Left Linear Theories

- A Generalization of Module Theory -

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Abstract

In this paper we introduce left linear theories of exponent N (a set) on the set L as maps $L \times L^N \ni (l, \lambda) \longrightarrow l \cdot \lambda \in L$ such that for all $l \in L$ and $\lambda, \mu \in L^N$ the relation $(l \cdot \lambda)\mu = l(\lambda \cdot \mu)$ holds, where $\lambda \cdot \mu \in L^N$ is given by $(\lambda \cdot \mu)(i) = \lambda(i)\mu, i \in N$. We assume that L has a unit, that is an element $\delta \in L^N$ with $l \cdot \delta = l$, for all $l \in L$, and $\delta \cdot \lambda = \lambda$, for all $\lambda \in L^N$. Next, left (resp. right) L-modules and L-M-bimodules and their homomorphisms are defined and lead to categories L-Mod, Mod-L, and L-M-Mod. These categories are algebraic categories and their free objects are described explicitly. Finally, $\mathrm{Hom}(X, Y)$ and $X \otimes Y$ are introduced and their properties are investigated.

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0. Introduction

The operations defining an R-module X over a ring R, a monoid module over a monoid, affine spaces, certain types of barycentric caculi, various convexity theories etc. have in common, that they form certain "linear combinations" subject to such laws as distributivity, associativity, or action of a unit. The general definition for such operations and their axioms can be derived in the case of R-modules as follows.

Let X be any unital module over a given ring R. Let furthermore r_* denote a sequence $(r_1, r_2, \ldots)$ of elements of R with finite support. Then we can associate with r_* the map $\sigma(r_*) : X^{I\!N} \longrightarrow X$ that is given by

$$\sigma(r_*)(m^*) = \sigma(r_*)(m^1, m^2, \ldots) := \sum_{i \in I\!N} r_i m^i, \qquad m^* \in X^{I\!N},$$

where this infinite sum is defined because the support of r_* is finite. We interpret $\sigma(r_*)$ as "operation on X". In particular, $\sigma(r_*)$ operates on $R^{(I\!N)}$, the set of all sequences of elements of R with finite support; that is

$$\sigma(r_*)(s^1_*, s^2_*, \ldots) = \sum_{i \in I\!N} r_i s^i_*.$$

With these notations the following laws are satisfied:

(A_0) if r_*, s^1_*, s^2_*, $\ldots$ are sequences as specified above and $m^* \in X^{I\!N}$, then

$$\sigma(\sigma(r_*)(s^1_*, s^2_*, \ldots))(m^*) = \sigma(r_*)(\sigma(s^1_*)(m^*), \sigma(s^2_*)(m^*), \ldots),$$

(A_1) if $\delta^j_* = (\delta^j_1, \delta^j_2, \ldots)$, with δ^j_i the usual Kronecker symbol, then

$$\sigma(\delta^j_*)(m^*) = m^j, \qquad\qquad m^* \in X^{I\!N}.$$

In terms of the initial module structure, (A_0) is

$$\sum_{i \in I\!N} (\sum_{j \in I\!N} r_j s^j_i) m^i = \sum_{j \in I\!N} r_j (\sum_{i \in I\!N} s^j_i m^i),$$

a combination of the associative law, the commutative law for the addition, and the distributive law.

Conversely, if a set X is given and if, for every r_* as above, $\sigma(r_*) : X^{I\!N} \longrightarrow X$ is a map such that A_0 and A_1 are satisfied, then X, equipped with the compositions

$$\begin{aligned} m^1 + m^2 &:= \sigma((1, 1, 0, \ldots))(m^1, m^2, \ldots), \\ rm^1 &:= \sigma((r, 0, 0, \ldots))(m^1, m^2, \ldots), \end{aligned}$$

is a unital R-module. A simple computation shows that the standard axioms for unital R-modules are equivalent to the axioms (A_0) and (A_1) (see 4.13).

(A_0) and (A_1) can be cast in a different way. The initially defined operation is expressed as a map

$$\sigma : R^{(I\!N)} \times X^{I\!N} \longrightarrow X.$$

Hence it gives rise to a map

$$(R^{(I\!N)})^{I\!N} \times X^{I\!N} \longrightarrow X^{I\!N}.$$

In particular we have

$$(R^{(I\!N)})^{I\!N} \times (R^{(I\!N)})^{I\!N} \longrightarrow (R^{(I\!N)})^{I\!N}.$$

Let us take this map as a "multiplication" on $(R^{(IN)})^{IN}$. Then associativity of this multiplication is expressed by (A_0), for $X = R^{(IN)}$, while unitality of the multiplication is given by (A_1), for $X = R^{(IN)}$. Thus $(R^{(IN)})^{IN}$ is a monoid. Moreover, (A_0) and (A_1), for X, state that X^{IN} is an $(R^{(IN)})^{IN}$-module.

This view of our axioms suggests that $R^{(IN)}$ be replaced by an arbitrary set $\mathcal{L}$ and that a monoid structure on $\mathcal{L}^{IN}$ be derived from a map $\mathcal{L} \times \mathcal{L}^{IN} \longrightarrow \mathcal{L}$. Then, of course, the monoid module structure on X^{IN} has to come from a map $\mathcal{L} \times X^{IN} \longrightarrow X$. It should be noted that this requirement puts a non-trivial restriction on the monoid structure on $\mathcal{L}^{IN}$, and similarly on the module structure on X^{IN}, namely the following:
the i^{th} "component" of a product of two elements in $\mathcal{L}^{IN}$ under the multiplication $\mathcal{L}^{IN} \times \mathcal{L}^{IN} \longrightarrow \mathcal{L}^{IN}$ depends only on the i^{th} "component" of the first factor and in general on the full second factor.

Several instances of this very general scheme have appeared in the literature. The Barycentric Calculus in [4] and [14] is a variant of case where R is a suitable ring and the elements of $\mathcal{L}$ are those $r_* \in R^{(IN)}$ for which $\sum_{i \in IN} r_i = 1$. This theory of "baryzentrischer Kalkül" was later elaborated in [5]. The theory of "Affine Räume" developed in [1], [2] is closely related to the barycentric calculus. In addition, various convexity theories (see [8], [9], [10], [12], [13], etc.) are special cases. In [3], applications to various physical problems are discussed. Additional examples are listed in 1.4.

We call left linear theory $\mathcal{L}$ the special type of monoid structure we described above on the set $\mathcal{L}^N$, where now the set IN of natural numbers is replaced by an arbitrary set N of cardinality greater than two. Furthermore a left $\mathcal{L}$-module is, by definition, a set X together with an operation $\mathcal{L} \times X^N \longrightarrow X$ subject to the laws $(\mathcal{L}_{0,X})$ and $(\mathcal{L}_{1,X})$ that are obtained from (A_0) and (A_1) by replacing IN by N. So we obtain a category of left $\mathcal{L}$-modules. We study this category in order to see which further structures on X can be derived from it. The category of $\mathcal{L}$-$\mathcal{L}$-bimodules turns out to be (almost) monoidal with a specific tensor product. One thing we could not find out is, whether the associativity homomorphism $\alpha : (X \otimes_{\mathcal{L}} Y) \otimes_{\mathcal{L}} Z \longrightarrow X \otimes_{\mathcal{L}} (Y \otimes_{\mathcal{L}} Z)$ is an isomorphism in general (cf. 6.17); however, it satisfies the usual coherence conditions for monoidal categories. Furthermore there is a right adjoint, i.e. an inner hom-functor for this tensor product.

In the first section we will give the general definition of a left linear theory and an extensive collection of examples. Then we introduce left, right, and bi-modules. In the third and forth sections we study properties of zero element, scalars, and addition derived from the given operations. We obtain that certain properties carry over from the left linear theory $\mathcal{L}$ to left $\mathcal{L}$-modules such as zero elements or some kind of additive structure: for example, if $\mathcal{L}$ becomes an abelian group by its operation, then every module over $\mathcal{L}$ carries the structure of an abelian group as well. It is interesting to see how free modules arise and how they can be embedded into $\mathcal{L}$ (section 5). In fact we will see, that the category of modules is an algebraic category. In the final section we study tensor products and inner hom-functors. The inner endomorphism monoid of a module will turn out to be again a left linear theory.

1. General left linear theories

In the following let N always be a fixed set of cardinality greater than two. For any set X let $\xi, \eta, \zeta, \ldots$ denote elements of the set X^N of maps from N to X and let the members of these families be denoted by $\xi(i), \eta(i), \zeta(i) \in X$, where $i \in N$. The constant map or constant family $N \longrightarrow X$ sending each $i \in N$ to a fixed $x \in X$ will be denoted by x^N.
We are going to consider sets X, Y, Z together with operations

$$X \times Y^N \ni (x, \eta) \mapsto x\eta \in Z$$

called a multiplication. This multiplication induces an operation

$$X^N \times Y^N \ni (\xi, \eta) \mapsto \xi\eta \in Z^N.$$

where $(\xi\eta)(i) := \xi(i)\eta \in Z$, $i \in N$.

1.1 Definition: A set $\mathcal{L}$ together with a multiplication

$$\mathcal{L} \times \mathcal{L}^N \ni (l, \lambda) \mapsto l\lambda \in \mathcal{L}$$

is called a *left linear theory* if

$$\forall l \in \mathcal{L}, \mu, \nu \in \mathcal{L}^N : (l\mu)\nu = l(\mu\nu). \tag{$\mathcal{L}_0$}$$

A left linear theory $\mathcal{L}$ is called *unital* if there is a $\delta \in \mathcal{L}^N$, called *unit*, such that

$$\forall l \in \mathcal{L}, \lambda \in \mathcal{L}^N : l\delta = l, \quad \delta\lambda = \lambda. \tag{$\mathcal{L}_1$}$$

1.2 Remark: The first condition for $\mathcal{L}$ to be unital is equivalent to $\lambda\delta = \lambda$. The second condition is equivalent to $\delta(i)\lambda = \lambda(i)$ for all $i \in N$.
The associative law ($\mathcal{L}_0$) for a left linear theory is equivalent to $(\lambda\mu)\nu = \lambda(\mu\nu)$.

If $\mathcal{L}$ is a unital left linear theory then $\mathcal{L}^N$ together with the multiplication $\mathcal{L}^N \times \mathcal{L}^N \longrightarrow \mathcal{L}^N$ is a monoid. The converse is not true, in general, because the i^{th} component of $\lambda\mu$ depends *only* on the i^{th} component of λ and is *independent* of the position of this component in the family λ, i.e. if $\lambda(i) = \nu(j)$ in the families $\lambda, \nu \in \mathcal{L}^N$ then the i^{th} component of $\lambda\mu$ and the j^{th} component of $\nu\mu$ are equal.

1.3 Lemma: *Let $\mathcal{L}$ be a unital left linear theory. Then*
(a) the unit δ is uniquely determined;
(b) if $\forall l \in \mathcal{L} : l\mu = l\nu$, then $\mu = \nu$;
(c) if $\mathcal{L}$ has at least two elements, then for all $i, j \in N$ with $i \neq j$ we have $\delta(i) \neq \delta(j)$.

PROOF: (a) From the above remark we have that $\mathcal{L}^N$ is a monoid with unit δ which is unique.
(b) $l\mu = l\nu$ implies $\delta(i)\mu = \delta(i)\nu$ for all $i \in N$ hence $\delta\mu = \delta\nu$ and $\mu = \nu$.
(c) Take $l \neq l'$ in $\mathcal{L}$ and form μ with components $\mu(k) = l$ if $k \neq j$ and $\mu(j) = l'$. Then $\delta(i)\mu = l \neq l' = \delta(j)\mu$ hence $\delta(i) \neq \delta(j)$. $\square$

We want to give examples for left linear theories. In the sequel *semiring* is meant to be a quintuple $(R, +, 0, \cdot, 1)$ consisting of a set R, two distinguished elements 0 and 1 of R, and two binary compositions of R such that

(o) $0 \neq 1, \quad 0 \cdot R = R \cdot 0 = 0,$
(i) $(R, +, 0)$ is a commutative monoid with neutral element 0,
(ii) $(R, \cdot, 1)$ is a monoid with neutral element 1,
(iii) $\forall x, y, z \in R : x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x.$

The multiplication of a semiring R will usually be written without the multiplication dot.

1.4 Examples: a) If $I\!N$ is the set of natural numbers, R is a semiring, and $\mathcal{L}$ is a set of (infinite) sequences $a^* = (a^1, a^2, \ldots)$ with entries in R such that

$(L1) \quad \forall a^* \in \mathcal{L}, \beta^* = (b_1^*, b_2^*, \ldots) \in \mathcal{L}^{I\!N} : a^*\beta^* := (\sum_i a^i\beta_i^1, \sum_i a^i\beta_i^2, \ldots) \in \mathcal{L}$

then $\mathcal{L}$ is a left linear theory, called a *sequential left linear theory over R*. We assume that the infinite sums are defined and obey the usual laws of addition and multiplication by requiring some weak kind of convergence. If, in addition,

$(L2) \quad \forall j \in I\!N : \delta^*(j) := (\delta_j^1, \delta_j^2, \ldots) \in \mathcal{L}$, where δ_j^i is the Kronecker symbol,

then $\mathcal{L}$ is a *unital* sequential left linear theory with unit δ^*. Observe, that this is not a definition but a fact and that not every unital sequential left linear theory will satisfy property $(L2)$, as can be seen in example h).
If the following additional axiom holds

$(L0)$ every $a^* \in \mathcal{L}$ has only finitely many non-zero entries,

then $\mathcal{L}$ is called a *(unital) finitely sequential left linear theory.*

An instance of what we have in mind in a) is the following

a') Let R be a semiring and denote by $R\{\{t\}\}$ the semiring of formal power series in t. For $s \in R\{\{t\}\}$ we denote by $\mathrm{ldeg}(s)$ the lower degree of s, that is if $s = \sum_{j\geq 0} r_j t^j$ then $r_j = 0$ for $j < \mathrm{ldeg}(s)$ and $r_j \neq 0$ for $j = \mathrm{ldeg}(s)$; obviously, $\mathrm{ldeg}(0) = +\infty$. Let $\Phi : I\!N \longrightarrow I\!N \cup \{+\infty\}$ be a function such that $\lim_{n\to\infty} \Phi(n) = +\infty$. Denote by $\mathcal{L}$ the set of all $\alpha \in R\{\{t\}\}^{I\!N}$ such that

$$\mathrm{ldeg}(\alpha(n)) \geq \Phi(n), \qquad n \in I\!N.$$

Then a simple computation shows that $\mathcal{L}$ is a sequential unital left linear theory.

b) The set $R^{(N)}$ of all elements of R^N with finite support over a semiring R is a unital left linear theory. If the index set $I\!N$ is the set of natural numbers then $R^{(I\!N)}$ is a unital finitely sequential left linear theory, called the *semiring R.*

c) The set $\mathrm{PRO}(R)$ of all elements x^* of $R^{(N)}$ with $\sum_{i\in N} x^i = 0$ is a left linear theory, called the *projective theory* over R.

d) The set $\mathrm{BAR}(R)$ of all elements x^* of $R^{(N)}$ with $\sum_{i\in N} x^i = 1$ is a unital left linear theory, called the *barycentric theory* over R ([1],[2],[5]).

e) The set $\mathrm{BAR}_e(R)$ of all elements x^* of $R^{(N)}$ with $\sum_{i\in N} x^i = e$ is a left linear theory.

f) The set $I^{(N)}$ of all elements of $R^{(N)}$ with entries in a one-sided ideal $I \subseteq R$ is a left linear theory.

g) The set $[S]^{(N)}$ of all elements of $R^{(N)}$ with entries in the commutator of a subset $S \subseteq R$ is a unital left linear theory.

h) If $I\!N$ is the set of natural numbers then the singleton set $\{\alpha^*\}$ in $R^{(I\!N)}$ with $\sum_{i\in I\!N} \alpha^i = 1$ is a unital sequential left linear theory.

i) If $I\!N$ is the set of natural numbers then the set of sequences in $R^{(I\!N)}$ with at most one non-zero coefficient is a unital sequential left linear theory.

For the following examples let R be the ring of real or complex numbers and $I\!N$ the set of natural numbers.

j) The set Ω_R of all elements x^* of $R^{I\!N}$ with $\sum_{i\in I\!N} \|x^i\| \leq 1$ is a unital sequential left linear theory, called *total convexity theory* [9].

k) The set Ω_{sc} of all elements x^* of $R^{I\!N}$ with $\sum_{i\in I\!N} x^i = 1$ and non-negative real entries is a unital sequential left linear theory, called *super convexity theory* [11].

l) The set $\Omega_{R,fin}$ of all elements x^* of $R^{(I\!N)}$ with $\sum_{i\in I\!N}\|x^i\| \leq 1$ is a unital sequential left linear theory, called *finite total convexity theory* [9].

m) The set Ω_c of all elements x^* of $R^{(I\!N)}$ with $\sum_{i\in I\!N} x^i = 1$ and non-negative real entries is a unital sequential left linear theory, called *convexity theory* [14].

n) The set $\mathcal{P}$ of all elements x^* of $R^{I\!N}$ with $\sum_{i\in I\!N} x^i \leq 1$ and non-negative real entries is a unital sequential left linear theory, called *positive convexity theory* [8], [15].

o) The set $\mathcal{P}^+$ of all elements x^* of $R^{I\!N}$ with $0 < \sum_{i\in I\!N} x^i \leq 1$ and non-negative real entries is a unital sequential left linear theory, called *strict positive convexity theory*.

p) Further examples arise from (6.15).

Most of these and other examples of convexity theories can be found in [12].

1.5 Definition: Let $\mathcal{L}$ and $\mathcal{M}$ be left linear theories. A map $\sigma : \mathcal{L} \longrightarrow \mathcal{M}$ is called a *homomorphisms of left linear theories*, if

$$\forall l \in \mathcal{L}, \lambda \in \mathcal{L}^N : \sigma(l\lambda) = \sigma(l)\sigma^N(\lambda).$$

If both theories are unital and σ satisfies in addition

$$\forall i \in N : \sigma(\delta(i)) = \delta(i),$$

then σ is called a *unital homomorphism* of left linear theories.

1.6 Remark: The left linear theories and the unital left linear theories form categories. These categories are equationally defined or algebraic in the sense of [6], hence the underlying functors from unital left linear theories to left linear theories to sets have left adjoints. Thus there are free left linear theories, free unital left linear theories, and the adjunction of a unit to a left linear theory can be performed.
In 1.2 we observed that each unital left linear theory is a monoid. This defines a functor from the category of unital left linear theories to the category of monoids which is not quite an algebraic functor in the classical sense [6], but it still has a left adjoint.

2. Modules

Let $\mathcal{L}$ be a unital left linear theory. We can define various different types of modules over $\mathcal{L}$. We will only consider unital modules.

Given a set X together with a multiplication

$$\mathcal{L} \times X^N \ni (l, \xi) \mapsto l\xi \in X.$$

As in section 1. we can define

$$\mathcal{L}^N \times X^N \ni (\lambda, \xi) \mapsto \lambda\xi \in X^N$$

by $(\lambda\xi)(i) = \lambda(i)\xi$.

2.1 Definition: The set X together with the given multiplication is called a *left $\mathcal{L}$-module* if

$$\forall l \in \mathcal{L}, \lambda \in \mathcal{L}^N, \xi \in X^N : (l\lambda)\xi = l(\lambda\xi) \qquad \text{and} \tag{$\mathcal{L}_{0,X}$}$$

$$\forall \xi \in X^N : \delta\xi = \xi. \tag{$\mathcal{L}_{1,X}$}$$

The multiplication of a left $\mathcal{L}$-module induces an multiplication $\mathcal{L}^N \times X^N \longrightarrow X^N$. So a left $\mathcal{L}$-module X induces an $\mathcal{L}^N$-set X^N over the monoid $\mathcal{L}^N$. The converse is not true. The remarks of 1.2 apply in a similar way.

2.2 Definition: Let X and Y be left $\mathcal{L}$-modules. A map $f : X \longrightarrow Y$ is called a *homomorphism of left $\mathcal{L}$-modules* if

$$\forall l \in \mathcal{L}, \xi \in X^N : f(l\xi) = lf^N(\xi).$$

Let $\mathcal{L}$-Mod denote the category of left $\mathcal{L}$-modules.

2.3 Remark: The category $\mathcal{L}$-Mod is an algebraic category. The underlying functor to the category of sets is an algebraic functor, hence has a left adjoint, the "free" left $\mathcal{L}$-module over a set. We will come back to the explicit construction of free modules in section 5.
The homomorphisms of $\mathcal{L}$-modules induce homomorphisms of $\mathcal{L}^N$-sets, so we get a functor from $\mathcal{L}$-modules to $\mathcal{L}^N$-sets.

2.4 Definition: A set X together with a multiplication

$$X \times \mathcal{L}^N \ni (x, \lambda) \mapsto x\lambda \in X$$

is called a *right $\mathcal{L}$-module* if

$$\forall x \in X, \lambda, \mu \in \mathcal{L}^N : (x\lambda)\mu = x(\lambda\mu) \qquad \text{and} \tag{$\mathcal{R}_{0,X}$}$$

$$\forall x \in X : x\delta = x. \tag{$\mathcal{R}_{1,X}$}$$

Again this multiplication induces a multiplication $X^N \times \mathcal{L}^N \ni (\xi, \lambda) \mapsto \xi\lambda \in X^N$, where $\xi\lambda(i) := \xi(i)\lambda$.

2.5 Definition: Let X and Y be right $\mathcal{L}$-modules. A map $f : X \longrightarrow Y$ is called a *homomorphism of right $\mathcal{L}$-modules* if

$$\forall x \in X, \lambda \in \mathcal{L}^N : f(x\lambda) = f(x)\lambda.$$

Let Mod-$\mathcal{L}$ denote the category of right $\mathcal{L}$-modules.

2.6 Definition: Let $\mathcal{L}$ and $\mathcal{M}$ be left linear theories. A left $\mathcal{L}$- and right $\mathcal{M}$-module X is called an *$\mathcal{L}$-$\mathcal{M}$-bimodule* if

$$\forall l \in \mathcal{L}, \xi \in X^N, \mu \in \mathcal{M}^N : (l\xi)\mu = l(\xi\mu).$$

2.7 Definition: Let X and Y be $\mathcal{L}$-$\mathcal{M}$-bimodules. A map $f : X \longrightarrow Y$ is called a *homomorphism of $\mathcal{L}$-$\mathcal{M}$-bimodules* if f is a homomorphism of left $\mathcal{L}$- and right $\mathcal{M}$-modules.
Let $\mathcal{L}$-Mod-$\mathcal{M}$ denote the category of $\mathcal{L}$-$\mathcal{M}$-bimodules.

2.8 Examples: a) $\mathcal{L}$ is a $\mathcal{L}$-$\mathcal{L}$-bimodule with the canonical operations.

b) If R is a ring, $\mathcal{L} = R^{(N)}$, and M a left R-module, then M is a left $\mathcal{L}$-module by

$$l\mu := \sum_i \alpha^i \mu(i) \in M$$

for $l = (\alpha^i) \in R^{(N)} = \mathcal{L}$, $\mu(i) \in M$, and $\mu \in M^N$. This defines a functor from left R-modules to left $\mathcal{L}$- modules. Similarly commutative R-monoids over semirings R define modules over $\mathcal{L} = R^{(N)}$.

c) If R, $\mathcal{L}$ and M are as above then $M^{(N)}$, the set of all elements of M^N with finite support, is a left $\mathcal{L}$-module by $l\xi := \sum_j \alpha^i \xi(i) \in M^{(N)}$ with $l = (\alpha^i) \in \mathcal{L}$, $\xi(i) \in M^{(N)}$ with componentwise operations and $\xi \in (M^{(N)})^N$. Again this defines a functor from left R-modules to left $\mathcal{L}$-modules.

d) If R, $\mathcal{L}$, and M are as above and M is an R-bimodule. Then $M^{(N)}$ is an $\mathcal{L}$-bimodule by the additional right operation $x\lambda = \sum_j m^j \lambda(j) \in M^{(N)}$ where $x = (m^i) \in M^{(N)}$ and $\lambda(j) = (\alpha^i_j) \in \mathcal{L} = R^{(N)}$. This defines a functor from R-bimodules to $\mathcal{L}$-bimodules. In 4.14 we show this functor to be an equivalence of categories.

e) For any index set I the product $\mathcal{L}^I$ is an $\mathcal{L}$-bimodule.

f) For N the set of natural numbers and $\mathcal{L} := R^{(N)}$ the set X of infinite sequences of power series $\sum_i r_i^j x^i$ ($X \subseteq R[[x]]^N$) such that $(r_i^j)_{j\in N}$ has finite support for all $i \in N$ is an $\mathcal{L}$-bimodule. The left multiplication is defined by $l\xi = (\alpha^k)(\sum_i r_{ik}^j x^i) := (\sum_{i,k} \alpha^k r_{ik}^j x^i)_j$. The right multiplication is $(\sum_i r_i^j x^i)(\alpha_j^k) := (\sum_{i,j} r_i^j \alpha_j^k x^i)_{k\in N}$. In fact there is a bimodule isomorphism $X \cong \mathcal{L}^N$ by $(\sum_i r_i^j x^i) \mapsto ((r_i^j)_{j\in N})_{i\in N}$.

g) Let $\mathcal{L}$ be the unital sequential left linear theory of 1.4 example i) consisting of sequences in $R^{(\mathbb{N})}$ with only one non-zero coefficient. Then $X := R\dot{\cup}R/(0=0)$ is a left $\mathcal{L}$-module, where $(0,\ldots,0,\alpha_i,0,\ldots)(x_j)_{j\in\mathbb{N}} = \alpha_i x_i$ viewed as element in the same component of X as x_i is from.

3. Zero Elements

3.1 Definition: Let $\mathcal{L}$ be a left linear theory.
a) Let X be a left $\mathcal{L}$-module. X *has a right zero* if there is a unique element $0 \in X$ such that

$$\forall l \in \mathcal{L} : l0^N = 0 \quad \text{(where } 0^N \text{ is the constant family).} \qquad (i)$$

b) Let X be a right $\mathcal{L}$-module. X *has a left zero* if there is a unique element $0 \in X$ such that

$$\forall \lambda \in \mathcal{L}^N : 0\lambda = 0. \qquad (ii)$$

c) Let X be an $\mathcal{L}$-bimodule. X *has a zero* if there is an element $0 \in X$ such that

(i) $\forall l \in \mathcal{L} : l0^N = 0$,
(ii) $\forall \lambda \in \mathcal{L}^N : 0\lambda = 0$.

3.2 Lemma: *Let X be an $\mathcal{L}$-bimodule.*
(a) If X has a right zero then $\forall \lambda \in \mathcal{L}^N : 0\lambda = 0$.
(b) If X has a left zero then $\forall l \in \mathcal{L} : l0^N = 0$.
Proof: (a) By associativity we have $\forall l \in \mathcal{L}, \lambda \in \mathcal{L}^N : l(0\lambda)^N = l(0^N\lambda) = (l0^N)\lambda = 0\lambda$ hence by uniqueness $0\lambda = 0$.
(b) By associativity we have $\forall l \in \mathcal{L}, \lambda \in \mathcal{L}^N : (l0^N)\lambda = l(0^N\lambda) = l0^N$ hence by uniqueness $l0^N = 0$. □

3.3 Corollary: *Let X be an $\mathcal{L}$-bimodule. If X has a right zero or a left zero, then X has a zero.*

3.4 Lemma: *The following are equivalent for the $\mathcal{L}$-bimodule $\mathcal{L}$:*

(a) $\mathcal{L}$ *has a right zero.*
(b) $\mathcal{L}$ *has a left zero.*
(c) $\mathcal{L}$ *has a zero.*

PROOF: (c) $\Rightarrow$ (a): We have to show uniqueness of the zero. Let $0' \in \mathcal{L}$ such that $\forall l \in \mathcal{L} : l0'^N = 0'$, then $0' = 00'^N = 0$ by (ii).

(c) $\Rightarrow$ (b): We have to show uniqueness of the zero. Let $0' \in \mathcal{L}$ such that $\forall \lambda \in \mathcal{L}^N : 0'\lambda = 0'$, then $0' = 0'0^N = 0$ by (i).

The converse is the previous corollary. $\square$

3.5 Lemma: *Let* $\mathcal{L}$ *have a zero* 0 *and let* X *be a left* $\mathcal{L}$*-module. Then we have*

$$\forall \xi, \eta \in X^N : 0\xi = 0\eta.$$

PROOF: Let $\xi, \eta \in X^N$ and $j \in N$. Define $\zeta \in X^N$ by

$$\zeta(i) := \begin{cases} \xi(j), & \text{for } i = j, \\ \eta(i), & \text{for } i \neq j. \end{cases}$$

Then with $k \neq j$ we get $0\xi = (0\delta(j)^N)\xi = 0(\delta(j)^N\xi) = 0\xi(j)^N = 0(\delta(j)^N\zeta) = (0\delta(j)^N)\zeta = (0\delta(k)^N)\zeta = 0(\delta(k)^N\zeta) = 0\eta(k)^N = 0(\delta(k)^N\eta) = (0\delta(k)^N)\eta = 0\eta$. $\square$

3.6 Corollary: *Let* $\mathcal{L}$ *have a zero* 0.
(a) If X *is a non-empty left* $\mathcal{L}$*-module then* X *has a right zero* $0' \in X$ *and* $\forall \xi \in X^N : 0\xi = 0'$.
(b) If X *is a non-empty* $\mathcal{L}$*-bimodule and has a left zero* $0'$ *then* $0'$ *is also the (unique) right zero of* X *and*

$$\forall \xi \in X^N : 0\xi = 0',$$

$$\forall x \in X : x0^N = 0'.$$

(c) If $f : X \longrightarrow Y$ *is a homomorphism of left* $\mathcal{L}$*-modules, then* $f(0) = 0$.
PROOF: (a) For some $\xi \in X$ define $0' := 0\xi \in X$. Then $l0'^N = l(0\xi)^N = l0^N\xi = 0\xi = 0'$ and uniqueness is obtained as follows. Let $0'' \in X$ be a right zero. Then $0'' = 0 \cdot 0''^N = 0 \cdot 0'^N = 0'$.
(b) X has a right zero $0''$ by part (a) which is a zero by 3.3 hence by uniqueness of the left zero we get $0' = 0'' = 0\xi$. Thus $x0^N = x(0^N\lambda) = (x0^N)\lambda$ implies by uniqueness $x0^N = 0'$.
(c) $f(0) = f(0\xi) = 0f^N(\xi) = 0$. $\square$

A non-empty right $\mathcal{L}$-module, however, will in general not have a left zero. The element $x0^N \in X$ satisfies (ii), but it will not be unique, e.g. $X = \{0, 0'\}$ where both elements satisfy (ii).

4. (Semi-)Additive Theories and Scalars

The elements of $\mathcal{L}$ can be considered as operators which produce allowable linear combinations of elements in a left $\mathcal{L}$-module. This is the essence of all the examples of convexity theories. In some sense the multiplication of elements in the module by certain elements (which we will call scalars and which will be discussed later on) and the addition in the module are hidden among these operators. Observe, however, that the addition proper is very often not allowable in convex sets.

In some cases, however, there will be a structure of an addition on the module. In section 3 we saw that a zero is carried over from $\mathcal{L}$ to modules over $\mathcal{L}$. We will study now how much of an additive structure will be transferred in a similar way. Throughout this section we shall assume that $\mathcal{L}$ is a unital left linear theory which has a zero: $0 \in \mathcal{L}$.

For $i, j \in N$ with $i \neq j$ and for elements x, y in a left $\mathcal{L}$-module X let $\varepsilon(x, y|i, j) \in X^N$ be given by

$$\varepsilon(x, y|i, j)(k) := \begin{cases} 0, & \text{if } k \neq i, j, \\ x, & \text{if } k = i, \\ y, & \text{if } k = j. \end{cases}$$

4.1 Definition: $\mathcal{L}$ is *semi-additive*, if there are $i, j \in N$, $i \neq j$ and $a \in \mathcal{L}$ with $a\varepsilon(\delta(i), \delta(j)|i, j) = a$. We call a an *addition* for $\mathcal{L}$.

For a left $\mathcal{L}$-module X we define an addition by $x + y := a\varepsilon(x, y|i, j)$.

4.2 Lemma: *Let $\mathcal{L}$ have an addition α. Then*

(a) $a = \delta(i) + \delta(j)$ in $\mathcal{L}$.

(b) If $f : X \longrightarrow Y$ is a homomorphism of left $\mathcal{L}$-modules then $\forall x, x' \in X$: $f(x + x') = f(x) + f(x')$.

(c) For every $\mathcal{L}$-bimodule X we have $\forall x, y \in X, \lambda \in \mathcal{L}^N : (x + y)\lambda = x\lambda + y\lambda$.

(d) $a\xi = a\varepsilon(\xi(i), \xi(j)|i, j)$.

Proof: (a) is clear from the definition of $x + y$.

(b) $f(x + x') = f(a\varepsilon(x, x'|i, j)) = af^N(\varepsilon(x, x'|i, j)) = a\varepsilon(f(x), f(x')|i, j)) = f(x) + f(x')$.

(c) follows from (b) since right multiplication by $\lambda \in \mathcal{L}^N$ is a homomorphism of left modules.

(d) $a\xi = a\varepsilon(\delta(i), \delta(j)|i, j)\xi = a\varepsilon(\xi(i), \xi(j)|i, j)$. □

4.3 Lemma: *Let $\mathcal{L}$ have an addition a. Let $i', j' \in N$ be two elements with $i' \neq j'$. Put $a' := \delta(i') + \delta(j')$. Then $a'\varepsilon(\delta(i'), \delta(j')|i', j') = a'$ and $x +_{a'} y := a'\varepsilon(x, y|i', j') = a\varepsilon(x, y|i, j) = x + y$. In particular the addition does not depend on the choice of $i, j \in N$ with $i \neq j$.*

PROOF: Since

$$\varepsilon(\delta(i'),\delta(j')|i,j)\varepsilon(\delta(i'),\delta(j')|i',j') = \varepsilon(\delta(i'),\delta(j')|i,j)$$

we have

$$\begin{aligned} a'\varepsilon(\delta(i'),\delta(j')|i',j') \quad &= (\delta(i')+\delta(j'))\varepsilon(\delta(i'),\delta(j')|i',j') \\ &= a\varepsilon(\delta(i'),\delta(j')|i,j)\varepsilon(\delta(i'),\delta(j')|i',j') \\ &= a\varepsilon(\delta(i'),\delta(j')|i,j) \\ &= a' \end{aligned}$$

and

$$a'\varepsilon(x,y|i',j') = a\varepsilon(\delta(i'),\delta(j')|i,j)\varepsilon(x,y|i',j') = a\varepsilon(x,y|i,j) = x+y.$$

□

From now on addition is written by means of the initially chosen i and j.

4.4 Definition: An addition a in $\mathcal{L}$ is called
associative if there are $i,j,k \in N$, mutually distinct, such that $\delta(i)+(\delta(j)+\delta(k)) = (\delta(i)+\delta(j))+\delta(k)$ holds,
commutative if $\delta(i)+\delta(j) = \delta(j)+\delta(i)$,
with zero element if $\delta(i)+0 = 0+\delta(i) = \delta(i)$,
with inverses if there exists $(-\delta(i)) \in \mathcal{L}$ with $\delta(i)+(-\delta(i)) = (-\delta(i))+\delta(i) = 0$.

4.5 Proposition: *Let $\mathcal{L}$ be a unital left linear theory with an addition a. If a is associative, commutative, with zero element, or with inverses then so is the addition of an $\mathcal{L}$-module X.*

PROOF: For $x \in X$ denote by $\varepsilon(x|i) \in X^N$ the map

$$\varepsilon(x|i)(t) := \begin{cases} 0, & \text{if } t \neq i, \\ x, & \text{if } t = i. \end{cases}$$

For $x,y,z \in X$ and mutually distinct $i,j,k \in N$ denote by $\varepsilon(x,y,z|i,j,k) \in X^N$ the map

$$\varepsilon(x,y,z|i,j,k)(t) := \begin{cases} 0, & \text{if } t \neq i,j,k, \\ x, & \text{if } t = i, \\ y, & \text{if } t = j, \\ z, & \text{if } t = k. \end{cases}$$

Then

$$\begin{aligned}(x+y)+z &= (\delta(i)+\delta(j))\varepsilon(x+y,z|i,j)\\ &= (\delta(i)+\delta(j))\varepsilon((\delta(i)+\delta(j))\varepsilon(x,y,z|i,j,k),\delta(k)\varepsilon(x,y,z|i,j,k)|i,j)\\ &= (\delta(i)+\delta(j))\varepsilon(\delta(i)+\delta(j),\delta(k)|i,j)\varepsilon(x,y,z|i,j,k)\\ &= ((\delta(i)+\delta(j))+\delta(k))\varepsilon(x,y,z|i,j,k)\end{aligned}$$

and similarly $x+(y+z) = (\delta(i)+(\delta(j)+\delta(k)))\varepsilon(x,y,z|i,j,k)$. Thus associativity is inherited by the modules.
Furthermore $x+y = (\delta(i)+\delta(j))\varepsilon(x,y|i,j)$ and $y+x = (\delta(i)+\delta(j))\varepsilon(y,x|i,j) = (\delta(i)+\delta(j))\varepsilon(\delta(j),\delta(i)|i,j)\varepsilon(x,y|i,j) = (\delta(j)+\delta(i))\varepsilon(x,y|i,j)$ show that commutativity is inherited.
For the other two laws we have $x+0 = (\delta(i)+\delta(j))\varepsilon(x,0|i,j) = (\delta(i)+\delta(j))\varepsilon(\delta(i),0|i,j)\varepsilon(x,0|i,j) = (\delta(i)+0)\varepsilon(x,0|i,j)$ and $x = \delta(i)\varepsilon(x,0|i,j)$ resp. $x+(-x) = (\delta(i)+\delta(j))\varepsilon(x,-x|i,j) = (\delta(i)+\delta(j))\varepsilon(\delta(i),-\delta(i)|i,j)\varepsilon(x,0|i,j) = (\delta(i)+(-\delta(i)))\varepsilon(x,0|i,j)$ and $0 = 0\varepsilon(x,0|i,j)$. □

4.6 Example: Let $\mathcal{L}$ be a unital sequential left linear theory as defined in example 1.4.a) with $0 = (0,0,\ldots) \in \mathcal{L}$. Any element of the form $(a_1,a_2,0,0,\ldots) \in \mathcal{L}$ defines an addition. The addition is commutative if and only if $a_1 = a_2$. It is associative if and only if $a_i^2 = a_i$ and $a_1a_2 = a_2a_1$. The addition is an addition with zero element if and only if $a_1 = a_2 = 1$. If $\mathcal{L}$ is defined over a *ring* R then the addition is an addition with inverses if and only if $(-1,0,0,\ldots) \in \mathcal{L}$.

Now we study somewhat more in detail sequences in $\mathcal{L}^N$ with entries $\delta(i)$, where δ is the unit in $\mathcal{L}$.

4.7 Definition: Let $\sigma : N \longrightarrow N$ be a map. Then we define $\delta^\sigma := \delta \circ \sigma$, i.e. $\delta^\sigma(i) = \delta(\sigma(i))$. A map $\sigma : N \longrightarrow N$ is called an *(i,j)-map* if $\sigma^{-1}(\{j\}) = \{i\}$. We call $b \in \mathcal{L}$ an *i-scalar* if

$$\forall j \in N\ \forall (i,j)\text{-maps } \sigma,\tau : b\delta^\sigma = b\delta^\tau.$$

A little calculation for a sequential left linear theory $\mathcal{L}$ with zero over the semiring R shows that the i-scalars are of the form $(0,\ldots,0,r_i,0,\ldots)$.

4.8 Lemma: *(a) For $\sigma,\tau : N \longrightarrow N$ we have $\delta^{\sigma\tau} = \delta^\tau\delta^\sigma \in \mathcal{L}$.*
(b) If σ is an (i,j)-map and τ is a (j,k)-map then $\tau\sigma$ is an (i,k)-map.
(c) $\delta(i)$ is an i-scalar for all $i \in N$.

(d) Let $l \in \mathcal{L}$. Then $l\delta(i)^N$ is an i-scalar.
(e) If $b \in \mathcal{L}$ is an i-scalar and σ is an (i,j)-map then $b\delta^\sigma$ is a j-scalar.
(f) If b is an i-scalar and $\xi, \eta \in X^N$ with $\xi(i) = \eta(i)$, then $b\xi = b\eta$. In particular $b\xi$ depends only on the i^{th} component of ξ.

PROOF: Straightforward substitutions of the definitions and simple calculations.

□

4.9 Definition: For $x \in X$ and an i-scalar $b \in \mathcal{L}$ we define $bx := b\varepsilon(x|i)$.

Now we want to study elements in X^{N^a} and in X^{N^b} for a set X, where a and b are non-negative integers. Let $\chi : N^a \longrightarrow N^b$ be a map. For $\Xi \in X^{N^b}$ we define $\Xi^\chi := \Xi \circ \chi \in X^{N^a}$. Then we have $\Xi^{\vartheta\chi} = (\Xi^\vartheta)^\chi$ for $\chi : N^a \longrightarrow N^b$ and $\vartheta : N^b \longrightarrow N^c$.

Now let a map $\mathcal{L} \times X^N \ni (l, \xi) \mapsto l\xi \in Z$ be given. We observe that $X^{N^b} \cong (X^N)^{N^{b-1}}$ and define

$$\mathcal{L}^{N^a} \times X^{N^b} \ni (\Lambda, \Xi) \mapsto \Lambda\Xi \in Z^{N^{a+b-1}}$$

by

$$(\Lambda\Xi)(i_1, \ldots, i_{a+b-1}) := \Lambda(i_1, \ldots, i_a)\Xi(-, i_{a+1}, \ldots, i_{a+b-1}).$$

For a left $\mathcal{L}$-module X one checks $\Xi^{\vartheta\chi} = \delta^\chi\Xi^\vartheta$, which is a generalization of Lemma 4.8 (a).

4.10 Lemma: *Let X be a left $\mathcal{L}$-module. If $\Lambda \in \mathcal{L}^{N^a}$, $M \in \mathcal{L}^{N^b}$, and $\Xi \in X^{N^c}$ then*

$$(\Lambda M)\Xi = \Lambda(M\Xi).$$

PROOF:

$$\begin{aligned}
(\Lambda(\mathrm{M}\Xi)) \quad & (i_1, \ldots, i_{a+b+c-2}) \\
& = \Lambda(i_1, \ldots, i_a)(\mathrm{M}\Xi)(-, i_{a+1}, \ldots, i_{a+b+c-2}) \\
& = \Lambda(i_1, \ldots, i_a)(\mathrm{M}(-, i_{a+1}, \ldots, i_{a+b-1})\Xi(-, i_{a+b}, \ldots, i_{a+b+c-2})) \\
& = (\Lambda(i_1, \ldots, i_a)\mathrm{M}(-, i_{a+1}, \ldots, i_{a+b-1}))\Xi(-, i_{a+b}, \ldots, i_{a+b+c-2}) \\
& = (\Lambda\mathrm{M}(i_1, \ldots, i_{a+b-1})\Xi(-, i_{a+b}, \ldots, i_{a+b+c-2}) \\
& = ((\Lambda\mathrm{M})\Xi)(i_1, \ldots, i_{a+b+c-2}).
\end{aligned}$$

□

We fix maps $\sigma : N \longrightarrow N \times N$ and $\tau : N \times N \longrightarrow N$ with $\tau\sigma = \mathrm{id}_N$ and $\sigma\tau = \mathrm{id}_{N\times N}$. Furthermore let $\pi : N \times N \longrightarrow N \times N$ be the map with $\pi(i,j) = (j,i)$.

4.11 Corollary: *Let $l, m \in \mathcal{L}$. If $l(m\delta^{\tau}) = m(l\delta^{\tau\pi})$ then $l(m\Xi) = m(l\Xi^{\pi})$ for all $\Xi \in X^{N\times N}$.*

PROOF:
$l(m\Xi) = l(m\Xi^{\sigma\tau}) = l(m(\delta^{\tau}\Xi^{\sigma})) = l((m\delta^{\tau})\Xi^{\sigma}) = (l(m\delta^{\tau}))\Xi^{\sigma} = (m(l\delta^{\tau\pi}))\Xi^{\sigma} = m((l\delta^{\tau\pi})\Xi^{\sigma}) = m(l(\delta^{\tau\pi}\Xi^{\sigma})) = m(l\Xi^{\sigma\tau\pi}) = m(l\Xi^{\pi})$. □

4.12 Lemma: *If $b \in \mathcal{L}$ is a k-scalar with $b(\delta(i) + \delta(j)) = b\delta(i) + b\delta(j)$ then for all $\mathcal{L}$-modules X and all $x, y \in X$ we have $b(x + y) = bx + by$.*

PROOF: This is a special case of Lemma 4.11 but can also be obtained by simply multiplying the equation

$$\begin{aligned} b\varepsilon(\delta(i) + \delta(j)|k) &= b(\delta(i) + \delta(j)) \\ &= b\delta(i) + b\delta(j) \\ &= (\delta(i) + \delta(j))\varepsilon(b\varepsilon(\delta(i)|k), b\varepsilon(\delta(j)|k)|i,j) \end{aligned}$$

from the right by $\varepsilon(x, y|i,j)$. □

In view of this result Lemma 4.11 can be seen as a general distributive law.

The tools developed by now let us construct some interesting examples of certain right modules and bimodules. For this purpose let $\Delta = \{\delta(i)|i \in N\} \subseteq \mathcal{L}$. Then by the unitary law $(\mathcal{L}_1)$ $\delta(i)\lambda = \lambda(i)$ it is clear that Δ is a unital left linear subtheory of $\mathcal{L}$. By Lemma 1.3.(c) there is a bijection between Δ^N and $\mathcal{D} = \mathrm{Map}(N, N)$. We observe that $\mathcal{D}$ is a monoid under the composition of maps. We will consider sets M, which are (left) $\mathcal{D}$-sets, i.e. on which $\mathcal{D}$ operates, such that the unital and associative laws hold: $1(m) = m$ and $\tau(\sigma(m)) = (\tau\sigma)(m)$. If M is a left $\mathcal{L}$-module, then an operation of $\mathcal{D}$ on M is called $\mathcal{L}$- linear, if all $\sigma \in \mathcal{D}$ define $\mathcal{L}$-module homomorphisms $\sigma : M \longrightarrow M$, i.e. $l\sigma^N(\mu) = \sigma(l\mu)$. Now we can prove

4.13 Proposition: *(a) A set M is a $\mathcal{D}$-set if and only if M is a Δ-right module.*
(b) A left $\mathcal{L}$-module M is an $\mathcal{L}$-Δ-bimodule if and only if there is an $\mathcal{L}$-linear operation of $\mathcal{D}$ on M.

PROOF: We identify the $\mathcal{D}$ and the Δ operation by $\sigma(m) = m\delta^{\sigma}$. Then by Lemma 4.8 (a) we have $m\delta^{\tau\sigma} = (m\delta^{\sigma})\delta^{\tau}$ if and only if $(\tau\sigma)(m) = \tau(\sigma(m))$. Furthermore we have $m\delta = m$ iff $1(m) = m$. For the $\mathcal{L}$-left operation we get $l(\mu\delta^\sigma) = (l\mu)\delta^\sigma$ iff $l\sigma^N(\mu) = \sigma(l\mu)$. $\square$

4.14 Theorem: *Let $\mathcal{L} = R^{(N)}$ be the semiring. Then the functor from R-Mod to $\mathcal{L}$-Mod (as defined in example 2.8.b) is an equivalence of categories.*

Proof: We describe the inverse functor. Let M be a left $\mathcal{L}$-module. Since the addition in $\mathcal{L}$ is described by $a = (1, 1, 0, 0, \ldots)$ and the addition is associative, commutative, with zero element and with inverses, M is an Abelian group by Prop. 4.5. Clearly $(r, 0, \ldots)$ is a 1-scalar in $\mathcal{L}$. For $m \in M$ and $r \in R$ we define $r \circ m := (r, 0, \ldots) \cdot m = (r, 0, \ldots)\varepsilon(m|1)$. By 4.12 we get $r \circ (m_1 + m_2) = (r, 0, \ldots) \cdot (m_1 + m_2) = (r, 0, \ldots)\varepsilon(m_1 + m_2|1) = (r, 0, \ldots)a\varepsilon(m_1, m_2|1, 2) = (r, 0, \ldots)a\varepsilon(\delta(1), \delta(2)|1, 2)\varepsilon(m_1, m_2|1, 2) = a\varepsilon(r\delta(1), r\delta(2)|1, 2)\varepsilon(m_1, m_2|1, 2) = r \circ m_1 + r \circ m_2$. Further by 4.11 we have

$$\begin{aligned} (r_1 + r_2) \circ m &= (r_1 + r_2, 0, \ldots) \circ m = a\varepsilon((r_1, 0, \ldots), (r_2, 0, \ldots)|1, 2)\varepsilon(m|1) \\ &= r_1 \circ m + r_2 \circ m. \end{aligned}$$

The associativity follows from

$$\begin{aligned} (rs) \circ m &= (rs, 0, \ldots) \cdot m = (r, 0, \ldots)\varepsilon((s, 0, \ldots)|1)\varepsilon(m|1) \\ &= (r, 0, \ldots)\varepsilon(s \circ m|1) = r \circ (s \circ m). \end{aligned}$$

Finally $1 \circ m = (1, 0, \ldots)\varepsilon(m|1) = \delta(1)\varepsilon(m|1) = m$. We leave to the reader to check that this describes an inverse to the functor of example 2.8.b).

$\square$

5. Free Modules

In this section we will explicitly construct free modules. Their existence is actually clear from the fact that we are considering algebraic categories and that the underlying functors are algebraic functors in the sense of [6] so they have left adjoint "free" functors. But we are also interested in the actual size of free modules and in their computational rules.

Let Set_N be the category of non-empty sets of cardinality less than or equal to the cardinality of N. We first want to construct free left $\mathcal{L}$-modules ${}_{\mathcal{L}}F(Y)$ over sets Y in Set_N. They will all turn out to be submodules of $\mathcal{L}$.

Let Y be an object in Set_N. Then by the cardinality assumption there are maps $\sigma : N \longrightarrow Y$ and $\tau : Y \longrightarrow N$ with σ surjective and τ injective. We call σ, τ a pair of matching maps for Y. For each Y in Set_N we choose a pair of matching maps. We can choose the matching maps in such a way that $\sigma\tau = \mathrm{id}_Y$ holds. We use elements δ^ρ as defined in 4.7 and write $\mathcal{L}\delta^\rho := \{l\delta^\rho | l \in \mathcal{L}\}$.

5.1 Lemma: *If (Y, σ, τ) and (Y', σ', τ') are sets in Set_N with matching maps and if $f : Y \longrightarrow Y'$ is a map, then there is a map $\rho : N \longrightarrow N$ with $\rho\tau = \tau' f$. Furthermore we have a homomorphism of left $\mathcal{L}$-modules*

$$_{\mathcal{L}}F(f) : \mathcal{L}\delta^{\tau\sigma} \ni l\delta^{\tau\sigma} \mapsto l\delta^{\tau\sigma}\delta^{\rho} \in \mathcal{L}\delta^{\tau'\sigma'},$$

which is independent of the choice of ρ.

Proof: Since σ' is surjective there is a map $\omega : N \longrightarrow N$ with $\sigma'\omega = f\sigma$. Since τ is injective there is a map $\rho : N \longrightarrow N$ such that $\rho\tau = \tau' f$:

$$\begin{array}{ccccc} N & \xrightarrow{\sigma} & Y & \xrightarrow{\tau} & N \\ {\scriptstyle\omega}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle\rho}\downarrow \\ N & \xrightarrow{\sigma'} & Y' & \xrightarrow{\tau'} & N. \end{array}$$

Hence we have $l\delta^{\tau\sigma}\delta^\rho = l\delta^{\rho\tau\sigma} = l\delta^{\tau' f\sigma} = l\delta^{\tau'\sigma'\omega} = l\delta^\omega\delta^{\tau'\sigma'} \in \mathcal{L}\delta^{\tau'\sigma'}$ so multiplication with δ^ρ from the right is a well defined map and indeed an $\mathcal{L}$-homomorphism. But according to the second description of $l\delta^{\tau\sigma}\delta^\rho = l\delta^\omega\delta^{\tau'\sigma'}$ the product is independent of the choice of ρ (and also independent of the choice of ω). □

5.2 Corollary: *The construction in Lemma 5.1 defines a functor $_{\mathcal{L}}F : Set_N \longrightarrow \mathcal{L}$-Mod.*

Proof: Let $_{\mathcal{L}}F(Y) := \mathcal{L}\delta^{\tau\sigma}$. The independence of the map $_{\mathcal{L}}F(f)$ of the choice of ρ and ω lets us use $\rho\rho'$ resp. $\omega\omega'$ for composites ff' of maps and $\rho = \omega = \mathrm{id}$ in case of an identity map. □

The functor $_{\mathcal{L}}F$ certainly depends on the choice of the matching maps for the various sets. It is clear, however, with the same considerations as above that another choice of matching maps does not change the isomorphism type of the $\mathcal{L}$-module $_{\mathcal{L}}F(Y)$ obtained.

5.3 Proposition: *The functor $_{\mathcal{L}}F : Set_N \longrightarrow \mathcal{L}$-Mod defines free left $\mathcal{L}$-modules, i.e. given $Y \in Set_N$, $M \in \mathcal{L}$-Mod, and a map $f : Y \longrightarrow M$ there is exactly one homomorphism of $\mathcal{L}$-modules $\tilde{f} : {_{\mathcal{L}}F(Y)} \longrightarrow M$ such that the diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\chi} & _{\mathcal{L}}F(Y) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \tilde{f}} \\ & & M \end{array}$$

with $\chi := \delta^\tau$ commutes.

PROOF: Since $\sigma\tau = \mathrm{id}_Y$ we have $\delta^{\tau\sigma}\delta^{\tau\sigma} = \delta^{\tau\sigma}$. So we get $\chi(y) = \delta^\tau(y) = \delta^{\tau\sigma\tau}(y) = \delta^\tau(y)\delta^{\tau\sigma} \in \mathcal{L}\delta^{\tau\sigma} = {}_{\mathcal{L}}F(Y)$. Now let us define $\tilde{f}(l\delta^{\tau\sigma}) := l\delta^{\tau\sigma}(f \circ \sigma)$. In order to show that $\tilde{f}$ is a homomorphism of $\mathcal{L}$-modules we compute $\tilde{f}(l\lambda\delta^{\tau\sigma}) = l\lambda\delta^{\tau\sigma}(f \circ \sigma) = l\tilde{f}^N(\lambda\delta^{\tau\sigma})$. The universal diagram commutes since $\tilde{f}\chi(y) = \tilde{f}(\delta^\tau(y)) = \tilde{f}(\delta^\tau(y)\delta^{\tau\sigma}) = \delta^\tau(y)f\sigma = f\sigma\tau(y) = f(y)$. In order to show the uniqueness of $\tilde{f}$ let $\tilde{g}$ be given with $\tilde{g}\chi = f$. Then $\tilde{g}(l\delta^{\tau\sigma}) = l\delta^{\tau\sigma}\tilde{g}^N(\delta^{\tau\sigma}) = l\delta^{\tau\sigma}(\tilde{g} \circ \delta^\tau \circ \sigma) = l\delta^{\tau\sigma}(\tilde{g} \circ \chi \circ \sigma)) = l\delta^{\tau\sigma}(f \circ \sigma) = \tilde{f}(l\delta^{\tau\sigma})$ hence $\tilde{f} = \tilde{g}$. $\square$

Now let X be an arbitrary set and let $X^{\langle N \rangle}$ be the category of those subsets $\emptyset \neq Y \subseteq X$ which are in Set_N, with inclusions as morphisms. (If N is infinite, this is a directed set.)

5.4 Proposition: *The free left $\mathcal{L}$-module ${}_{\mathcal{L}}F(X)$ over the set X is*

$$ {}_{\mathcal{L}}F(X) = \varinjlim\nolimits_{Y \in X^{\langle N \rangle}} {}_{\mathcal{L}}F(Y). $$

PROOF: Observe that

$$ X = \varinjlim\nolimits_{Y \in X^{\langle N \rangle}} Y. $$

Since ${}_{\mathcal{L}}F$ was a functor on Set_N the following diagram shows the claim

$$
\begin{tikzcd}
Y \arrow[r, "\chi_\iota"] \arrow[d, "\iota"'] & {}_{\mathcal{L}}F(Y) \arrow[d, "\iota"] \arrow[ddr, "\tilde{f}_\iota"] & \\
X \arrow[r, "\chi"] \arrow[drr, "f"'] & {}_{\mathcal{L}}F(X) \arrow[dr, "\tilde{f}"] & \\
& & M
\end{tikzcd}
$$

$\square$

Now we consider the construction of free right $\mathcal{L}$-modules.

5.5 Proposition: *Let X be a set. Then $X \times \mathcal{L}^N$ is the free right $\mathcal{L}$-module over X.*

PROOF: $X \times \mathcal{L}^N$ carries a componentwise right $\mathcal{L}$-structure. We define a map $\chi : X \longrightarrow X \times \mathcal{L}^N$ by $\chi(x) := (x, \delta)$. Let $f : X \longrightarrow M$ be a map into a right $\mathcal{L}$-module M. We define $\tilde{f} : X \times \mathcal{L}^N \longrightarrow M$ by $\tilde{f}(x, \lambda) := f(x)\lambda$. This is obviously a homomorphism of right $\mathcal{L}$-modules and satisfies $\tilde{f}\chi = f$. For any homomorphism $\tilde{g} : X \times \mathcal{L}^N \longrightarrow M$ with $\tilde{g}\chi = f$ we have $\tilde{g}(x, \lambda) = \tilde{g}(x, \delta)l = \tilde{g}\chi(x)l = f(x)l = \tilde{f}(x, \lambda)$ hence $\tilde{f} = \tilde{g}$. $\square$

5.6 Proposition: *Let X be a set. Then the free $\mathcal{L}$-$\mathcal{M}$-bimodule over X is*

$$_{\mathcal{L}}F_{\mathcal{M}}(X) := {_{\mathcal{L}}F}(X \times \mathcal{M}^N).$$

PROOF: In the following universal problem diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[drr, "f"'] & X \times \mathcal{M}^N \arrow[r] \arrow[dr, "f'"] & {_{\mathcal{L}}F}(X \times \mathcal{M}^N) \arrow[d, "f''"] \\
 & & M
\end{tikzcd}$$

the map f'' is a homomorphism of right $\mathcal{M}$-modules since f' is compatible with multiplication from the right with elements of $\mathcal{M}^N$. $\square$

5.7 Definition: Let X be a left, a right, or a bi-module. A subset $\mathrm{Rel} \subseteq X \times X$ is called a (left, right, or bi-) *congruence relation* if Rel is an equivalence relation and a left, right, or bi-sub-module.

5.8 Proposition: *Let X be a (left, right, or bi-) module.*
(a) For each subset $U \in X \times X$ there is a smallest (left, right, or bi-) congruence relation Rel with $U \subseteq$ Rel.
(b) For any (left, right, or bi-) congruence relation Rel the set of equivalence classes X/Rel is a (left, right, or bi-) module.

PROOF: (a) Take Rel as the intersection of all congruence relations containing U.
(b) The proof of the module properties is straightforward. $\square$

6. TENSOR PRODUCTS

For a map $f : X \times Y^N \longrightarrow Z$ we define $f^{N-} : X^N \times Y^N \longrightarrow Z^N$ by

$$f^{N-}(\xi, \eta)(i) := f(\xi(i), \eta), \qquad i \in N.$$

6.1 Definition: Let $_{\mathcal{L}}X_{\mathcal{M}}$, $_{\mathcal{M}}Y_{\mathcal{K}}$, and $_{\mathcal{L}}Z_{\mathcal{K}}$ be bimodules. A map $f : X \times Y^N \longrightarrow Z$ is called *bilinear* if for all $x \in X, \xi \in X^N, \eta \in Y^N, l \in \mathcal{L}, \mu \in \mathcal{M}^N, \kappa \in \mathcal{K}^N$:

$$\begin{aligned}
f(l\xi, \eta) &= l f^{N-}(\xi, \eta), \\
f(x\mu, \eta) &= f(x, \mu\eta), \\
f(x, \eta\kappa) &= f(x, \eta)\kappa.
\end{aligned}$$

An $\mathcal{L}$-$\mathcal{K}$-bimodule $X \otimes_{\mathcal{M}} Y$ together with a bilinear map

$$\otimes : X \times Y^N \ni (x, \eta) \mapsto x \otimes_{\mathcal{M}} \eta \in X \otimes_{\mathcal{M}} Y$$

is called a *tensor product* if for every $\mathcal{L}$-$\mathcal{K}$-bimodule Z and for every bilinear map $f : X \times Y^N \longrightarrow Z$ there is exactly one homomorphism of $\mathcal{L}$-$\mathcal{K}$-bimodules $g : X \otimes_{\mathcal{M}} Y \longrightarrow Z$ such that the diagram

$$\begin{tikzcd} X \times Y^N \arrow[r, "\otimes"] \arrow[dr, "f"'] & X \otimes_{\mathcal{M}} Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

commutes.

As is customary we speak of $X \otimes_{\mathcal{M}} Y$ as the tensor product of X and Y and omit reference to the map $X \times Y^N \longrightarrow X \otimes_{\mathcal{M}} Y$. For $\xi \in X^N$, $\eta \in Y^N$ let $\xi \otimes_{\mathcal{M}} \eta \in (X \otimes_{\mathcal{M}} Y^N)^N$ be defined by $(\xi \otimes_{\mathcal{M}} \eta)(i) := \xi(i) \otimes_{\mathcal{M}} \eta \in X \times Y^N$. The following rules of calculation follow immediately:

$$\begin{array}{rcl} (l\xi) \otimes_{\mathcal{M}} \eta & = & l(\xi \otimes_{\mathcal{M}} \eta), \\ (x\mu) \otimes_{\mathcal{M}} \eta & = & x \otimes_{\mathcal{M}} (\mu\eta), \\ x \otimes_{\mathcal{M}} (\eta\kappa) & = & (x \otimes_{\mathcal{M}} \eta)\kappa, \\ (\lambda\xi) \otimes_{\mathcal{M}} \eta & = & \lambda(\xi \otimes_{\mathcal{M}} \eta), \\ \xi\mu \otimes_{\mathcal{M}} \eta & = & \xi \otimes_{\mathcal{M}} \mu\eta, \\ \xi \otimes_{\mathcal{M}} (\eta\kappa) & = & (\xi \otimes_{\mathcal{M}} \eta)\kappa. \end{array}$$

Observe that $l(\xi \otimes_{\mathcal{M}} \eta) = (l\xi) \otimes_{\mathcal{M}} \eta$, but that expressions of the form $l(x_i \otimes_{\mathcal{M}} \eta_i)_{i \in N}$ are defined as well (cf. Proposition 6.16).

6.2 Proposition: *For any bimodules ${}_{\mathcal{L}}X_{\mathcal{M}}$ and ${}_{\mathcal{M}}Y_{\mathcal{K}}$ the tensor product $X \otimes_{\mathcal{M}} Y$ exists.*

PROOF: Let ${}_{\mathcal{L}}F_{\mathcal{K}}(X \times Y^N)$ denote the free $\mathcal{L}$-$\mathcal{K}$-bimodule over $X \times Y^N$. We form the smallest congruence relation $(\mathrm{Rel} \subseteq {}_{\mathcal{L}}F_{\mathcal{K}}(X \times Y^N) \times {}_{\mathcal{L}}F_{\mathcal{K}}(X \times Y^N))$ on ${}_{\mathcal{L}}F_{\mathcal{K}}(X \times Y^N)$ which contains the elements

$$\begin{array}{l} ((l\xi, \eta), l(\xi, \eta)), \\ ((x\mu, \eta), (x, \mu\eta)), \\ ((x, \eta\kappa), (x, \eta)\kappa). \end{array}$$

Then the diagram

$$\begin{tikzcd} X \times Y^N \arrow[r, "\chi"] \arrow[drr, "f"'] & {}_{\mathcal{L}}F_{\mathcal{K}}(X \times Y^N) \arrow[r, "\nu"] \arrow[dr, "f'"'] & {}_{\mathcal{L}}F_{\mathcal{K}}(X \times Y^N)/\mathrm{Rel} \arrow[d, "g"] \\ & & Z \end{tikzcd}$$

provides a universal solution for the universal problem of tensor products. The details are straightforward to check. $\square$

6.3 Corollary: *The tensor product is a covariant functor*

$$\mathcal{L}\text{-}Mod\text{-}\mathcal{M} \times \mathcal{M}\text{-}Mod\text{-}\mathcal{K} \longrightarrow \mathcal{L}\text{-}Mod\text{-}\mathcal{K}.$$

PROOF: The claim follows immediately from the fact that bimodule homomorphisms $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ induce a map $f \times g^N : X \times Y^N \longrightarrow X' \times Y'^N$ which commutes with the operations in a way as to preserve bilinear maps, i.e. the map $\otimes_{\mathcal{M}}(f \times g^N)$ is a bilinear map:

$$\begin{array}{rcl} f(l\xi) \otimes_{\mathcal{M}} g^N(\eta) & = & (lf^{N-}(\xi)) \otimes_{\mathcal{M}} g^N(\eta) = l(f^{N-}(\xi) \otimes_{\mathcal{M}} g^N(\eta)), \\ f(x\mu) \otimes_{\mathcal{M}} g^N(\eta) & = & f(x)\mu \otimes_{\mathcal{M}} g^N(\eta) = f(x) \otimes_{\mathcal{M}} \mu g^N(\eta) \\ & = & f(x) \otimes_{\mathcal{M}} g^N(\mu\eta), \\ f(x) \otimes_{\mathcal{M}} g^N(\eta\kappa) & = & f(x) \otimes_{\mathcal{M}} (g^N(\eta)\kappa) = (f(x) \otimes_{\mathcal{M}} g^N(\eta))\kappa. \end{array}$$

So $f \times g^N$ induces a unique bimodule homomorphism

$$f \otimes_{\mathcal{M}} g : X \otimes_{\mathcal{M}} Y \longrightarrow X' \otimes_{\mathcal{M}} Y',$$

which defines the functor. Observe, that $(f \otimes_{\mathcal{M}} g)(x \otimes_{\mathcal{M}} \eta) = f(x) \otimes_{\mathcal{M}} g^N(\eta)$. $\square$

6.4 Definition: For bimodules ${}_{\mathcal{L}}X_{\mathcal{K}}$ and ${}_{\mathcal{M}}Y_{\mathcal{K}}$ define

$$\mathcal{H}om(X, Y) := \{f : X^N \longrightarrow Y | \forall \kappa \in \mathcal{K}^N, \xi \in X^N : f(\xi\kappa) = f(\xi)\kappa\}.$$

6.5 Lemma: *$\mathcal{H}om(X, Y)$ is a $\mathcal{M}$-$\mathcal{L}$-bimodule by the operations*

$$\begin{array}{rcl} (\gamma f)(\xi) & := & \gamma(f(\xi)) \\ (f\lambda)(\xi) & := & f(\lambda\xi) \end{array}$$

and a bifunctor contravariant in X and covariant in Y in $\mathcal{M}$-Mod-$\mathcal{L}$.

PROOF: Straightforward. $\square$

6.6 Proposition: *Let ${}_{\mathcal{L}}X_{\mathcal{M}}$, ${}_{\mathcal{M}}Y_{\mathcal{K}}$, ${}_{\mathcal{L}}Z_{\mathcal{K}}$ be bimodules. Then there is a natural isomorphism*

$$\mathcal{L}\text{-}Mod\text{-}\mathcal{K}(X \otimes_{\mathcal{M}} Y, Z) \cong \mathcal{L}\text{-}Mod\text{-}\mathcal{M}(X, \mathcal{H}om(Y, Z)).$$

PROOF: We define $\sigma : \mathcal{L}\text{-Mod-}\mathcal{K}(X \otimes_{\mathcal{M}} Y, Z) \longrightarrow \mathcal{L}\text{-Mod-}\mathcal{M}(X, \mathcal{H}om(Y, Z))$ by $\sigma(f)(x)(\eta) := f(x \otimes_{\mathcal{M}} \eta)$. Then

$$\begin{aligned}
\sigma(f)(x)(\eta\kappa) &= f(x \otimes_{\mathcal{M}} (\eta\kappa)) = f(x \otimes_{\mathcal{M}} \eta)\kappa = \sigma(f)(x)(\eta)\kappa, \\
(\sigma(f)(x)\mu)(\eta) &= \sigma(f)(x)(\mu\eta) = f(x \otimes_{\mathcal{M}} \mu\eta) = f(x\mu \otimes_{\mathcal{M}} \eta) \\
&= \sigma(f)(x\mu)(\eta), \\
(l\sigma(f)^N(\xi))(\eta) &= l(f^{N-}(\xi \otimes_{\mathcal{M}} \eta)) = f(l\xi \otimes_{\mathcal{M}} \eta) = \sigma(f)(l\xi)(\eta).
\end{aligned}$$

Thus σ is well-defined. The inverse map is defined by

$$\sigma^{-1} : \mathcal{L}\text{-Mod-}\mathcal{M}(X, \mathcal{H}om(Y, Z)) \longrightarrow \mathcal{L}\text{-Mod-}\mathcal{K}(X \otimes_{\mathcal{M}} Y, Z)$$

by $\sigma^{-1}(g)(x \otimes_{\mathcal{M}} \eta) := g(x)(\eta)$. Since $g(x\mu)(\eta) = g(x)(\mu\eta)$, $g(l\xi)(\eta) = lg^N(\xi)(\eta)$, and $g(x)(\eta\kappa) = g(x)(\eta)\kappa$, the map $\sigma^{-1}(g)$ is a well-defined bimodule homomorphism on $X \otimes_{\mathcal{M}} Y$. Obviously σ^{-1} is the inverse map to σ. It is easily seen that these maps are natural transformations. $\square$

6.7 Proposition: *Let ${}_{\mathcal{L}}X_{\mathcal{M}}$, ${}_{\mathcal{M}}Y_{\mathcal{K}}$, ${}_{\mathcal{K}}Z_{\mathcal{I}}$ be bimodules. Then the map*

$$\alpha : (X \otimes_{\mathcal{M}} Y) \otimes_{\mathcal{K}} Z \ni (x \otimes_{\mathcal{M}} \eta) \otimes_{\mathcal{K}} \zeta \mapsto x \otimes_{\mathcal{M}} (\eta \otimes_{\mathcal{K}} \zeta) \in X \otimes_{\mathcal{M}} (Y \otimes_{\mathcal{K}} Z)$$

is a bimodule homomorphism, which is coherent in the sense of monoidal categories.

PROOF: We first show that the map in the proposition is well defined. For that purpose we consider the map

$$\beta : X \otimes_{\mathcal{M}} Y \longrightarrow \mathcal{H}om(Z, X \otimes_{\mathcal{M}} (Y \otimes_{\mathcal{K}} Z))$$

with $\beta(x \otimes_{\mathcal{M}} \eta)(\zeta) := x \otimes_{\mathcal{M}} (\eta \otimes_{\mathcal{K}} \zeta)$. The map β is well-defined since

$$\begin{aligned}
(l\xi) \otimes_{\mathcal{M}} (\eta \otimes_{\mathcal{K}} \zeta) &= l(\xi \otimes_{\mathcal{M}} (\eta \otimes_{\mathcal{K}} \zeta)), \\
x\mu \otimes_{\mathcal{M}} (\eta \otimes_{\mathcal{K}} \zeta) &= x \otimes_{\mathcal{M}} (\mu\eta \otimes_{\mathcal{K}} \zeta), \\
x \otimes_{\mathcal{M}} (\eta\kappa \otimes_{\mathcal{K}} \zeta) &= x \otimes_{\mathcal{M}} (\eta \otimes_{\mathcal{K}} \kappa\zeta), \\
x \otimes_{\mathcal{M}} (\eta \otimes_{\mathcal{K}} \zeta\vartheta) &= (x \otimes_{\mathcal{M}} (\eta \otimes_{\mathcal{K}} \zeta))\vartheta.
\end{aligned}$$

The adjoint map is $\alpha : (X \otimes_{\mathcal{M}} Y) \otimes_{\mathcal{K}} Z \longrightarrow X \otimes_{\mathcal{M}} (Y \otimes_{\mathcal{K}} Z)$ with $\alpha((x \otimes_{\mathcal{M}} \eta) \otimes_{\mathcal{K}} \zeta) = x \otimes_{\mathcal{M}} (\eta \otimes_{\mathcal{K}} \zeta)$.
The coherence diagram is

$$\begin{tikzcd}
((U \otimes X) \otimes Y) \otimes Z \arrow[r, "\alpha\otimes 1"] \arrow[d, "\alpha"'] & (U \otimes (X \otimes Y)) \otimes Z \arrow[r, "\alpha"] & U \otimes ((X \otimes Y) \otimes Z) \arrow[d, "\alpha\otimes 1"] \\
(U \otimes X) \otimes (Y \otimes Z) \arrow[rr, "\alpha"] & & U \otimes (X \otimes (Y \otimes Z)).
\end{tikzcd}$$

By elementwise computations it is easy to see that this diagram commutes. $\square$

We will see in Cor. 6.17 that α is surjective.

6.8 Remark: We cannot show that α is an isomorphism. In fact we conjecture that it is not in general. However, how close it is to being bijective, can be seen by propositions 6.16 and 6.17. Nevertheless $\mathcal{L}$-Mod-$\mathcal{L}$ behaves like a closed monoidal category. The above pentagon diagram suffices to generate coherence. And the inner hom-functors have the usual properties. Before we prove this, we show that there is a two-sided unit for the tensor product.

6.9 Proposition: *There are natural isomorphisms*

$$s : \mathcal{L} \otimes_{\mathcal{L}} X \cong X \qquad \text{and} \qquad d : X \otimes_{\mathcal{M}} \mathcal{M} \cong X$$

which satisfy the coherence diagrams for monoidal categories

$$\begin{tikzcd} (X \otimes_{\mathcal{L}} \mathcal{L}) \otimes_{\mathcal{L}} Y \arrow[rr, "\alpha"] \arrow[dr, "d \otimes_{\mathcal{L}} Y"'] & & X \otimes_{\mathcal{L}} (\mathcal{L} \otimes_{\mathcal{L}} Y) \arrow[dl, "X \otimes_{\mathcal{L}} s"] \\ & X \otimes_{\mathcal{L}} Y & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L} \otimes_{\mathcal{L}} \mathcal{L} \arrow[r, equal] \arrow[dr, "d"'] & \mathcal{L} \otimes_{\mathcal{L}} \mathcal{L} \arrow[d, "s"] \\ & \mathcal{L} \end{tikzcd} .$$

Proof: We define $s' : \mathcal{L} \times X^N \longrightarrow X$ by $s'(l, \xi) := l\xi$. Obviously s' is bilinear, so it factors uniquely through a bimodule homomorphism s on the tensor product. We define $s^{-1} : X \longrightarrow \mathcal{L} \otimes_{\mathcal{L}} X$ by $s^{-1}(x) := \delta(1) \otimes_{\mathcal{L}} x^N$. Then $ss^{-1}(x) = s(\delta(1) \otimes_{\mathcal{L}} x^N) = \delta(1)x^N = x$ hence $ss^{-1} = \mathrm{id}|_X$ and $s^{-1}s(l \otimes_{\mathcal{L}} \xi) = s^{-1}(l\xi) = \delta(1) \otimes_{\mathcal{L}} (l\xi)^N = \delta(1) \otimes_{\mathcal{L}} l^N\xi = \delta(1)l^N \otimes_{\mathcal{L}} \xi = l \otimes_{\mathcal{L}} \xi$. Obviously it is sufficient to show that $s^{-1}s$ and id agree on elements of the form $l \otimes_{\mathcal{L}} \xi$, if we know that d^{-1} is an $\mathcal{L}$-bimodule homomorphism. We have $s^{-1}(l\xi) = \delta(1) \otimes_{\mathcal{L}} (l\xi)^N = \delta(1) \otimes_{\mathcal{L}} l^N \xi = l \otimes_{\mathcal{L}} \xi = l\delta \otimes_{\mathcal{L}} \xi = l(\delta(1) \otimes_{\mathcal{L}} x_i^N) = l(s^{-1})^N(\xi)$, where we used

$$\delta(i) \otimes_{\mathcal{L}} \xi = \delta(j)\delta(i)^N \otimes_{\mathcal{L}} \xi = \delta(j) \otimes_{\mathcal{L}} \delta(i)^N \xi = \delta(j) \otimes_{\mathcal{L}} x_i^N$$

for all choices of $i, j \in N$. Furthermore we have $s^{-1}(x\lambda) = \delta(1) \otimes_{\mathcal{L}} (x\lambda)^N = \delta(1) \otimes_{\mathcal{L}} x^N\lambda = s^{-1}(x)\lambda$. Consequently s and s^{-1} are inverse to each other. They are obviously natural transformations.

We define now $d' : X \times \mathcal{M}^N \longrightarrow X$ by $d'(x, \mu) := x\mu$. d' is obviously bilinear, so it factors uniquely through an $\mathcal{L}$-bimodule homomorphism d on the tensor product. We define $d^{-1} : X \longrightarrow X \otimes_{\mathcal{M}} \mathcal{M}$ by $d^{-1}(x) := x \otimes_{\mathcal{M}} \delta$. Then $dd^{-1}(x) = x\delta = x$ and $d^{-1}d(x \otimes_{\mathcal{M}} \mu) = x\mu \otimes_{\mathcal{M}} \delta = x \otimes_{\mathcal{M}} \mu$. Again we must show that d^{-1} is an $\mathcal{L}$-bimodule homomorphism: $d^{-1}(\gamma\xi) = \gamma\xi \otimes_{\mathcal{M}} \delta = \gamma(d^{-1})^N(\xi)$ and $d^{-1}(x\mu) = x\mu \otimes_{\mathcal{M}} \delta = x \otimes_{\mathcal{M}} \mu = x \otimes_{\mathcal{M}} \delta\mu = d^{-1}(x)\mu$. $\square$

6.10 Definition: $E \in \mathcal{L}$-Mod-$\mathcal{L}$ together with $\mathcal{L}$-bimodule homomorphisms $\nabla : E \otimes_{\mathcal{L}} E \longrightarrow E$ and $\eta : \mathcal{L} \longrightarrow E$ is called a *monoid*, if the following diagrams commute

$$\begin{tikzcd}
(E \otimes_{\mathcal{L}} E) \otimes_{\mathcal{L}} E \arrow[r, "\nabla\otimes_{\mathcal{L}}1"] \arrow[d, "\alpha"'] & E \otimes_{\mathcal{L}} E \arrow[dd, "\nabla"] \\
E \otimes_{\mathcal{L}} (E \otimes_{\mathcal{L}} E) \arrow[d, "1\otimes_{\mathcal{L}}\nabla"'] & \\
E \otimes_{\mathcal{L}} E \arrow[r, "\nabla"] & E
\end{tikzcd}$$

and

$$\begin{tikzcd}
E \arrow[r, "(1\otimes_{\mathcal{L}}\eta)d^{-1}"] \arrow[d, "(\eta\otimes_{\mathcal{L}}1)s^{-1}"'] \arrow[dr, "\mathrm{id}"] & E \otimes_{\mathcal{L}} E \arrow[d, "\nabla"] \\
E \otimes_{\mathcal{L}} E \arrow[r, "\nabla"] & E.
\end{tikzcd}$$

6.11 Proposition: *An $\mathcal{L}$-$\mathcal{L}$-bimodule E is a monoid in $\mathcal{L}$-Mod-$\mathcal{L}$, if and only if E is a left linear theory and $\eta : \mathcal{L} \longrightarrow E$ is a homomorphism of left linear theories inducing the $\mathcal{L}$-bimodule structure on E.*

PROOF: Let E be a left linear theory and $\eta : \mathcal{L} \longrightarrow E$ be a homomorphism of left linear theories. Then the multiplication $m : E \times E^N \longrightarrow E$ defines a homomorphism $\nabla : E \otimes_{\mathcal{L}} E \longrightarrow E$ in $\mathcal{L}$-Mod-$\mathcal{L}$ since

$$\begin{aligned}
&m(e\lambda, \epsilon') = (e \circ \eta^N(\lambda)) \circ \epsilon' = e \circ (\eta^N(\lambda) \circ \epsilon') = m(e, \lambda\epsilon'),\\
&m(l\epsilon, \epsilon') = (\eta(l) \circ \epsilon) \circ \epsilon') = \eta(l) \circ (\epsilon \circ \epsilon') = lm^N(\epsilon, \epsilon'),\\
&m(e, \epsilon'\lambda) = e \circ (\epsilon' \circ \eta^N(\lambda)) = (e \circ \epsilon') \circ \eta^N(\lambda) = m(e, \epsilon')\lambda.
\end{aligned}$$

The induced map ∇ is associative, since the multiplication is associative. We use the same symbol for the units in $\mathcal{L}$ and in E, hence $\eta(\delta) = \delta$. The unit property for a monoid follows from $\delta(1) \circ x^N = \delta(1)x^N = x$ and $x \circ \delta = x\delta = x$. Hence (E, ∇, η) is a monoid.

Conversely let E be a monoid. Then there is a multiplication

$$m : E \times E^N \xrightarrow{\otimes} E \otimes_{\mathcal{L}} E \xrightarrow{\nabla} E$$

which obviously is associative. From the unit property of a monoid we get $\eta(\delta(i)) \circ \epsilon = \nabla(\eta \otimes_{\mathcal{L}} 1)(\delta(i) \otimes_{\mathcal{L}} \epsilon) = \nabla(\eta \otimes_{\mathcal{L}} 1)(\delta(1) \otimes_{\mathcal{L}} e_i^N) = e_i = \delta(i)\epsilon$, using an equality from the proof of the previous proposition and $e \circ \eta^N(\delta) = \nabla(E \otimes_{\mathcal{L}} \eta)(e \otimes_{\mathcal{L}} \delta) = e = e\delta$, where we abbreviated $\eta^N(\delta) =: \delta$. Then δ is the unit of E and E is a left linear theory. The map η preserves the unit and induces the bimodules structure of E, since $\eta(l) = \eta(l\delta) = l\eta(\delta) = l\delta$, hence $\eta(l) \circ \epsilon = (l\delta) \circ \epsilon = l\epsilon$ and

$e \circ \eta^N(\lambda) = e \circ (\lambda\delta) = e\lambda$. Finally η is a homomorphism of left linear theories since $\eta(l\lambda) = \eta(l\delta\lambda) = l\delta\lambda = (l\delta\lambda) \circ \delta = (l\delta) \circ (\lambda\delta) = \eta(l) \circ \eta^N(\lambda)$. □

6.12 Remark: Let X and Y be $\mathcal{L}$-bimodules. The evaluation

$$ev : \mathcal{H}om(X, Y) \otimes_{\mathcal{L}} X \longrightarrow Y$$

with $ev(f \otimes_{\mathcal{L}} \xi) := f(\xi)$ is the counit for the pair of adjoint functors $- \otimes_{\mathcal{L}} X$ and $\mathcal{H}om(X, -)$.

6.13 Corollary: *Let ${}_{\mathcal{L}}X_{\mathcal{M}}$, ${}_{\mathcal{M}}Y_{\mathcal{K}}$, ${}_{\mathcal{L}}Z_{\mathcal{K}}$ be bimodules. For each bimodule homomorphism $f : X \otimes_{\mathcal{M}} Y \longrightarrow Z$ there is a unique bimodule homomorphism $g : X \longrightarrow \mathcal{H}om(Y, Z)$ such that the diagram commutes:*

$$\begin{tikzcd}
X \otimes_{\mathcal{M}} Y \arrow[d, "g \otimes_{\mathcal{M}} Y"'] \arrow[dr, "f"] & \\
\mathcal{H}om(Y, Z) \otimes_{\mathcal{M}} Y \arrow[r, "ev"] & Z.
\end{tikzcd}$$

Proof: This is a standard consequence of the fact that $- \otimes_{\mathcal{M}} Y$ is left-adjoint to $\mathcal{H}om(Y, -)$. □

6.14 Corollary: *The evaluation $ev : \mathcal{H}om(X, Y) \otimes_{\mathcal{M}} X \longrightarrow Y$ defines a composition of maps*

$$\mathcal{H}om(Y, Z) \otimes_{\mathcal{M}} \mathcal{H}om(X, Y) \ni f \otimes_{\mathcal{M}} \psi \mapsto (x \mapsto f(\psi(\xi)) \in \mathcal{H}om(X, Z),$$

which is associative in the sense that the diagram

$$\begin{tikzcd}
(\mathcal{H}om(Z, U) \otimes \mathcal{H}om(Y, Z)) \otimes \mathcal{H}om(X, Y) \arrow[r] \arrow[d] & \mathcal{H}om(Y, U) \otimes \mathcal{H}om(X, Y) \arrow[dd] \\
\mathcal{H}om(Z, U) \otimes (\mathcal{H}om(Y, Z) \otimes \mathcal{H}om(X, Y)) \arrow[d] & \\
\mathcal{H}om(Z, U) \otimes \mathcal{H}om(X, Z) \arrow[r] & \mathcal{H}om(X, U)
\end{tikzcd}$$

commutes.

Proof: The map $\mathcal{H}om(Y, Z) \otimes_{\mathcal{M}} \mathcal{H}om(X, Y) \longrightarrow \mathcal{H}om(X, Z)$ induced by the evaluation is uniquely determined by the following diagram

$$\begin{tikzcd}
(\mathcal{H}om(Y, Z) \otimes \mathcal{H}om(X, Y)) \otimes X \arrow[r] \arrow[dd, "g \otimes 1"'] & \mathcal{H}om(Y, Z) \otimes (\mathcal{H}om(X, Y) \otimes X) \arrow[d, "1 \otimes ev"] \\
& \mathcal{H}om(Y, Z) \otimes Y \arrow[d, "ev"] \\
\mathcal{H}om(X, Z) \otimes X \arrow[r] & Z.
\end{tikzcd}$$

It is easy to see that the composition is described as given in the Corollary. $\square$

6.15 Corollary: *For a bimodule ${}_{\mathcal{L}}X_{\mathcal{M}}$ the set of inner endomorphisms $\mathcal{E}nd(X) := \mathcal{H}om(X, X)$ is a left linear theory.*

PROOF: By Corollary 6.14 $\mathcal{E}nd(X)$ is a monoid in the quasi-monoidal category $\mathcal{L}$-Mod-$\mathcal{L}$. The unitary law is given by the map $\mathcal{L} \ni l \mapsto (\xi \mapsto l\xi) \in \mathcal{E}nd(X)$. By Proposition 6.11 it is a left linear theory. $\square$

6.16 Lemma: *Let N be an infinite set. Let $(x_i \otimes_{\mathcal{M}} \eta_i)_{i \in N} \in (X \otimes_{\mathcal{M}} Y)^N$. Then there are elements $\xi' \in X^N$, $\eta' \in Y^N$ such that*

$$(x_i \otimes_{\mathcal{M}} \eta_i)_{i \in N} = (\xi'(i) \otimes_{\mathcal{M}} \eta')_{i \in N}.$$

PROOF: Since N is infinite there is a bijection $\tau : N \times N \longrightarrow N$ with inverse map σ. We define $\Delta := \delta\tau$ such that $\Delta(i, -) \in \mathcal{M}^N$, $\eta' := \eta\sigma$, and $\xi'(i) := \xi(i)\Delta(i, -)$. Then we have $\xi'(i) \otimes_{\mathcal{M}} \eta' = \xi(i)\Delta(i, -) \otimes_{\mathcal{M}} \eta' = \xi(i) \otimes_{\mathcal{M}} \Delta(i, -)\eta' = \xi(i) \otimes_{\mathcal{M}} \eta_i$, since $\Delta(i, j)\eta = \delta\tau(i, j)\eta = \eta'\tau(i, j) = \eta_i(j)$. $\square$

6.17 Theorem: *The associativity map α is surjective.*

PROOF: By 6.16 the generators $x \otimes_{\mathcal{M}} (\eta(i) \otimes_{\mathcal{K}} \zeta_i)_{i \in N}$ are in the image of α. $\square$

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