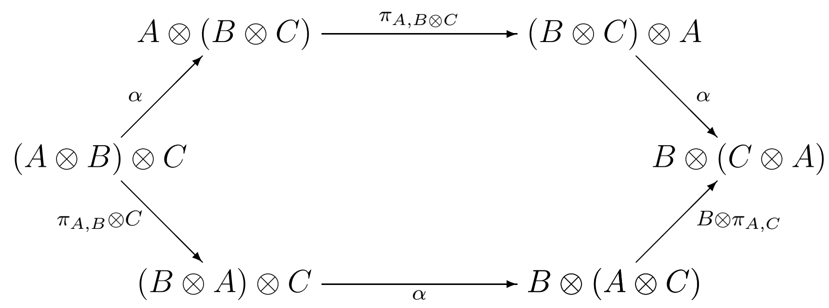
A monoidal functor $(\mathcal{F}, \xi, \xi_0)$ is called *strict*, if ξ and ξ_0 are identities.

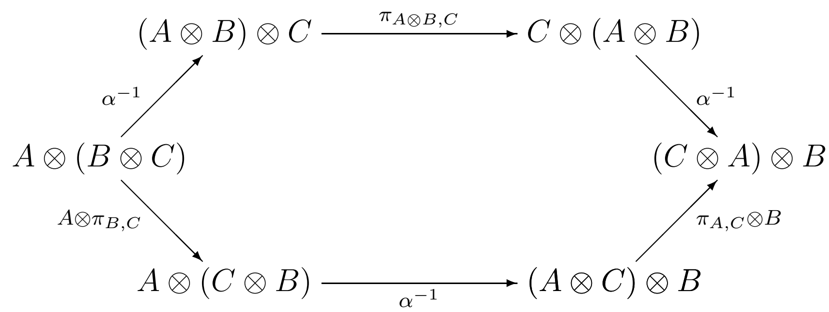
For a bialgebra B, the forgetful functors ${}_B\mathcal{M} \longrightarrow \mathcal{M}_k$ and $\mathcal{M}^B \longrightarrow \mathcal{M}_k$ are both monoidal and in fact strict. This property can be used to characterize bialgebras, as was shown in [31]: A k-algebra A is a bialgebra if and only if the category ${}_A\mathcal{M}$ of A-modules is monoidal such that the forgetful functor ${}_A\mathcal{M} \longrightarrow \mathcal{M}_k$ is a monoidal functor.

Definition A.3 A *prebraiding* for a monoidal category $(\mathcal{C}, \otimes, \alpha, I, \lambda, \rho)$ is a natural transformation

$$\sigma_{A,B} : A \otimes B \xrightarrow{\cong} B \otimes A \ ,$$

that satisfies $\sigma_{A,I} = \sigma_{I,A} = \mathrm{id}_A$ for the unit object I, and makes the following diagrams commute:





A *braiding* for $\mathcal{C}$ is a prebraiding which is a natural isomorphism. A monoidal category with a (pre-)braiding is called a *(pre-)braided monoidal category.* A braiding σ with the property $\sigma^2 = \mathrm{id}$ is called a *symmetry.*

The category $\mathcal{M}_k$ is braided, and in fact symmetric, with the twist

$$\sigma : V \otimes W \longrightarrow W \otimes V \ , \ v \otimes w \mapsto w \otimes v \ .$$

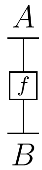
The category ${}_B\mathcal{M}$ of modules over a bialgebra B is in general not braided. However, one can construct a braiding that is based on the symmetry of the underlying category $\mathcal{M}_k$, if B is a so-called quasitriangular bialgebra (see for instance [28]).

Another important example of braided monoidal category is discussed in this thesis. It is the category of Yetter-Drinfeld modules over a Hopf algebra H introduced by Yetter in [60].

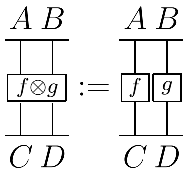
B Graphical Calculus

For computations in monoidal categories one can employ the graphical notation. This way of denoting morphisms in monoidal categories goes back to Penrose [34] and was also applied by Yetter in [60]. It has the advantage of not using "elements" in objects, and is thus valid in any monoidal category.

Let $\mathcal{C}$ be a strict monoidal category, and let $A, B \in \mathcal{C}$. We denote a morphism $f : A \longrightarrow B$ as

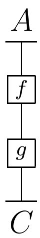
$f =$  .

Given two morphisms $f : A \longrightarrow C$, $g : B \longrightarrow D$, we denote their tensor product $f \otimes g : A \otimes B \longrightarrow C \otimes D$ by

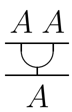
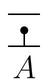
$f \otimes g =$  .

The symbol $\otimes$ is understood implicitly by placing objects A, B next to each other in the top and bottom row. Morphisms are denoted by boxed (or circled) symbols that are connected with their respective domain and target. Due to the existence of coherence isomorphisms, the unit object of the category is not denoted at all.

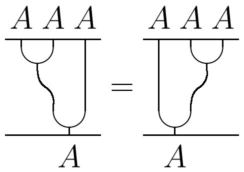
The composition of morphisms $f : A \longrightarrow B$, $g : B \longrightarrow C$ is expressed by

$g \circ f =$  .

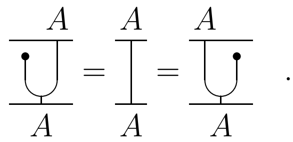
We give a few examples of especially defined morphisms:
Let (A, ∇, η) be an algebra in $\mathcal{C}$. Then the multiplication $\nabla : A \otimes A \longrightarrow A$ and the unit $\eta : I \longrightarrow A$ are denoted by

$\nabla =$  and $\eta =$  .

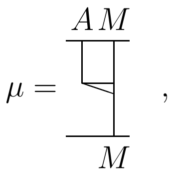
Associativity of ∇ and the unit axioms read as



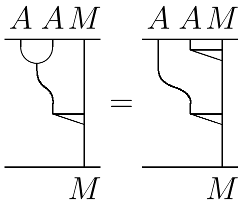
and



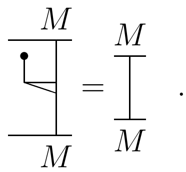
Let M be a left A-module. Then the module structure map $\mu : A \otimes M \longrightarrow M$ is denoted as



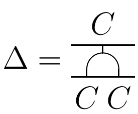
and it satisfies



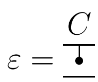
and



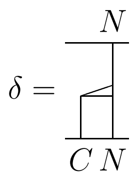
Let (C, Δ, ε) be a coalgebra in the category $\mathcal{C}$. The graphical symbols for the comultiplication and the counit are obtained by turning the symbols for ∇ and η upside down:



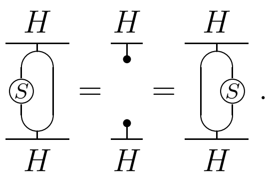
and



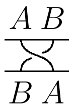
We do the same in order to obtain a notation for the comodule structure map $\delta : N \longrightarrow C \otimes N$ of a left C-comodule N:

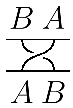


For a Hopf algebra H with the antipode $S : H \longrightarrow H$, the antipode axiom reads as



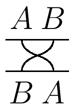
In a braided monoidal category $\mathcal{C}$, the braiding $\sigma_{A,B} : A \otimes B \longrightarrow B \otimes A$ is denoted by

$\sigma_{A,B} =$ 

with inverse $\sigma_{B,A} =$ 

.

If σ is moreover a symmetry, we use the notation

$\sigma_{A,B} =$ 

to express the fact that the inverse of σ is given by the same symbol.

C Duals and inner Hom-Functors

Definition C.1 Let $\mathcal{C}$ be a monoidal category and $X \in \mathcal{C}$ an object. An object $X^* \in \mathcal{C}$ together with a morphism $\mathrm{ev} : X \otimes X^* \longrightarrow I$ is called a *right dual* for M if there exists a morphism $\mathrm{db} : I \longrightarrow X^* \otimes X$ such that

$$(X \xrightarrow{X \otimes \mathrm{db}} X \otimes X^* \otimes X \xrightarrow{\mathrm{ev} \otimes X} X) = \mathrm{id}_X$$

$$(X^* \xrightarrow{\mathrm{db} \otimes X^*} X^* \otimes X \otimes X^* \xrightarrow{X^* \otimes \mathrm{ev}} X^*) = \mathrm{id}_{X^*}$$

The category $\mathcal{C}$ is called *right rigid*, if each object $X \in \mathcal{C}$ has a right dual. The morphisms ev and db are called the *evaluation* resp. the *dual basis*.

Let k be a commutative ring. The category $\mathcal{M}_k$ is in general not rigid. However, if a k-module V is finitely generated and projective then we know by the dual basis lemma that there exist elements $v_1, \ldots, v_n \in V$ and $v^1, \ldots, v^n \in \mathrm{Hom}_k(V, k)$ such that $v = \sum v^i(v) v_i$ for all $v \in V$. Then $\mathrm{Hom}_k(V, k) =: V^*$ is a right dual for V with the evaluation map and dual basis given by

$$\mathrm{ev} : V \otimes \mathrm{Hom}_k(V, k) \longrightarrow k \ , \ v \otimes f \mapsto f(v)$$
$$\mathrm{db} : k \longrightarrow \mathrm{Hom}_k(V, k) \otimes V \ , \ 1 \mapsto \sum v^i \otimes v_i \ .$$

Hence, the category $\mathcal{M}_k^f$ of finitely generated projective k-modules is rigid.

It follows in the same way that an R-bimodule $M \in {}_R\mathcal{M}_R$ over an arbitrary ring R has a right dual if it is finitely generated and projective as a left R-module. The right dual is $M^* = \mathrm{Hom}_R(M, R)$.

Let H be a Hopf algebra over k, and let N be a left H-comodule that is finitely generated and projective over k. We declare a left H-comodule structure on the dual $N^* = \mathrm{Hom}_k(N, K)$ by $\delta(f) := f_{(-1)} \otimes f_{(0)}$ for $f \in \mathrm{Hom}_k(N, K)$ such that $f_{(-1)} f_{(0)}(n) = S(n_{(-1)}) f(n_{(0)})$ for all $n \in N$. Then the evaluation and dual basis maps are H-colinear, and N^* is a dual for N in ${}^H\mathcal{M}$. In particular it was shown in [56] that Hopf algebras can be characterized through this property.

Let X^* be a right dual for $X \in \mathcal{C}$. Then it is easy to see that for all $Y, Z \in \mathcal{C}$ the maps

$$\mathrm{Hom}(X \otimes Y, Z) \longrightarrow \mathrm{Hom}(Y, X^* \otimes Z) \ , \ f \mapsto (X^* \otimes f)(\mathrm{db} \otimes Y)$$
$$\mathrm{Hom}(Y, X^* \otimes Z) \longrightarrow \mathrm{Hom}(X \otimes Y, Z) \ , \ g \mapsto (\mathrm{ev} \otimes Z)(X \otimes g)$$

are inverse to each other. This implies that the functor $\mathcal{C} \longrightarrow \mathcal{C}\ ,\ Z \mapsto X^* \otimes Z$ is right adjoint to the functor $\mathcal{C} \longrightarrow \mathcal{C}\ ,\ Y \mapsto X \otimes Y$.

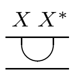
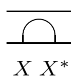
If $X \in \mathcal{C}$ has a right dual, then it is uniquely determined up to isomorphism. This follows from applying the above adjunction to the unit object $I \in \mathcal{C}$.

Let $\mathcal{C}$ be a rigid monoidal category. Then we can define for each morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ the *transposed morphism*

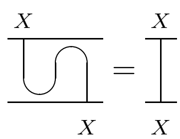
$$(f^* : Y^* \longrightarrow X^*) :=$$
$$(Y^* \xrightarrow{\mathrm{db} \otimes Y^*} X^* \otimes X \otimes Y^* \xrightarrow{X^* \otimes f \otimes Y^*} X^* \otimes Y \otimes Y^* \xrightarrow{X^* \otimes \mathrm{ev}} X^*) \ .$$

This gives rise to a contravariant functor $\mathcal{C} \longrightarrow \mathcal{C}\ ,\ X \mapsto X^*$, the so-called *duality functor*.

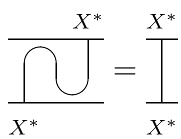
Graphically, the evaluation and dual basis in a rigid monoidal category are denoted by

$\mathrm{ev} =$  and $\mathrm{db} =$  .

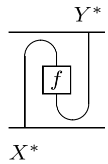
Then the two equations in the definition of a right dual read as

$(\mathrm{ev} \otimes X)(X \otimes \mathrm{db}) =$  $= \mathrm{id}_X$

and

$(X^* \otimes \mathrm{ev})(\mathrm{db} \otimes X^*) =$  $= \mathrm{id}_{X^*}$.

The transposed morphism of $f : X \longrightarrow Y$ is obtained as

$f^* =$ 

.

In a rigid monoidal category $\mathcal{C}$ the functor $\mathcal{C} \longrightarrow \mathcal{C}$, $Y \mapsto X \otimes Y$ is left adjoint for each object $X \in \mathcal{C}$. For categories that are not rigid, this property can be used to obtain the following weaker concept of duality:

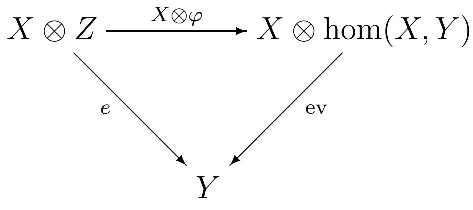
Definition C.2 Let $\mathcal{C}$ be a monoidal category and let X be an object of $\mathcal{C}$. If the functor $\mathcal{C} \longrightarrow \mathcal{C}$, $Y \mapsto X \otimes Y$ is left adjoint, we denote its right adjoint by $\mathcal{C} \longrightarrow \mathcal{C}$, $Y \mapsto \mathrm{hom}(X, Y)$ and call it a *(left) inner hom-functor*.
If the functor $X \otimes -$ is left adjoint for each $X \in \mathcal{C}$, then the category $\mathcal{C}$ is called *left closed*.

According to this definition, a right rigid monoidal category is called left closed. Although this might seem a bit confusing and there exist in fact also different names in the literature, we choose this definition to be consistent with [43].

Since an inner hom-functor is always right adjoint by definition, it comes together with an adjunction morphism

$$\mathrm{ev} : X \otimes \mathrm{hom}(X, Y) \longrightarrow Y$$

and has the following universal property: For each $Z \in \mathcal{C}$ and $e : X \otimes Z \longrightarrow Y$ there exists a unique morphism $\varphi : Z \longrightarrow \mathrm{hom}(X, Y)$ such that the diagram



commutes.

Let H be a Hopf algebra over k. Then for two H-modules $M, N \in {}_H\mathcal{M}$, the k-module $\mathrm{Hom}_k(M, N)$ of k-linear maps has an H-module structure defined by

$$(h \cdot f)(m) := h_{(1)} f(S(h_{(2)})m)$$

for $h \in H$, $f \in \mathrm{Hom}_k(M, N)$ and $m \in M$. Thus, applying the standard hom-tensor adjunction, we see that $\mathrm{Hom}_k(M, N)$ defines a left inner hom-functor in the category ${}_H\mathcal{M}$. This is also the left inner hom-functor in the underlying category ${}_k\mathcal{M}$ of k-modules. Such a situation can be expressed by saying that the underlying functor ${}_H\mathcal{M} \longrightarrow {}_k\mathcal{M}$ preserves left inner hom-functors:

Definition C.3 Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ be a monoidal functor between left closed monoidal categories. By the universal property of the inner hom-functor in $\mathcal{D}$, there exists a unique morphism

$$\zeta : \mathcal{F}(\mathrm{hom}(X, Y)) \longrightarrow \mathrm{hom}(\mathcal{F}(X), \mathcal{F}(Y))$$

for $X, Y \in \mathcal{C}$, for which the diagram

$$\begin{array}{ccc}
\mathcal{F}(X) \otimes \mathcal{F}(\mathrm{hom}(X,Y)) & \xrightarrow{\mathcal{F}(X)\otimes\zeta} & \mathcal{F}(X) \otimes \mathrm{hom}(\mathcal{F}(X), \mathcal{F}(Y)) \\
\cong \downarrow & & \downarrow \mathrm{ev} \\
\mathcal{F}(X \otimes \mathrm{hom}(X,Y)) & \xrightarrow[\mathcal{F}(\mathrm{ev})]{} & \mathcal{F}(Y)
\end{array}$$

commutes. We say that the functor $\mathcal{F}$ *preserves left inner hom-functors* if ζ is an isomorphism for all $X, Y \in \mathcal{C}$.

D Cohomomorphism Objects of Diagrams

We summarize the definitions and main results about cohomomorphism objects and coendomorphism objects that are used in Chapter 2. We follow the exposition given by Pareigis in [32].

Let $\mathcal{M}$ be a small symmetric monoidal abelian category with tensor product $\otimes : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$, and assume that $\mathcal{M}$ is cocomplete and that colimits commute with tensor products.
We consider two diagrams $\omega : \mathcal{C} \longrightarrow \mathcal{M}$ and $\nu : \mathcal{C} \longrightarrow \mathcal{M}$, where $\mathcal{C}$ is another small category. Such a diagram is called finite if it factors through the full subcategory $\mathcal{M}_0$ of objects in $\mathcal{M}$ that possess duals as in Definition C.1.

Theorem and Definition D.1 *Let $\omega, \nu : \mathcal{C} \longrightarrow \mathcal{M}$ be two diagrams in $\mathcal{M}$, and let ν be finite. Then the set of all natural transformations $\mathrm{Nat}(\omega, M \otimes \nu)$ is representable as a functor in M.*
The representing object is given as the difference cokernel of

$$\coprod_{f \in \mathrm{Mor}(\mathcal{C})} \omega(\mathrm{dom}(f)) \otimes \nu(\mathrm{cod}(f))^* \underset{\mathrm{id} \otimes \nu(f)^*}{\overset{\omega(f) \otimes \mathrm{id}}{\rightrightarrows}} \coprod_{X \in \mathrm{Ob}(\mathcal{C})} \omega(X) \otimes \nu(X)^* \ ,$$

where $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$ denote the domain and codomain of the respective morphism f. It is called the cohomomorphism object of ω and ν, and denoted as $\mathrm{cohom}(\nu, \omega)$.
In case $\omega = \nu$, the representing object of $\mathrm{Nat}(\omega, M \otimes \omega)$ is called $\mathrm{coend}(\omega)$, the coendomorphism object of ω.

We remark that the assumption on finiteness of the diagram ν is really just needed to show the existence of the cohomomorphism object $\mathrm{cohom}(\nu, \omega)$. Whenever a coendomorphism exists, it will satisfy the properties that we are going to show in the sequel even without the condition that ν be finite.

If necessary, we are going to abbreviate the symbol $\mathrm{cohom}(\nu, \omega)$ by (ν, ω).

Since every representable functor gives rise to a universal problem, we obtain that a cohomomorphism object satisfies the following universal property:

Corollary D.2 *Let $\omega, \nu : \mathcal{C} \longrightarrow \mathcal{M}$ be two diagrams and let ν be finite. There is a natural transformation $\delta : \omega \longrightarrow \mathrm{cohom}(\nu, \omega) \otimes \nu$ such that for each object $M \in \mathcal{M}$ and each natural transformation $\varphi : \omega \longrightarrow M \otimes \nu$ there exists a unique morphism $\tilde{\varphi} : \mathrm{cohom}(\nu, \omega) \longrightarrow M$ such that the diagram*

$$\begin{array}{ccc} \omega & \xrightarrow{\ \delta\ } & \mathrm{cohom}(\nu, \omega) \otimes \nu \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \tilde{\varphi}} \\ & & M \otimes \nu \end{array}$$

commutes.

Proof. The natural transformation $\delta : \omega \longrightarrow \mathrm{cohom}(\nu, \omega) \otimes \nu$ arises from the composition

$$\omega(X) \xrightarrow{\mathrm{id} \otimes \mathrm{db}} \omega(X) \otimes \nu(X)^* \otimes \nu(X) \xrightarrow{\iota(X,X) \otimes \mathrm{id}} \mathrm{cohom}(\nu, \omega) \otimes \nu(X) \ .$$

for each $X \in \mathcal{C}$. Now the universal property is just a translation of the properties of a difference cokernel. $\square$

Proposition D.3 *Let $\omega, \nu, \tau : \mathcal{C} \longrightarrow \mathcal{M}$ be diagrams in $\mathcal{M}$, and let ν and τ be finite. There is a comultiplication*

$$\Delta : \mathrm{cohom}(\nu, \omega) \longrightarrow \mathrm{cohom}(\tau, \omega) \otimes \mathrm{cohom}(\nu, \tau)$$

which is coassociative in the sense that, given another finite diagram $\varsigma : \mathcal{C} \longrightarrow \mathcal{M}$, the square

$$\begin{array}{ccc} \mathrm{cohom}(\nu,\omega) & \xrightarrow{\Delta} & \mathrm{cohom}(\tau,\omega)\otimes\mathrm{cohom}(\nu,\tau) \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta\otimes\mathrm{id}} \\ \mathrm{cohom}(\varsigma,\omega)\otimes\mathrm{cohom}(\nu,\varsigma) & \xrightarrow{\mathrm{id}\otimes\Delta} & \mathrm{cohom}(\varsigma,\omega)\otimes\mathrm{cohom}(\tau,\varsigma)\otimes\mathrm{cohom}(\nu,\tau) \end{array}$$

commutes.

Proof. We define the morphism Δ via the universal property described in Proposition D.2. It is induced by the diagram

$$\begin{array}{ccc} \omega & \xrightarrow{\delta} & \mathrm{cohom}(\nu,\omega)\otimes\nu \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\Delta\otimes\mathrm{id}} \\ \mathrm{cohom}(\tau,\omega)\otimes\tau & \xrightarrow{\mathrm{id}\otimes\delta} & \mathrm{cohom}(\tau,\omega)\otimes\mathrm{cohom}(\nu,\tau)\otimes\nu \end{array}$$

For coassociativity, we consider the following two commutative diagrams (note that we abbreviate $\mathrm{cohom}(\nu, \omega) := (\nu, \omega)$):

$$\begin{array}{ccc} \omega & \xrightarrow{\delta} & (\nu,\omega)\otimes\nu \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\Delta\otimes\mathrm{id}} \\ (\tau,\omega)\otimes\tau & \xrightarrow{\mathrm{id}\otimes\delta} & (\tau,\omega)\otimes(\nu,\tau)\otimes\nu \\ \downarrow{\scriptstyle\Delta\otimes\mathrm{id}} & & \downarrow{\scriptstyle\Delta\otimes\mathrm{id}\otimes\mathrm{id}} \\ (\varsigma,\omega)\otimes(\tau,\varsigma)\otimes\tau & \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\delta} & (\varsigma,\omega)\otimes(\tau,\varsigma)\otimes(\nu,\tau)\otimes\nu \end{array}$$

$$\begin{array}{ccc}
\omega & \xrightarrow{\delta} & (\nu,\omega)\otimes\nu \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \Delta\otimes \mathrm{id}} \\
(\varsigma,\omega)\otimes\varsigma & \xrightarrow{\mathrm{id}\otimes\delta} & (\varsigma,\omega)\otimes(\nu,\varsigma)\otimes\nu \\
\downarrow{\scriptstyle \mathrm{id}\otimes\delta} & & \downarrow{\scriptstyle \mathrm{id}\otimes\Delta\otimes\mathrm{id}} \\
(\varsigma,\omega)\otimes(\tau,\varsigma)\otimes\tau & \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\delta} & (\varsigma,\omega)\otimes(\tau,\varsigma)\otimes(\nu,\tau)\otimes\nu
\end{array}$$

By the universal property of cohomomorphism objects, we just have to show that the morphisms on the left hand sides of both squares are equal, i.e. that the diagram

$$\begin{array}{ccc}
\omega & \xrightarrow{\delta} & \mathrm{cohom}(\tau,\omega)\otimes\tau \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \Delta\otimes\mathrm{id}} \\
\mathrm{cohom}(\varsigma,\omega)\otimes\varsigma & \xrightarrow[\mathrm{id}\otimes\delta]{} & \mathrm{cohom}(\varsigma,\omega)\otimes\mathrm{cohom}(\tau,\varsigma)\otimes\tau
\end{array}$$

commutes. But this follows from the definition of Δ. □

Remark D.4 If $\nu = \omega = \tau$, then the above proposition shows that $\mathrm{coend}(\omega)$ has a coassociative comultiplication $\Delta : \mathrm{coend}(\omega) \longrightarrow \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega)$. There also exists a counit $\varepsilon : \mathrm{coend}(\omega) \longrightarrow I$, which is defined by

$$\begin{array}{ccc}
\omega & \xrightarrow{\delta} & \mathrm{coend}(\omega)\otimes\omega \\
& \searrow{\scriptstyle \cong} & \downarrow{\scriptstyle \varepsilon\otimes\mathrm{id}} \\
& & I\otimes\omega
\end{array}$$

such that Δ is counitary. Hence, $\mathrm{coend}(\omega)$ has the structure of a coalgebra. It can easily be verified that $\mathrm{cohom}(\nu,\omega)$ is a $(\mathrm{coend}(\omega),\mathrm{coend}(\nu))$-bicomodule with the following multiplications from Proposition D.3:

$$\begin{aligned}
\Delta_\omega &: \ \mathrm{cohom}(\nu,\omega) \longrightarrow \mathrm{coend}(\omega)\otimes\mathrm{cohom}(\nu,\omega) \\
\Delta_\nu &: \ \mathrm{cohom}(\nu,\omega) \longrightarrow \mathrm{cohom}(\nu,\omega)\otimes\mathrm{coend}(\nu)
\end{aligned}$$

It can be shown that $\mathrm{coend}(\omega)$ is the universal coalgebra such that all objects $\omega(X)$ are comodules over $\mathrm{coend}(\omega)$ and all morphisms $\omega(f)$ are comodule homomorphisms.
This leads to the following reconstruction theorem:

Theorem D.5 *Let C be a coalgebra in $\mathcal{M}$ and let $\upsilon : {}^{C}\mathcal{M} \longrightarrow \mathcal{M}$ be the forgetful functor. Then $C \cong \mathrm{coend}(\upsilon)$ as coalgebras.*

Morphisms of cohomomorphism objects arise as follows:

Proposition D.6 *Let $\omega, \nu : \mathcal{C} \longrightarrow \mathcal{M}$ be finite diagrams and let $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ be a functor. Then $\mathcal{F}$ induces a morphism $\phi : \mathrm{cohom}(\nu\mathcal{F}, \omega\mathcal{F}) \longrightarrow \mathrm{cohom}(\nu, \omega)$ that is compatible with the comultiplication on cohoms described in Proposition D.3.*
In particular, $\mathcal{F}$ induces a coalgebra morphism $\mathrm{coend}(\omega\mathcal{F}) \longrightarrow \mathrm{coend}(\omega)$.

Proof. The morphism $\phi : \mathrm{cohom}(\nu\mathcal{F}, \omega\mathcal{F}) \longrightarrow \mathrm{cohom}(\nu, \omega)$ is obtained by the universal property of $\mathrm{cohom}(\nu\mathcal{F}, \omega\mathcal{F})$ from

$$\begin{array}{ccc} \omega\mathcal{F} & \xrightarrow{\delta} & \mathrm{cohom}(\nu\mathcal{F}, \omega\mathcal{F}) \otimes \nu\mathcal{F} \\ & \searrow^{\delta\mathcal{F}} & \downarrow{\scriptstyle \phi\otimes\mathrm{id}} \\ & & \mathrm{cohom}(\nu, \omega) \otimes \nu\mathcal{F} \end{array}$$

Its properties follow from the universal properties of the respective cohomomorphism objects. □

Corollary D.7 *Let $\omega, \nu : \mathcal{C} \longrightarrow \mathcal{M}$ be finite diagrams and let $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{C}$ be an equivalence. Then the map $\phi : \mathrm{cohom}(\nu\mathcal{F}, \omega\mathcal{F}) \longrightarrow \mathrm{cohom}(\nu, \omega)$ is an isomorphism.*

For the rest of this section we assume that the base category $\mathcal{M}$ is symmetric. This additional structure provides additional properties for the cohomomorphism objects $\mathrm{cohom}(\nu, \omega)$.

Definition D.8 Let $\omega : \mathcal{C} \longrightarrow \mathcal{M}$ be a diagram in $\mathcal{M}$. Assume that $\mathcal{C}$ is a monoidal category and that ω is a monoidal functor. Then ω is called a *monoidal diagram.*

Definition D.9 Let $\omega, \nu : \mathcal{C} \longrightarrow \mathcal{M}$ be monoidal diagrams in $\mathcal{M}$ and let $A \in \mathcal{C}$ be an algebra. A natural transformation $\varphi : \omega \longrightarrow A \otimes \nu$ is called *monoidal*, if the diagrams

$$\begin{array}{ccc} \omega(X) \otimes \omega(Y) & \xrightarrow{\varphi(X)\otimes\varphi(Y)} & A \otimes A \otimes \nu(X) \otimes \nu(Y) \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \nabla\otimes\xi} \\ \omega(X \otimes Y) & \xrightarrow{\varphi(X\otimes Y)} & A \otimes \nu(X \otimes Y) \end{array}$$

and

$$\begin{array}{ccc} I & \xrightarrow{\cong} & I \otimes I \\ {\scriptstyle \xi_0}\downarrow & & \downarrow{\scriptstyle \eta\otimes\xi_0} \\ \omega(I) & \xrightarrow{\varphi(I)} & A \otimes \nu(I) \end{array}$$

commute.

We define the tensor product of two diagrams $\omega : \mathcal{C} \longrightarrow \mathcal{M}$ and $\nu : \mathcal{D} \longrightarrow \mathcal{M}$ in $\mathcal{M}$ by

$$\omega \otimes \nu : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{M} \ , \ (\omega \otimes \nu)(X, Y) := \omega(X) \otimes \nu(Y) \ .$$

The tensor product of two diagrams can be viewed as the diagram consisting of all the tensor products of all the objects of the first diagram and all the objects of the second diagram.
For the corresponding cohoms we have:

Proposition D.10 *Let $\omega, \nu : \mathcal{C} \longrightarrow \mathcal{M}$ and $\varsigma, \tau : \mathcal{D} \longrightarrow \mathcal{M}$ be diagrams in $\mathcal{M}$ and let ν and τ be finite. Then*

$$\mathrm{cohom}(\nu \otimes \tau, \omega \otimes \varsigma) \cong \mathrm{cohom}(\nu, \omega) \otimes \mathrm{cohom}(\tau, \varsigma) \ .$$

Corollary D.11 *Under the assumptions of Proposition D.10 there is a natural transformation*

$$\delta : \omega \otimes \varsigma \longrightarrow \mathrm{cohom}(\nu, \omega) \otimes \mathrm{cohom}(\tau, \varsigma) \otimes \nu \otimes \tau \ ,$$

such that for each object $M \in \mathcal{C}$ and each natural transformation $\varphi : \omega \otimes \varsigma \longrightarrow M \otimes \nu \otimes \tau$ there exists a unique morphism $\tilde{\varphi} : \mathrm{cohom}(\nu, \omega) \otimes \mathrm{cohom}(\tau, \varsigma) \longrightarrow M$ such that the diagram

$$\begin{array}{ccc} \omega \otimes \varsigma & \xrightarrow{\delta} & \mathrm{cohom}(\nu, \omega) \otimes \mathrm{cohom}(\tau, \varsigma) \otimes \nu \otimes \tau \\ & {\scriptstyle \varphi}\searrow & \downarrow{\scriptstyle \tilde{\varphi}\otimes \mathrm{id}} \\ & & M \otimes \nu \otimes \tau \end{array}$$

commutes.

Theorem D.12 *Let $\omega, \nu : \mathcal{C} \longrightarrow \mathcal{M}$ be diagrams in $\mathcal{M}$ and let ν be finite. Then $\mathrm{cohom}(\nu, \omega)$ is an algebra and $\delta : \omega \longrightarrow \mathrm{cohom}(\nu, \omega) \otimes \nu$ is a monoidal natural transformation..*

Proof. The multiplication on $\mathrm{cohom}(\nu,\omega)$ results from the diagram

$$\begin{array}{ccc} \omega(X)\otimes\omega(Y) & \xrightarrow{\delta\otimes\delta} & \mathrm{cohom}(\nu,\omega)\otimes\mathrm{cohom}(\nu,\omega)\otimes\nu(X)\otimes\nu(Y) \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \nabla\otimes\xi} \\ \omega(X\otimes Y) & \xrightarrow{\delta} & \mathrm{cohom}(\nu,\omega)\otimes\nu(X\otimes Y) \end{array}$$

together with Proposition D.10. For the unit we consider the diagram $\omega_0 : \mathcal{C}_0 \longrightarrow \mathcal{M}$, where $\mathcal{C}_0$ is the category with just one object J and just one morphism $\mathrm{id} : J \longrightarrow J$, such that $\omega_0(J) = I$. Then the universal map $\delta_0 : \omega_0(J) \longrightarrow \mathrm{coend}(\omega_0)\otimes\omega_0(J)$ is equal to the isomorphism $I \longrightarrow I\otimes I$ and the following diagram induces a morphism $\eta : I \longrightarrow \mathrm{cohom}(\nu,\omega)$:

$$\begin{array}{ccc} I & \xrightarrow{\delta_0} & I\otimes I \\ \downarrow{\scriptstyle \xi_0} & & \downarrow{\scriptstyle \eta\otimes\xi_0} \\ \omega(I) & \xrightarrow{\delta(I)} & \mathrm{cohom}(\nu,\omega)\otimes\nu(I) \end{array}$$

Now we deduce from the commutative diagram

$$\begin{array}{ccc} \omega(X)\otimes I & \xrightarrow{\delta\otimes\delta_0} & \mathrm{cohom}(\nu,\omega)\otimes I\otimes\nu(X)\otimes I \\ \downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}\otimes\eta\otimes\mathrm{id}\otimes\mathrm{id}} \\ \omega(X)\otimes\omega_0(J) & \xrightarrow{\delta\otimes\delta_0} & \mathrm{cohom}(\nu,\omega)\otimes\mathrm{cohom}(\nu,\omega)\otimes\nu(X)\otimes I \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \nabla\otimes\cong} \\ \omega(X) & \xrightarrow{\delta} & \mathrm{cohom}(\nu,\omega)\otimes\nu(X) \end{array}$$

that η is a right unit for ∇. In the same way we can see that η is a left unit and that ∇ is associative. Now it is clear that $\delta : \omega \longrightarrow \mathrm{cohom}(\nu,\omega)\otimes\nu$ is a monoidal natural transformation. □

Corollary D.13 *Let $\omega,\nu,\tau : \mathcal{C} \longrightarrow \mathcal{M}$ be monoidal diagrams in $\mathcal{M}$ and let ν and τ be finite. Then the comultiplication*

$$\Delta : \mathrm{cohom}(\nu,\omega) \longrightarrow \mathrm{cohom}(\tau,\omega)\otimes\mathrm{cohom}(\nu,\tau)$$

from Proposition D.3 and the counit $\varepsilon : \mathrm{cohom}(\nu,\omega) \longrightarrow I$ from Remark D.4 are algebra morphisms.
Thus, if ω is finite, $\mathrm{coend}(\omega)$ *and* $\mathrm{coend}(\nu)$ *are bialgebras and* $\mathrm{cohom}(\nu,\omega)$ *is a* $(\mathrm{coend}(\omega),\mathrm{coend}(\nu))$*-bicomodule algebra.*

It turns out that the universal property of cohomomorphism objects holds even with respect to comodule algebras and algebra morphisms:

Corollary D.14 *Let $\omega, \nu : \mathcal{C} \longrightarrow \mathcal{M}$ be monoidal diagrams and let ν be finite. There is a monoidal natural transformation $\delta : \omega \longrightarrow \mathrm{cohom}(\nu, \omega) \otimes \nu$ such that for each algebra $A \in \mathcal{M}$ and each monoidal natural transformation $\varphi : \omega \longrightarrow A \otimes \nu$ there exists a unique algebra morphism $\tilde{\varphi} : \mathrm{cohom}(\nu, \omega) \longrightarrow A$ such that the diagram*

$$\begin{array}{ccc} \omega & \xrightarrow{\ \delta\ } & \mathrm{cohom}(\nu, \omega) \otimes \nu \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \tilde{\varphi}} \\ & & A \otimes \nu \end{array}$$

commutes.

Altogether we conclude:

Corollary D.15 *For diagrams $\omega, \nu : \mathcal{C} \longrightarrow \mathcal{M}$, with ν finite and $M \in \mathcal{M}$, there is a natural isomorphism*

$$\mathrm{Nat}(\omega, M \otimes \nu) \cong \mathrm{Mor}(\mathrm{cohom}(\nu, \omega), M)\ .$$

If the diagrams are monoidal and $A \in \mathcal{M}$ is an algebra, then there is a natural isomorphism

$$\mathrm{Nat}^{\otimes}(\omega, A \otimes \nu) \cong \mathrm{Alg}(\mathrm{cohom}(\nu, \omega), A)\ .$$

The essence of these results lies in the Tannaka-duality from Theorem D.5. It says that if all representations of a coalgebra resp. bialgebra are known, then the coalgebra resp. bialgebra itself can be recovered:

Theorem D.16 (Tannaka duality) *Let C be a coalgebra in $\mathcal{M}$ and let $\upsilon : \mathcal{M}^C \longrightarrow \mathcal{M}$ be the forgetful functor. Then $C \cong \mathrm{coend}(\upsilon)$ as coalgebras. If moreover C is a bialgebra and υ is a monoidal functor, then $C \cong \mathrm{coend}(\upsilon)$ as bialgebras.*

E Faithfully Flat Descent

We give a brief overview of the mechanism of faithfully flat descent for extensions of noncommutative algebras. We following [41], which itself refers to [2]. Below, we concentrate on left descent data. In the main part of the text we will also apply their right version.

Definition E.1 Let $\eta : R \subset S$ be a ring extension. A (left) descent data from S to R is a left S-module M together with an S-module homomorphism $D : M \longrightarrow S \otimes_R M$ making the diagrams

$$\begin{array}{ccc} M & \xrightarrow{\ D\ } & S \otimes_R M \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle S\otimes_R D} \\ S \otimes_R M & \xrightarrow[S\otimes_R \eta \otimes_R M]{} & S \otimes_R S \otimes_R M \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\ D\ } & S \otimes_R M \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle m} \\ & & M \end{array}$$

commute, where the map $m : S \otimes_R M \longrightarrow M$ is induced by the left S-module structure on M.

Descent data (M, D) from S to R form a category. If N is a left R-module, then the map

$$D : S \otimes_R N \longrightarrow S \otimes_R S \otimes_R N \ , \ \ s \otimes n \mapsto s \otimes 1 \otimes n$$

defines a left descent data on the induced S-module $S \otimes_R N$. This leads to a functor from ${}_R\mathcal{M}$ to the category of descent data from S to R.

Theorem E.2 (Faithfully flat descent) *Let $\eta : R \subset S$ be an inclusion of rings such that S is faithfully flat as a right R-module.*

Then the canonical functor from ${}_R\mathcal{M}$ to the category of descent data from S to R is an equivalence of categories. The inverse equivalence maps a descent data (M, D) to

$${}^D M := \{m \in M \mid D(m) = 1 \otimes m\} \ .$$

In particular, for every descent data (M, D), the map

$$f : S \otimes_R M^D \longrightarrow M \ , \ \ s \otimes m \mapsto sm$$

is an isomorphism with inverse induced by D. That is $f^{-1}(m) = D(m) \in S \otimes_R M^D \subset S \otimes_R M$.

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