TENSOR PRODUCTS AND FORGETFUL FUNCTORS OF ENTWINED MODULES

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ABSTRACT. Let A and B be a K-algebras and let $\mathcal{M}$ be the category of K-modules. It is well known that any algebra-homomorphism $f : A \longrightarrow B$ induces a forgetful functor $U_f : \mathcal{M}_B \longrightarrow \mathcal{M}_A$ that commutes with the two underlying functors $U_A : \mathcal{M}_A \longrightarrow \mathcal{M}$ and $U_B : \mathcal{M}_B \longrightarrow \mathcal{M}$ resp. i.e. $U_A U_f = U_B$. Conversely any functor $\mathcal{F} : \mathcal{M}_B \longrightarrow \mathcal{M}_A$ satisfying $U_A U_f = U_B$ is of the form $\mathcal{F} = U_f$ for a uniquely determined algebra-homomorphism $f : A \longrightarrow B$ [7].

Similarly there is a bijection between the monoidal (tensor) stuctures on $\mathcal{M}_A$ and $U_A : \mathcal{M}_A \longrightarrow \mathcal{M}$ and the bialgebra structures on A.

In this paper we extend these facts to the category $\mathcal{M}_A^C(\psi)$ of entwined modules over an algebra A and a coalgebra C of an entwined structure (A, C, ψ). This leads to the definition of morphisms $(f, g) : (A, C, \psi) \longrightarrow (A', C', \psi')$ where $f : C' \otimes A \longrightarrow A'$ is an entwined measuring of algebras and $g : C' \longrightarrow A' \otimes C$ is an entwined comeasuring of coalgebras. In the case of tensor categories and functors on $\mathcal{M}_A^C(\psi)$ we obtain as diagonal on A an entwined double measuring $\widetilde{\Delta}_A : C \otimes C \otimes A \longrightarrow A \otimes A$ and as multiplication on C an entwined double comeasuring $\widetilde{\nabla}_C : C \otimes C \longrightarrow A \otimes A \otimes C$ satisfying certain compatibility conditions.

INTRODUCTION

In [3] and [4] the notion of an entwined module was introduced and studied under the auspices of Galois theory. In [6] we studied the structure of the category of entwined modules.

Let C be a coalgebra and A be an algebra. Let $\psi : C \otimes A \longrightarrow A \otimes C$ be an entwining, i.e. a linear map satisfying the axioms of a certain distributivity [1, 2] given in section 1. Such a triple (A, C, ψ) is called an entwined structure. Then we can define an entwined module P to be a right C-comodule and a right A-module such that the two structures are compatible, i.e. the comodule structure map is a homomorphism of A-modules and the module structure morphism is a homomorphism of C-comodules. Morphisms of entwined modules are linear maps which are simultaneously homomorphisms of A-modules and of C-comodules. So we get the category of entwined modules $\mathcal{M}_A^C(\psi)$. They generalize e.g. Doi-Hopf modules [5].

If A and C are bialgebras then the tensor product of A-modules as well as of C-comodules is again an A-module resp. a C-comodule. In [6] we studied necessary and sufficient conditions for the entwined structure (A, C, ψ) so that the category of entwined modules with this tensor product becomes a monoidal category. We also studied conditions for this category to become a braided category.

This is, however, not the most general situation. We will investigate under which conditions the category of entwined modules becomes a monoidal category such that the underlying functor $\omega : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}$ to the category of vector spaces preserves the tensor product. In

2000 *Mathematics Subject Classification.* Primary 16A05.

this paper we will develop necessary and sufficient conditions on (A, C, ψ) for $\mathcal{M}_A^C(\psi)$ and $\omega : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}$ to be monoidal.

In [3] Brzeziński defined and studied morphisms between entwined structures (A, C, ψ) and (A', C', ψ'). They were defined as a pair of morphisms $f : A \longrightarrow A'$, $g : C \longrightarrow C'$ satisfying certain compatibility conditions. These morphisms induced some interesting new functors on entwined modules.

In this paper we require that a morphism of entwined structures $\Phi : (A, C, \psi) \longrightarrow (A', C', \psi')$ induces a forgetful $U_\Phi : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}_{A'}^{C'}(\psi')$ (or $U_\Phi : \mathcal{M}_{A'}^{C'}(\psi') \longrightarrow \mathcal{M}_A^C(\psi)$) that is compatible with the underlying functors to the category of vector spaces, or more generally $\mathbb{K}$-modules, $\mathcal{M}$. This is not the case for Brzezinski's definition of morphisms of entwined structures. So we will get a different definition of such a morphism in this paper.

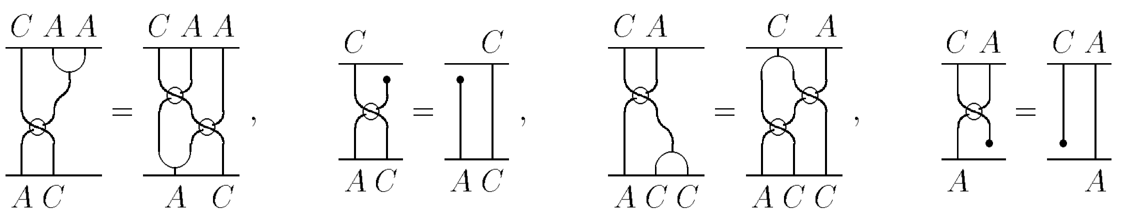
1. Basic definitions and tools

Let $\mathcal{M} = (\mathcal{M}, \otimes, I, \alpha, \lambda, \mu, \sigma)$ be a symmetric monoidal category. The main application we envision is $\mathcal{M} = \mathbb{K}$-Mod, the category of $\mathbb{K}$-modules over a commutative ring $\mathbb{K}$.

Let A be an algebra and C a coalgebra in $\mathcal{M}$. An *entwining* is a linear map $\psi : C \otimes A \longrightarrow A \otimes C$ such that

$$\psi(1_C \otimes \nabla_A) = (\nabla_A \otimes 1_C)(1_A \otimes \psi)(\psi \otimes 1_A),$$
$$\psi(1_C \otimes \eta_A) = \eta_A \otimes 1_C,$$
$$(1_A \otimes \Delta_C)\psi = (\psi \otimes 1_C)(1_C \otimes \psi)(\Delta_C \otimes 1_A),$$
$$(1_A \otimes \varepsilon_C)\psi = \varepsilon_C \otimes 1_A,$$

or in terms of braid diagrams



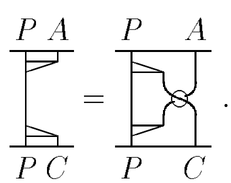
(1)

The triple (A, C, ψ) is called an *entwined structure.*

Let P in $\mathcal{M}$ be a right A-module (with structure map $\mu_P : P \otimes A \longrightarrow P$) and simultaneously a right C-comodule (with structure map $\delta_P : P \longrightarrow P \otimes C$). It is called an *entwined module* if

$$\delta_P \mu_P = (\mu_P \otimes 1_C)(1_P \otimes \psi)(\delta_P \otimes 1_A)$$

or in terms of a braid diagram



The conditions (1) for an entwining structure are necessary and sufficient for the structure map μ_P to be a comodule homomorphism and the structure map δ_P to be a module homomorphism [6]. A morphism of entwined modules is a morphism in $\mathcal{M}$ which is a homomorphism of A-modules and of C-comodules. The category of entwined modules is denoted by $\mathcal{M}_A^C(\psi)$.

A category $\mathcal{C}$ together with a functor $\otimes : \mathcal{M} \times \mathcal{C} \longrightarrow \mathcal{C}$ and natural isomorphisms $\beta : (V \otimes W) \otimes P \longrightarrow V \otimes (W \otimes P)$ and $\zeta : I \otimes P \longrightarrow P$ is called an $\mathcal{M}$*-category* if the morphisms $\alpha, \lambda, \rho, \beta, \zeta$ are coherent in the obvious sense.

Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{M}$-categories. A functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ together with a natural isomorphism $\xi : \mathcal{F}(V \otimes P) \longrightarrow V \otimes \mathcal{F}(P)$ is called an $\mathcal{M}$*-functor* if all associated morphisms are coherent.

Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \longrightarrow \mathcal{D}$ be $\mathcal{M}$-functors. A natural transformation $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is called an $\mathcal{M}$*-transformation* if the diagrams

$$\begin{tikzcd}
\mathcal{F}(X \otimes P) \arrow[r, "\varphi(X \otimes P)"] \arrow[d, "\xi_{\mathcal{F}}"'] & \mathcal{G}(X \otimes P) \arrow[d, "\xi_G"] \\
X \otimes \mathcal{F}(P) \arrow[r, "X \otimes \varphi(P)"] & X \otimes \mathcal{G}(P)
\end{tikzcd}$$

commute.

Observe that $\mathcal{M}$ and $\mathcal{M}_A^C(\psi)$ are $\mathcal{M}$-categories and the underlying functor $\omega : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}$ is an $\mathcal{M}$-functor. If U and V are in $\mathcal{M}$ then $\omega \otimes U$ (with $(\omega \otimes U)(P) := \omega(P) \otimes U$) and $\omega \otimes V$ are $\mathcal{M}$-functors. We are interested in the $\mathcal{M}$-transformations $\varphi : \omega \otimes U \longrightarrow \omega \otimes V$.

Since $\mathcal{M}$ is symmetric we can similarly define an $\mathcal{M}^n$-functor in n-variables, in particular functors $\omega^n \otimes U$ given by $(\omega^n \otimes U)(P_1, \ldots, P_n) := \omega(P_1) \otimes \ldots \otimes \omega(P_n) \otimes U$. Furthermore we can define $\mathcal{M}^n$-transformations. The set of $\mathcal{M}^n$-transformations from $\omega^n \otimes U$ to $\omega^n \otimes V$ is denoted by $\mathrm{Nat}_{\mathcal{M}^n}(\omega^n \otimes U, \omega^n \otimes V)$. If $n = 0$, then $\mathcal{M}^0$ is the one point category and $\omega^0(\emptyset) \cong I$.

In [6, Theorem 3.1] we proved the following

Theorem 1.1. *Let (A, C, ψ) be an entwined structure. Then there is a natural (in U and V) isomorphism*

$$Nat_{\mathcal{M}^n}(\omega^n \otimes U, \omega^n \otimes V) \cong \mathcal{M}(C^{\otimes n} \otimes U, A^{\otimes n} \otimes V).$$

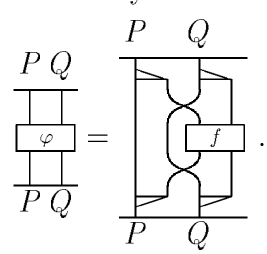
Proof. We gave the proof only for the functor ω^2, but it extends easily to functors of the form $\omega^n \otimes U$ as remarked in [6] after the proof of Theorem 2.1. We only give the map and its inverse without checking the details.

If $\varphi : \omega^n \otimes U \longrightarrow \omega^n \otimes V$ is an $\mathcal{M}^n$-transformation then it induces a morphism in $\mathcal{M}$

$$((1_A \otimes \varepsilon_C)^n \otimes 1_V)\varphi(A \otimes C, \ldots, A \otimes C)((\eta_A \otimes 1_C)^n \otimes 1_U) : C^{\otimes n} \otimes U \longrightarrow A^{\otimes n} \otimes V.$$

If $f : C^{\otimes n} \otimes U \longrightarrow A^{\otimes n} \otimes V$ is a morphism in $\mathcal{M}$ then $(\mu_{P_1} \otimes \ldots \otimes \mu_{P_n} \otimes 1_V)\pi(1_{P_1} \otimes \ldots \otimes 1_{P_n} \otimes f)(\delta_{P_1} \otimes \ldots \otimes \delta_{P_n} \otimes 1_U)$ is an $\mathcal{M}^n$-transformation (where π is a suitable permutation of the tensor factors in $\mathcal{M}$). These two maps are inverses of each other.

The last construction can be better viewed by the following braid diagram (for $n = 2$)



It says that every $\mathcal{M}^n$-transformation φ is induced by a unique morphism f in the described way. $\square$

2. Morphisms of entwined structures and forgetful functors

Let (A, C, ψ) and (A', C', ψ') be entwined structures. In this section we want to study "forgetful" functors $\nu : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}_{A'}^{C'}(\psi')$ such that the diagram

$$\begin{tikzcd}
\mathcal{M}_A^C(\psi) \arrow[rr, "\nu"] \arrow[dr, "\omega"'] & & \mathcal{M}_{A'}^{C'}(\psi') \arrow[dl, "\omega'"] \\
& \mathcal{M} &
\end{tikzcd} \tag{2}$$

commutes. We want to define morphisms between the entwined structures in such a way that the induced forgetful functor has this property.

Assume that such a functor ν satisfying (2) is given. For $P = (P, \mu_P, \delta_P) \in \mathcal{M}_A^C(\psi)$ we have $\nu(P) = (P, \mu'_P, \delta'_P) \in \mathcal{M}_{A'}^{C'}(\psi')$. In particular the A'-module structure on P induces a natural transformation $\mu' : \omega \otimes A' \longrightarrow \omega$. This transformation is an $\mathcal{M}$-transformation. So by Theorem 1.1 there is a unique morphism $f : C \otimes A' \longrightarrow A$ in $\mathcal{M}$ that induces μ'. Similarly there is a unique morphism $g : C \longrightarrow A \otimes C'$ in $\mathcal{M}$ that induces $\delta' : \omega \longrightarrow \omega \otimes C'$.

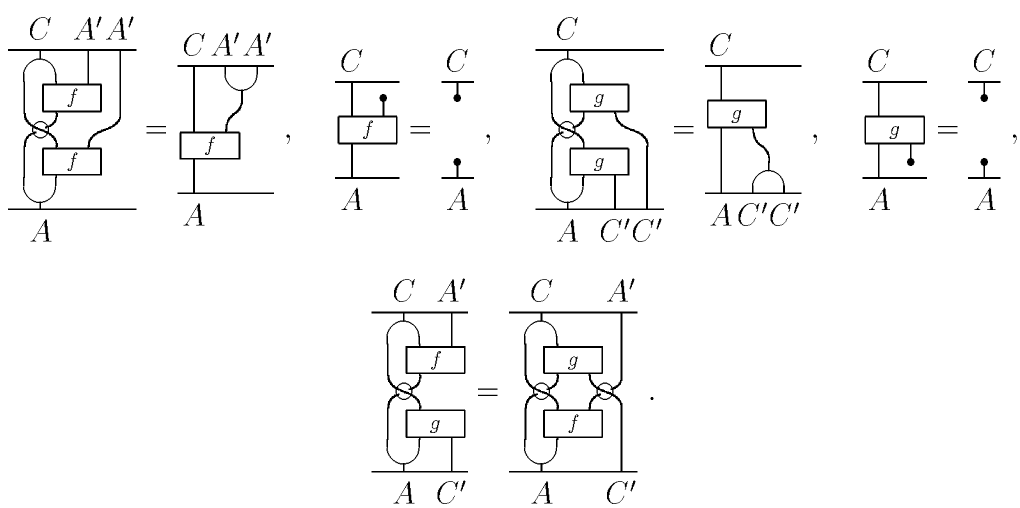
Definition 2.1. Let (A, C, ψ) and (A', C', ψ') be entwined structures. A pair of morphisms $(f : C \otimes A' \longrightarrow A, g : C \longrightarrow A \otimes C')$ is called a *morphism of entwined structures*

$$(f, g) : (A, C, \psi) \longrightarrow (A', C', \psi')$$

if the following hold:

(1) $\nabla_A(1_A \otimes f)(\psi \otimes 1_{A'})(1_C \otimes f \otimes 1_{A'})(\Delta_C \otimes 1_{A'} \otimes 1_{A'}) = f(1_C \otimes \nabla_{A'})$,
(2) $f(1_C \otimes \eta_{A'}) = \eta_A \varepsilon_C$,
(3) $(\nabla_A \otimes 1_{C'} \otimes 1_{C'})(1_A \otimes g \otimes 1_{C'})(\psi \otimes 1_{C'})(1_C \otimes g)\Delta_C = (1_A \otimes \Delta_{C'})g$,
(4) $(1_A \otimes \varepsilon_{C'})g = \eta_A \varepsilon_C$,
(5) $(\nabla_A \otimes 1_{C'})(1_A \otimes g)\psi(1_C \otimes f)(\Delta_C \otimes 1'_A) = (\nabla_A \otimes 1_{C'})(1_A \otimes f \otimes 1_{C'})(\psi \otimes \psi')(1_C \otimes g \otimes 1_{A'}))(\Delta_C \otimes 1'_A)$.

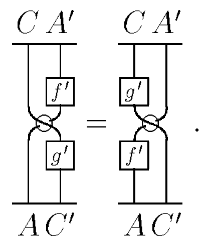
The associated braid diagrams (to be read from top to bottom) are



Remark 2.2. (1) In case of $f = \varepsilon_C \otimes f'$ and $g = \eta_A \otimes g'$ for morphisms $f' : A' \longrightarrow A$ and $g' : C \longrightarrow C'$ the conditions (1) - (5) reduce to the requirement that f' is a morphism of algebras, g' is a morphism of coalgebras and that

$$(1_A \otimes g')\psi(1_C \otimes f') = (f' \otimes 1_{C'})\psi'(g' \otimes 1_{A'})$$

or as braid diagram

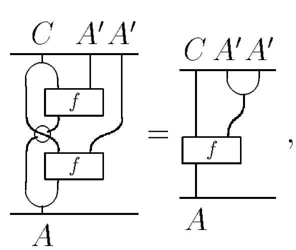


(2) The conditions (1) and (2) say essentially, that $f : C \otimes A' \longrightarrow A$ is an entwined measuring from A' to A by C. The conditions (3) and (4) say, that $g : C \longrightarrow A \otimes C'$ is an entwined comeasuring from C to C' by A. Condition (5) has to be considered as a compatibility condition for the measuring and the comeasuring.

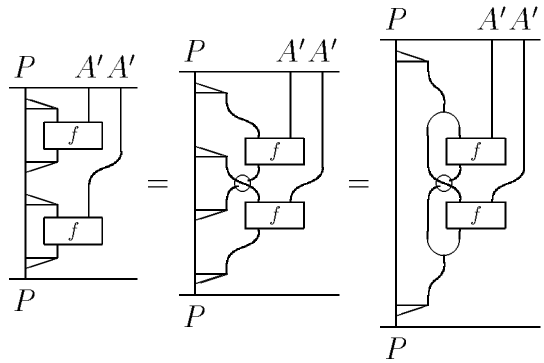
Theorem 2.3. *Let (A, C, ψ) and (A', C', ψ') be entwined structures. There is a bijection between forgetful $\mathcal{M}$-functors $\nu : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}_{A'}^{C'}(\psi')$ satisfying (2) and morphisms $(f, g) : (A, C, \psi) \longrightarrow (A', C', \psi')$.*

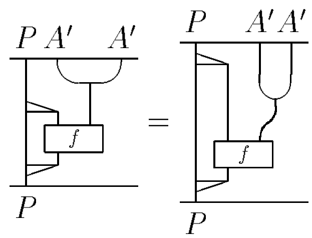
Proof. A functor $\nu : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}_{A'}^{C'}(\psi')$ making the diagram (2) commutative is given by defining functorially the structure of an entwined module over (A', C', ψ') for each module $P \in \mathcal{M}_A^C(\psi)$. That means that we have to construct natural $\mathcal{M}$-transformations $\mu'_P : \omega(P) \otimes A' \longrightarrow \omega(P)$ and $\delta'_P : \omega(P) \longrightarrow \omega(P) \otimes C'$ that make each $P \in \mathcal{M}_A^C(\psi)$ into an entwined module over (A', C', ψ'). By Theorem 1.1 they are completely described by morphisms $f : C \otimes A' \longrightarrow A$ and $g : C \longrightarrow A \otimes C'$. We check now all the conditions for P to become an entwined module over (A', C', ψ') and give equivalent conditions in terms of f and g.

1) The associativity of the A' operation on P leads to two $\mathcal{M}$-transformations $\omega \otimes A' \otimes A' \longrightarrow \omega$ that are equal. So we get



We will only show in this instance how to get to this equivalent condition in terms of braid diagrams. The following two $\mathcal{M}$-transformations are equal





iff the A'-operation on P is associative iff the two morphism that occur on the right hand side of these equations and define the $\mathcal{M}$-transformations are equal. This is equation (1) in the definition of morphisms of entwined structures.

2) With a similar argument one shows that the operation of A' on P is unary iff equation (2) holds in the definition of morphisms of entwined structures.

3) The cooperation of C' on P is coassociative iff equation (3) holds.

4) The cooperation of C' on P is counary iff equation (4) holds.

5) The A'-module and C'-comodule structure on P define an entwined module iff equation (5) holds. $\square$

We remark that in the case $\mathcal{M} = \mathbb{K}$-Mod, the category of modules over a commutative ring $\mathbb{K}$, the requirements to be $\mathcal{M}$-functors resp. $\mathcal{M}$-transformations are automatically satisfied for any $\mathbb{K}$-additive functor and any natural transformation by [8] Proposition A.3 and Theorem A.4.

3. Tensor products on entwined modules

In this section we use again Theorem 1.1 and the same techniques as in the preceding section to study monoidal structures on the category of entwined modules that are defined on the underlying tensor product in $\mathcal{M}$.

Theorem 3.1. *Let (A, C, ψ) be an entwined structure. There is a bijection between $\mathcal{M}^2$-monoidal structures on $\mathcal{M}_A^C(\psi)$ preserved by the underling $\mathcal{M}$-functor $\omega : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}$ and quadruples of morphisms*

- $\widetilde{\Delta}_A : C \otimes C \otimes A \longrightarrow A \otimes A$,
- $\widetilde{\nabla}_C : C \otimes C \longrightarrow A \otimes A \otimes C$,
- $\widetilde{\varepsilon}_A : A \longrightarrow I$,

- $\widetilde{\eta}_C : I \longrightarrow C$,

satisfying the following identities

(1) $(\nabla_A \otimes \nabla_A)(1_A \otimes \sigma \otimes 1_A)(1_A \otimes 1_A \otimes \widetilde{\Delta}_A)(1_A \otimes \sigma \otimes 1_C \otimes 1_A)(\psi \otimes \psi \otimes 1_A)(1_C \otimes \sigma \otimes 1_A \otimes 1_A)(1_C \otimes 1_C \otimes \widetilde{\Delta}_A \otimes 1_A)(1_C \otimes \sigma \otimes 1_C \otimes 1_A \otimes 1_A)(\Delta_C \otimes \Delta_C \otimes 1_A \otimes 1_A)$
$= \widetilde{\Delta}_A(1_C \otimes 1_C \otimes \nabla_A)$,

(2) $\widetilde{\Delta}_A(1_C \otimes 1_C \otimes \eta_A) = (\eta_A \otimes \eta_A)(\varepsilon_C \otimes \varepsilon_C)$,

(3) $(\nabla_A \otimes \nabla_A \otimes 1_C \otimes 1_C)(1_A \otimes \sigma \otimes 1_A \otimes 1_C \otimes 1_C)(1_A \otimes 1_A \otimes \widetilde{\nabla}_C \otimes 1_C)(1_A \otimes \sigma \otimes 1_C \otimes 1_C)(\psi \otimes \psi \otimes 1_C)(1_C \otimes \sigma \otimes 1_A \otimes 1_C)(1_C \otimes 1_C \otimes \widetilde{\nabla}_C)(1_C \otimes \sigma \otimes 1_C)(\Delta_C \otimes \Delta_C)$
$= (1_A \otimes 1_A \otimes \Delta_C)\widetilde{\nabla}_C$

(4) $(1_A \otimes 1_A \otimes \varepsilon_C)\widetilde{\nabla}_C = (\eta_A \otimes \eta_A)(\varepsilon_C \otimes \varepsilon_C)$,

(5) $(\nabla_A \otimes \nabla_A \otimes 1_C)(1_A \otimes \sigma \otimes 1_A \otimes 1_C)(1_A \otimes 1_A \otimes \widetilde{\nabla}_C)(1_A \otimes \sigma \otimes 1_C)(\psi \otimes \psi)(1_C \otimes \sigma \otimes 1_A)(1_C \otimes 1_C \otimes \widetilde{\Delta}_A)(1_C \otimes \sigma \otimes 1_C \otimes 1_A)(\Delta_C \otimes \Delta_C \otimes 1_A)$
$= (\nabla_A \otimes \nabla_A \otimes 1_C)(1_A \otimes \sigma \otimes 1_A \otimes 1_C)(1_A \otimes 1_A \otimes \widetilde{\Delta}_A \otimes 1_C)(1_A \otimes \sigma \otimes 1_C \otimes 1_A \otimes 1_C)(\psi \otimes \psi \otimes \psi)(1_C \otimes \sigma \otimes 1_A \otimes 1_C \otimes 1_A)(1_C \otimes 1_C \otimes \widetilde{\nabla}_C)(1_C \otimes \sigma \otimes 1_C \otimes 1_A)(\Delta_C \otimes \Delta_C \otimes 1_A)$,

(6) $\widetilde{\varepsilon}_A \otimes \widetilde{\varepsilon}_A = \widetilde{\varepsilon}_A \nabla_A$,

(7) $\widetilde{\varepsilon}_A \eta_A = 1_I$,

(8) $\widetilde{\eta}_C \otimes \widetilde{\eta}_C = \Delta_C \widetilde{\eta}_C$,

(9) $\varepsilon_C \widetilde{\eta}_C = 1_I$,

(10) $(\widetilde{\varepsilon}_A \otimes 1_C)\psi(\widetilde{\eta}_C \otimes 1_A) = \widetilde{\eta}_C \widetilde{\varepsilon}_A$,

(11) $\varepsilon_C \otimes 1_A = (1_A \otimes \widetilde{\varepsilon}_A)\widetilde{\Delta}_A(1_C \otimes \widetilde{\eta}_C \otimes 1_A) = (\widetilde{\varepsilon}_A \otimes 1_A)\widetilde{\Delta}_A(\widetilde{\eta}_C \otimes 1_C \otimes 1_A)$,

(12) $\eta_A \otimes 1_C = (1_A \otimes \widetilde{\varepsilon}_A \otimes 1_C)\widetilde{\nabla}_C(1_C \otimes \widetilde{\eta}_C) = (\widetilde{\varepsilon}_A \otimes 1_A \otimes 1_C)\widetilde{\nabla}_C(\widetilde{\eta}_C \otimes 1_C)$,

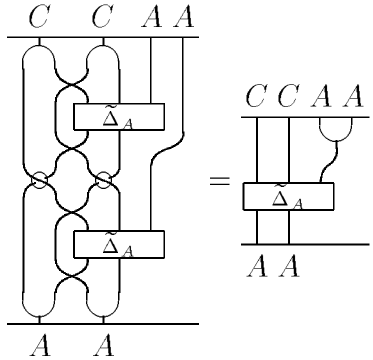
(13) $(\nabla_A \otimes \nabla_A \otimes 1_A)(1_A \otimes \sigma \otimes 1_A \otimes 1_A)(1_A \otimes 1_A \otimes \widetilde{\Delta}_A \otimes 1_A)(1_A \otimes \sigma \otimes 1_C \otimes 1_A \otimes 1_A)(\psi \otimes \psi \otimes \widetilde{\Delta}_A)(1_C \otimes \sigma \otimes 1_A \otimes 1_C \otimes 1_C \otimes 1_A)(1_C \otimes 1_C \otimes \widetilde{\nabla}_C \otimes 1_C \otimes 1_A)(1_C \otimes \sigma \otimes 1_C \otimes 1_C \otimes 1_A)(\Delta_C \otimes \Delta_C \otimes 1_C \otimes 1_A)$
$= (1_A \otimes \nabla_A \otimes \nabla_A)(\sigma \otimes \sigma \otimes 1_A)(1_A \otimes \sigma \otimes \widetilde{\Delta}_A)(1_A \otimes 1_A \otimes \sigma \otimes 1_A)(1_A \otimes 1_A \otimes 1_A \otimes \sigma \otimes 1_A)(1_A \otimes 1_A \otimes 1_A \otimes 1_A \otimes \widetilde{\Delta}_A)(1_A \otimes 1_A \otimes 1_A \otimes \sigma \otimes 1_C \otimes 1_A)(1_A \otimes 1_A \otimes \sigma \otimes 1_C \otimes 1_C \otimes 1_A)(1_A \otimes \sigma \otimes 1_C \otimes 1_C \otimes 1_C \otimes 1_A)(\sigma \otimes \sigma \otimes 1_C \otimes 1_C \otimes 1_A)(1_C \otimes \psi \otimes \psi \otimes 1_C \otimes 1_A)(1_C \otimes 1_C \otimes \sigma \otimes 1_A \otimes 1_C \otimes 1_A)(1_C \otimes 1_C \otimes 1_C \otimes \widetilde{\nabla}_C \otimes 1_A)(1_C \otimes \Delta_C \otimes \Delta_C \otimes 1_A)$,

(14) $(\nabla_A \otimes \nabla_A \otimes 1_A \otimes 1_C)(1_A \otimes \sigma \otimes 1_A \otimes 1_A \otimes 1_C)(1_A \otimes 1_A \otimes \widetilde{\Delta}_A \otimes 1_A \otimes 1_C)(1_A \otimes \sigma \otimes 1_C \otimes 1_A \otimes 1_A \otimes 1_C)(\psi \otimes \psi \otimes \widetilde{\nabla}_C)(1_C \otimes \sigma \otimes 1_A \otimes 1_C \otimes 1_C)(1_C \otimes 1_C \otimes \widetilde{\nabla}_C \otimes 1_C)(1_C \otimes \sigma \otimes 1_C \otimes 1_C)(\Delta_C \otimes \Delta_C \otimes 1_C)$
$= (1_A \otimes \nabla_A \otimes \nabla_A \otimes 1_C)(1_A \otimes 1_A \otimes 1_A \otimes \widetilde{\Delta}_A \otimes 1_C)(1_A \otimes 1_A \otimes \sigma \otimes 1_C \otimes 1_A \otimes 1_C)(1_A \otimes \psi \otimes \psi \otimes 1_A \otimes 1_C)(\sigma \otimes \sigma \otimes 1_A \otimes 1_A \otimes 1_C)(1_C \otimes \sigma \otimes 1_A \otimes 1_A \otimes 1_A \otimes 1_C)(1_C \otimes 1_C \otimes \sigma \otimes 1_A \otimes 1_A \otimes 1_C)(1_C \otimes 1_C \otimes 1_C \otimes \sigma \otimes 1_A \otimes 1_C)(1_C \otimes 1_C \otimes 1_C \otimes 1_C \otimes \widetilde{\nabla}_C)(1_C \otimes 1_C \otimes 1_C \otimes \sigma \otimes 1_C)(1_C \otimes 1_C \otimes \sigma \otimes 1_C)(1_C \otimes \sigma \otimes \widetilde{\nabla}_C)(\sigma \otimes \sigma \otimes 1_C)(1_C \otimes \Delta_C \otimes \Delta_C)$.

Proof. As in the proof of Theorem 2.3 we determine the conditions in terms of braid diagrams which can then be translated into the given formulas.

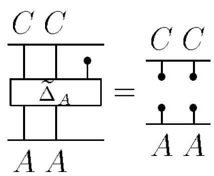
By Theorem 1.1 we have $\mathrm{Nat}_{\mathcal{M}^2}(\omega \otimes \omega \otimes A, \omega \otimes \omega) \cong \mathcal{M}(C \otimes C \otimes A, A \otimes A)$. So the tensor product of two modules P and Q in $\mathcal{M}_A^C(\psi)$ gets an A-structure by a uniquely determined morphism $\widetilde{\Delta}_A : C \otimes C \otimes A \longrightarrow A \otimes A$.

The A-operation on $P \otimes Q$ is associative iff



(1)

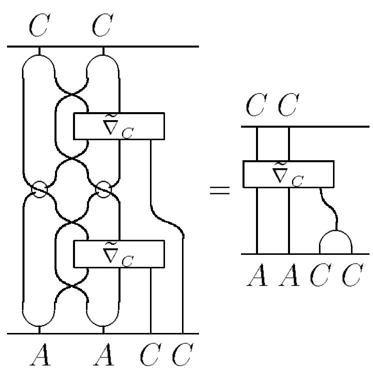
The A-operation on $P \otimes Q$ is unary iff



(2)

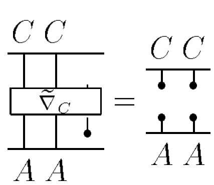
By Theorem 1.1 we also have $\mathrm{Nat}_{\mathcal{M}^2}(\omega \otimes \omega, \omega \otimes \omega \otimes C) \cong \mathcal{M}(C \otimes C, A \otimes A \otimes C)$. So the tensor product of two modules P and Q in $\mathcal{M}_A^C(\psi)$ gets a C-comodule structure by a morphism $\widetilde{\nabla}_C : C \otimes C \to A \otimes A \otimes C$.

The C-cooperation on $P \otimes Q$ is coassociative iff



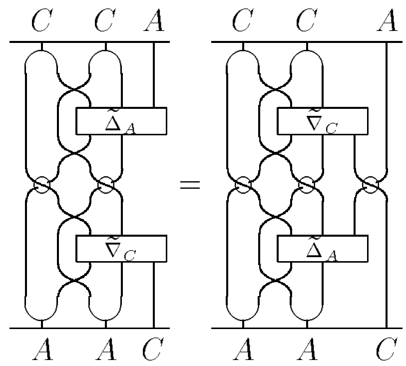
(3)

The C-cooperation on $P \otimes Q$ is counary iff



(4)

The tensor product $P \otimes Q$ is entwined with the given A-module and C-comodule structures iff

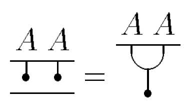


(5)

So equations (1)-(5) describe the tensor functor $\otimes : \mathcal{M}_A^C(\psi) \times \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}_A^C(\psi)$.

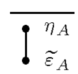
Now we study the special structures on the neutral object I. By Theorem 1.1 we have $\mathrm{Nat}(\omega^0 \otimes A, \omega^0) \cong \mathcal{M}(A, I)$. So the A-operation on I is described by a morphism $\widetilde{\varepsilon}_A : A \longrightarrow I$.

The A-operation on I is associative iff



(6)

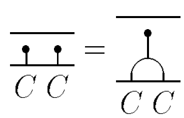
The A-operation on I is unary iff



$= \mathrm{id}_I$ (7)

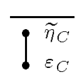
We also have $\mathrm{Nat}(\omega^0, \omega^0 \otimes C) \cong \mathcal{M}(I, C)$. So the C-cooperation on I is described by a morphism $\widetilde{\eta}_C : I \longrightarrow C$.

The C-cooperation on I is coassociative iff



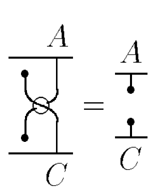
(8)

The C-cooperation on I is counary iff



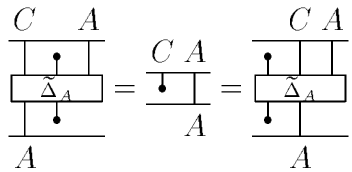
$= \mathrm{id}_I$ (9)

The object I is entwined with the given A-module and C-comodule structures iff



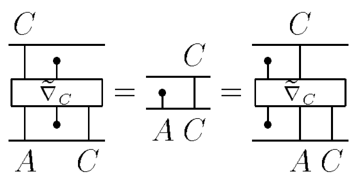
(10)

The object I acts as a left and right neutral object in $\mathcal{M}_A^C(\psi)$, i.e. λ and ρ in $\mathcal{M}$ are morphisms (for the A-module structure and the C-comodule structure) of entwined modules iff



(11)

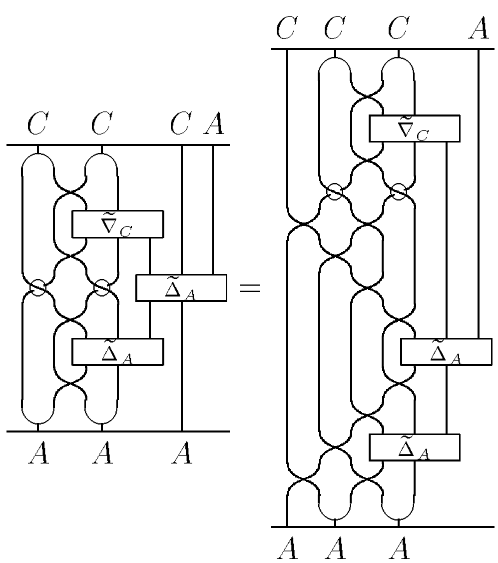
and



(12)

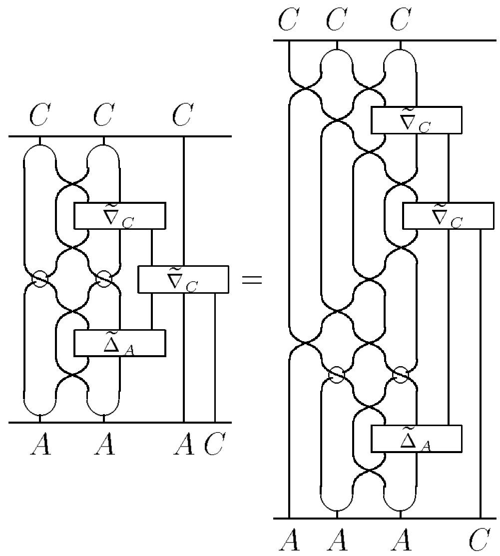
where the first equations comes from $\rho : \omega \otimes I \cong \omega$ and the second equations from $\lambda^{-1} : \omega \cong I \otimes \omega$.

Finally we study the associativity morphism $\alpha : (P \otimes Q) \otimes R \longrightarrow P \otimes (Q \otimes R)$. It is a morphism of A-modules iff



(13)

α is a morphism of C-comodules iff



(14)

$\square$

Corollary 3.2. *Let (A, C, ψ) be an entwined structure. An $\mathcal{M}^2$-monoidal structure on $\mathcal{M}_A^C(\psi)$ preserved by the underlying $\mathcal{M}$-functor $\omega : \mathcal{M}_A^C(\psi) \longrightarrow \mathcal{M}$ is given by morphisms of the form*

- - $\widetilde{\Delta}_A = \varepsilon_C \otimes \varepsilon_C \otimes \Delta_A$ *with* $\Delta_A : A \longrightarrow A \otimes A$,
- - $\widetilde{\nabla}_C = \eta_A \otimes \eta_A \otimes \nabla_C$ *with* $\nabla_C : C \otimes C \longrightarrow C$,
- - $\widetilde{\varepsilon}_A = \varepsilon_A : A \longrightarrow I$,
- - $\widetilde{\eta}_C = \eta_C : I \longrightarrow C$,

iff the following identities hold:

(1) $(\nabla_A \otimes \nabla_A)(1_A \otimes \sigma \otimes 1_A)(\Delta_A \otimes \Delta_A) = \Delta_A \nabla_A$,
(2) $\Delta_A \eta_A = \eta_A \otimes \eta_A$,
(3) $(\nabla_C \otimes \nabla_C)(1_C \otimes \sigma \otimes 1_C)(\Delta_C \otimes \Delta_C) = \Delta_C \nabla_C$,
(4) $\varepsilon_C \nabla_C = \varepsilon_C \otimes \varepsilon_C$,
(5) $(1_A \otimes 1_A \otimes \nabla_C)(1_A \otimes \sigma \otimes 1_C)(\psi \otimes \psi)(1_C \otimes \sigma \otimes 1_A)(1_C \otimes 1_C \otimes \Delta_A) = (\Delta_A \otimes 1_C)\psi(\nabla_C \otimes 1_A)$,
(6) $\varepsilon_A \otimes \varepsilon_A = \varepsilon_A \nabla_A$,
(7) $\varepsilon_A \eta_A = \mathrm{id}_I$,
(8) $\eta_C \otimes \eta_C = \Delta_C \eta_C$,
(9) $\varepsilon_C \eta_C = \mathrm{id}_I$,
(10) $(\varepsilon_A \otimes 1_C)\psi(\eta_C \otimes 1_A) = \eta_C \varepsilon_A$,
(11) $(1_A \otimes \varepsilon_A)\Delta_A = 1_A = (\varepsilon_A \otimes 1_A)\Delta_A$,
(12) $\nabla_C(1_C \otimes \eta_C) = 1_C = \nabla_C(\eta_C \otimes 1_C)$,
(13) $(\Delta_A \otimes 1_A)\Delta_A = (1_A \otimes \Delta_A)\Delta_A$,
(14) $\nabla_C(\nabla_C \otimes 1_C) = \nabla_C(1_C \otimes \nabla_C)$.

Remark 3.3. These are essentially the conditions worked out in [6, Theorem 4.1] where we assumed that A and C are bialgebras in the first place without any additional (co-)action upon each other. The conditions arise from Theorem 3.1 by substituting the particular form of the morphisms $\widetilde{\Delta}_A$ and $\widetilde{\nabla}_C$ and using the properties of an entwined structure. So in this situation we get from the monoidal structure that A and C are bialgebras (conditions (1) –

(4), (6) – (9), and (11) – (14)). The conditions (5) and (10) describe the compatibility with the given entwining and are exactly the conditions of [6, Theorem 4.1].

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