FOURIER TRANSFORMS OVER FINITE QUANTUM GROUPS

BODO PAREIGIS

1. Introduction

In this note we want to clarify the notion of an integral for arbitrary Hopf algebras that has been introduced a long time ago [2, 6]. The relation between the integral on a Hopf algebra and integrals in functional analysis has only been hinted at in several publications. With the strong interest in quantum groups, i.e. non-commutative and non-cocommutative Hopf algebras, we wish to show in which form certain transformation rules for integrals occur in quantum groups.

Our point of view will be the following. Let G be a quantum group in the sense of non-commutative algebraic geometry, that is a space whose function algebra is given by an arbitrary Hopf algebra H over some base field $\mathbb{K}$. We will also have to use the algebra of linear functionals $H^* = \mathrm{Hom}(H, \mathbb{K})$ with the multiplication induced by the diagonal of H (called the bialgebra of G in the French literature). For most of this paper we will assume that H is finite dimensional. Observe that the functions in H do not commute under multiplication and that they usually have no general commutation formula.

The model for this setup can be found in functional analysis. There the group G is a locally compact group, H the space of representative functions on G, and H^* the space of generalized functions or distributions. Then the functions commute under multiplication.

We will also consider two special examples of our setup. For an arbitrary finite group G the Hopf algebra $H = \mathbb{K}^G$ is defined to be the algebra of functions on G. Then $H^* = \mathbb{K}G$, the group algebra, is the linear dual of H.

If the finite group G is Abelian and if $\mathbb{K}$ is algebraicly closed with $\mathrm{char}(\mathbb{K}) \neq |G|$ then the corresponding Hopf algebra is as above $H = \mathbb{K}^G$ and $H^* = \mathbb{K}G$. By Pontryagin duality there is the group $\widehat{G}$ of characters on G such that $H = \mathbb{K}^G = \mathbb{K}\widehat{G}$ and $H^* = \mathbb{K}G = \mathbb{K}^{\hat{G}}$.

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2. Integrals

Let H be an arbitrary Hopf algebra [3, 6]. The linear functionals $a \in H^*$ will be considered as *generalized integrals on* H ([5] p.123). We have an operation $H^* \otimes H \ni a \otimes f \mapsto \langle a, f \rangle \in \mathbb{K}$ that is nondegenerate on both sides.

We denote the elements of H by $f, g, h \in H$, the elements of H^* by $a, b, c \in H^*$, the (non existing) elements of the quantum group G by $x, y, z \in G$.

We will be interested in a special generalized integral $\int \in H^*$ satisfying

$$a \textstyle\int = \langle a, 1_H \rangle \int \tag{1}$$

or $a \int = \varepsilon(a) \int$. Such an integral is called a *left invariant integral.*

In the case of a locally compact group G such an element is given by the Haar integral with respect to a left invariant Haar measure [1]

$$\int_G f(x) \mu dx = \langle \textstyle\int, f \rangle.$$

Therefore we write in the general quantum group situation

$$\int f(x) dx := \langle \textstyle\int, f \rangle. \tag{2}$$

This notation has two parentheses, $\int$ and dx, so that the integrand f is clearly separated. We also use the notation

$$\int f(x) g(x) dx := \langle \textstyle\int, fg \rangle. \tag{3}$$

Observe that $f(x)$ and $g(x)$ are just parts of the whole symbol and in particular that they do not commute.

In the case of a finite group G a left invariant integral *in* $H^* = \mathbb{K}G$ *on* $H = \mathbb{K}^G$ is known to be

$$\textstyle\int = \sum_{x \in G} x \tag{4}$$

since $y \sum_{x \in G} x = \sum_{x \in G} yx = \sum_{x \in G} x = \langle y, 1_H \rangle \sum_{x \in G} x$. For arbitrary $a \in \mathbb{K}G$ we have $a \sum_{x \in G} x = \varepsilon(a) \sum_{x \in G} x$. So our integral notation turns out to be

$$\int f(x) dx = \sum_{x \in G} f(x) \tag{5}$$

and has the property

$$\int f(x) dx = \sum_{x \in G} f(x) = \sum_{x \in G} f(yx) = \int f(yx) dx \tag{6}$$

for all $y \in G$. This left invariant integral turns out to be also right invariant $\int a = \int \varepsilon(a)$.

3. v. Neumann Transforms

We return to the arbitrary Hopf algebra H of a quantum group. Since $H^* = \mathrm{Hom}(H, \mathbb{K})$ and $S : H \to H$ is an algebra antihomomorphism, the dual H^* is an H-module in four different ways:

$$\begin{array}{ll} \langle (f \rightharpoonup a), g\rangle := \langle a, gf\rangle, & \langle (a \leftharpoonup f), g\rangle := \langle a, fg\rangle, \\ \langle (f \rightharpoondown a), g\rangle := \langle a, S(f)g\rangle, & \langle (a \leftharpoondown f), g\rangle := \langle a, gS(f)\rangle. \end{array} \tag{7}$$

If H is finite dimensional then H^* is a Hopf algebra. The equality $\langle (f \rightharpoonup a), g\rangle = \langle a, gf\rangle = \sum \langle a_{(1)}, g\rangle\langle a_{(2)}, f\rangle$ implies

$$(f \rightharpoonup a) = \sum a_{(1)}\langle a_{(2)}, f\rangle. \tag{8}$$

Analogously we have

$$(a \leftharpoonup f) = \sum \langle a_{(1)}, f\rangle a_{(2)}. \tag{9}$$

An easy observation about left invariant integrals on H is

Lemma 1. *The set* $\mathrm{Int}_l(H^*)$ *of left invariant integrals is a two sided ideal in* H^**.*

The integral $\int$ is left invariant iff $\forall y \in H^* : y\int = \varepsilon(y)\int$ iff $\forall y \in H^*, f \in H : \langle y\int, f\rangle = \langle \int, (f \leftharpoonup y)\rangle = \varepsilon(y)\langle \int, f\rangle$. Since $\langle x, f\rangle = f(x)$ and $\langle x, (f \leftharpoonup y)\rangle = \langle yx, f\rangle = f(yx)$, the integral $\int$ is left invariant iff

$$\int f(yx)dx = \varepsilon(y)\int f(x)dx. \tag{10}$$

Theorem 2. *If there exists* $0 \neq \int \in \mathrm{Int}_l(H^*)$ *then the map* $H \ni f \mapsto (\int \leftharpoondown f) \in H^*$ *is injective.*

Proof. By [6] theorem 5.1.3 the following homomorphism $\mathrm{Int}_l(H^*) \otimes H \ni \int \otimes f \mapsto (\int \leftharpoondown f) \in H^{*rat}(\subseteq H^*)$ is bijective. ☐

Corollary 3. *If there exists* $0 \neq \int \in \mathrm{Int}_l(H^*)$ *then the antipode* $S : H \to H$ *is injective.*

Proof. The monomorphism $H \ni f \mapsto (\int \leftharpoondown f) \in H^*$ is composed of $S : H \to H$ and $H \ni f \mapsto (f \rightharpoonup \int) \in H^*$. ☐

We call a generalized integral $a \in H^*$ a *rational integral* if a is of the form $a = \sum \int_i \leftharpoondown f_i$.

Corollary 4. *For every rational integral* $a \in H^*$ *there is a unique* $g \in H$ *such that*

$$\langle a, f\rangle = \int f(x)S(g)(x)dx$$

for all $f \in H$*.*

Proof. For every rational integral a there is a unique function $g \in H$ with $a = (\int \leftharpoondown g)$, hence $\langle a, f\rangle = \langle \int \leftharpoondown g, f\rangle = \langle S(g) \rightharpoonup \int, f\rangle = \langle \int, fS(g)\rangle = \int f(x)S(g)(x)dx$. ☐

One of the first to study this property of the integral $\int$ to represent other linear functionals was J. v. Neumann in [4].

If H is finite dimensional then the isomorphism $\mathrm{Int}_l(H^*) \otimes H \ni \int \otimes f \mapsto (\int \leftharpoondown f) \in H^{*rat} (\subseteq H^*)$ shows $\mathrm{Int}_l(H^*)$ has dimension 1.

We choose for the rest of this paper a non zero left invariant integral $\int$ whenever we are in the situation of H finite dimensional.

Let H be finite dimensional. Since $\int a$ is a left invariant integral and $\dim(\mathrm{Int}_l(H^*)) = 1$ there is a unique $\mathrm{mod}(a) \in \mathbb{K}$ with

$$\int a = \mathrm{mod}(a) \int .$$

One checks that $\mathrm{mod} : H^* \to \mathbb{K}$ is an algebra homomorphism called the *modulus of* H^*. If $\mathrm{mod} = \varepsilon = 1_{H^*}$ then H^* is called *unimodular*. This is equivalent to $\int$ also being right invariant or $\mathrm{Int}_l(H^*) = \mathrm{Int}_r(H^*)$.

Corollary 5. *If H is finite dimensional then for every $a \in H^*$ there is a unique $g \in H$ such that $\langle a, f\rangle = \int f(x)S(g)(x)dx$ for all $f \in H$.*

Corollary 6. *If H is finite dimensional then $S : H \to H$ and $H \ni f \mapsto (f \rightharpoonup \int) \in H^*$ are bijective.*

If G is a finite group then every generalized integral $a \in \mathbb{K}^G$ can be written with a uniquely determined $g \in H$ as

$$\langle a, f\rangle = \int f(x)S(g)(x)dx = \sum_{x\in G} f(x)g(x^{-1}) \tag{11}$$

for all $f \in H$.

If G is a finite Abelian group then each group element (rational integral) $y \in G \subseteq \mathbb{K}G$ can be written as

$$y = \sum_{x\in G} \sum_{\chi \in \hat{G}} \beta_\chi \langle x^{-1}, \chi\rangle x$$

since $\langle y, f\rangle = \langle (\int \leftharpoondown \sum_{\chi\in\hat{G}} \beta_\chi \chi), f\rangle = \langle \int, fS(\sum_{\chi\in\hat{G}} \beta_\chi\chi)\rangle = \sum_{x\in G}\langle x, f\rangle \sum_{\chi\in\hat{G}} \beta_\chi \langle x, S(\chi)\rangle = \langle \sum_{x\in G}\sum_{\chi\in\hat{G}} \beta_\chi \langle x^{-1}, \chi\rangle x, f\rangle$. In particular the matrix $(\langle x^{-1}, \chi\rangle)$ is invertible.

4. The Nakayama Automorphism

Let H be finite dimensional. Since $\langle \int, fg\rangle = \langle (\int \leftharpoondown f), g\rangle$ as a functional on g is a generalized integral, there is a unique $\nu(f) \in H$ such that

$$\langle \textstyle\int, fg\rangle = \langle \textstyle\int, g\nu(f)\rangle \tag{12}$$

or

$$\int f(x)g(x)dx = \int g(x)\nu(f)(x)dx. \tag{13}$$

Although the functions $f, g \in H$ of the quantum group do not commute under multiplication, there is a simple commutation rule if the product is integrated.

Proposition 7. *The map $\nu : H \to H$ is an algebra automorphism, called the* Nakayama automorphism*.*

Proof. It is clear that ν is a linear map. We have $\int f\nu(gh) = \int ghf = \int hf\nu(g) = \int f\nu(g)\nu(h)$ hence $\nu(gh) = \nu(g)\nu(h)$ and $\int f\nu(1) = \int f$ hence $\nu(1) = 1$. Furthermore if $\nu(g) = 0$ then $0 = \langle \int, f\nu(g)\rangle = \langle \int, gf\rangle = \langle (f \rightharpoonup \int), g\rangle$ for all $f \in H$ hence $\langle a, g\rangle = 0$ for all $a \in H^*$ hence $g = 0$. So ν is injective hence bijective. $\square$

Corollary 8. *The map $H \ni f \mapsto (\int \leftharpoonup f) \in H^*$ is an isomorphism.*

Proof. We have
$$(\textstyle\int \leftharpoonup f) = (\nu(f) \rightharpoonup \textstyle\int)$$
since $\langle (\int \leftharpoonup f), g\rangle = \langle \int, fg\rangle = \langle \int, g\nu(f)\rangle = \langle (\nu(f) \rightharpoonup \int), g\rangle$. This implies the corollary. $\square$

If G is a finite group and $H = \mathbb{K}^G$ then H is commutative hence $\nu = \mathrm{id}$.

5. The Dirac Delta Function

An element $\delta \in H$ is called a *Dirac δ-function* if δ is a left invariant integral in H with $\langle \int, \delta\rangle = 1$, i.e. if δ satisfies
$$f\delta = \varepsilon(f)\delta \qquad \text{and} \qquad \int \delta(x)dx = 1$$
for all $f \in H$. If H has a Dirac δ-function then we write
$$\int^* a(x)dx = \textstyle\int^* a := \langle a, \delta\rangle. \tag{14}$$

Proposition 9.

1. *If H is finite dimensional then there exists a unique Dirac δ-function δ.*
2. *If H is infinite dimensional then there exists no Dirac δ-function.*

Proof. 1. Since $H \ni f \mapsto (f \rightharpoonup \int) \in H^*$ is an isomorphism there is a $\delta \in H$ such that $(\delta \rightharpoonup \int) = \varepsilon$. Then $(f\delta \rightharpoonup \int) = (f \rightharpoonup (\delta \rightharpoonup \int)) = (f \rightharpoonup \varepsilon) = \varepsilon(f)\varepsilon = \varepsilon(f)(\delta \rightharpoonup \int)$ which implies $f\delta = \varepsilon(f)\delta$. Furthermore we have $\langle \int, \delta\rangle = \langle \int, 1_H\delta\rangle = \langle (\delta \rightharpoonup \int), 1_H\rangle = \varepsilon(1_H) = 1_{\mathbb{K}}$.

2. is [6] exercise V.4. $\square$

Lemma 10. *Let H be a finite dimensional Hopf algebra. Then $\int \in H^*$ is a left integral iff*
$$a(\sum \textstyle\int_{(1)} \otimes S(\int_{(2)})) = (\sum \int_{(1)} \otimes S(\int_{(2)}))a \tag{15}$$

iff

$$\sum S(a)\textstyle\int_{(1)} \otimes \int_{(2)} = \sum \int_{(1)} \otimes a\int_{(2)} \tag{16}$$

iff

$$\sum f_{(1)}\langle \textstyle\int, f_{(2)}\rangle = \langle \int, f\rangle 1_H. \tag{17}$$

Proof. Let $\int$ be a left integral. Then

$$\sum a_{(1)}\textstyle\int_{(1)} \otimes S(\int_{(2)})S(a_{(2)}) = \sum (a\int)_{(1)} \otimes S((a\int)_{(2)}) = \varepsilon(a)(\sum \int_{(1)} \otimes S(\int_{(2)}))$$

for all $a \in H$. Hence

$$\begin{aligned}(\textstyle\sum \int_{(1)} \otimes S(\int_{(2)}))a &= \textstyle\sum \varepsilon(a_{(1)})(\int_{(1)} \otimes S(\int_{(2)}))a_{(2)} \\ &= \textstyle\sum a_{(1)}\int_{(1)} \otimes S(\int_{(2)})S(a_{(2)})a_{(3)} \\ &= \textstyle\sum a_{(1)}\int_{(1)} \otimes S(\int_{(2)})\varepsilon(a_{(2)}) = a(\sum \int_{(1)} \otimes S(\int_{(2)})).\end{aligned}$$

Conversely $a(\sum \int_{(1)} \varepsilon(S(\int_{(2)}))) = (\sum \int_{(1)} \varepsilon(S(\int_{(2)})a)) = \varepsilon(a)(\sum \int_{(1)} \varepsilon(S(\int_{(2)})))$, hence $\int = \sum \int_{(1)} \varepsilon(S(\int_{(2)}))$ is a left integral.

Since S is bijective the following holds

$$\begin{aligned}&\textstyle\sum S(a)\int_{(1)} \otimes \int_{(2)} = \sum S(a)\int_{(1)} \otimes S^{-1}(S(\int_{(2)})) \\ &= \textstyle\sum \int_{(1)} \otimes S^{-1}(S(\int_{(2)})S(a)) = \sum \int_{(1)} \otimes a\int_{(2)}.\end{aligned}$$

The converse follows easily.

If $\int \in \mathrm{Int}_l(H)$ is a left integral then $\sum\langle a, f_{(1)}\rangle\langle \int, f_{(2)}\rangle = \langle a\int, f\rangle = \langle a, 1_H\rangle\langle \int, f\rangle$.

Conversely if $\lambda \in H^*$ with (17) is given then $\langle a\lambda, f\rangle = \sum\langle a, f_{(1)}\rangle\langle \lambda, f_{(2)}\rangle = \langle a, 1_H\rangle\langle \lambda, f\rangle$ hence $a\lambda = \varepsilon(a)\lambda$. $\square$

If G is a finite group then

$$\delta(x) = \begin{cases} 0 \text{ if } x \neq e; \\ 1 \text{ if } x = e. \end{cases} \tag{18}$$

In fact since δ is left invariant we get $f(x)\delta(x) = f(e)\delta(x)$ for all $x \in G$ and $f \in \mathbb{K}^G$. Since $G \subset H^* = \mathbb{K}G$ is a basis, we get $\delta(x) = 0$ if $x \neq e$. Furthermore $\int \delta(x)dx = \sum_{x\in G} \delta(x) = 1$ implies $f(e) = 1$.

If G is a finite Abelian group we get $\delta = \alpha \sum_{\chi\in\hat{G}} \chi$ for some $\alpha \in \mathbb{K}$. The evaluation gives $1 = \alpha\langle \int, \delta\rangle = \alpha \sum_{x\in G,\chi\in\hat{G}}\langle \chi, x\rangle$. Now let $\lambda \in \widehat{G}$. Then $\sum_{\chi\in\hat{G}}\langle \chi, x\rangle = \sum_{\chi\in\hat{G}}\langle \lambda\chi, x\rangle = \langle \lambda, x\rangle \sum_{\chi\in\hat{G}}\langle \chi, x\rangle$. Since for each $x \in G \setminus \{e\}$ there is a λ such that $\langle \lambda, x\rangle \neq 1$ and we get

$$\sum_{\chi\in\hat{G}}\langle \chi, x\rangle = |G|\delta_{e,x}.$$

Hence $\sum_{x\in G,\chi\in\hat{G}}\langle\chi,x\rangle = |G| = \alpha^{-1}$ and

$$\delta = |G|^{-1}\sum_{\chi\in\hat{G}}\chi. \tag{19}$$

6. Fourier Transforms

Let H be finite dimensional for the rest of this paper. In Corollary 8 we have seen that the map $H \ni f \mapsto (\int \leftharpoonup f) \in H^*$ is an isomorphism. This map will be called the *Fourier transform.*

Theorem 11. *The Fourier transform $H \ni f \mapsto \widetilde{f} \in H^*$ is bijective with*

$$\widetilde{f} = (\textstyle\int \leftharpoonup f) = \sum\langle \int_{(1)}, f\rangle \int_{(2)} \tag{20}$$

The inverse Fourier transform is defined by

$$\widetilde{a} = \sum S^{-1}(\delta_{(1)})\langle a, \delta_{(2)}\rangle. \tag{21}$$

Since these maps are inverses of each other the following formulas hold

$$\begin{array}{ll} \langle\widetilde{f}, g\rangle = \displaystyle\int f(x)g(x)dx & \langle a, \widetilde{b}\rangle = \displaystyle\int^* S^{-1}(a)(x)b(x)dx \\ f = \sum S^{-1}(\delta_{(1)})\langle\widetilde{f}, \delta_{(2)}\rangle & a = \sum\langle \int_{(1)}, \widetilde{a}\rangle \int_{(2)}. \end{array} \tag{22}$$

Proof. We use the isomorphisms $H \to H^*$ defined by $\widehat{f} := \widetilde{f} = (\int \leftharpoonup f) = \sum\langle \int_{(1)}, f\rangle \int_{(2)}$ and $H^* \to H$ defined by $\widehat{a} := (a \rightharpoonup \delta) = \sum \delta_{(1)}\langle a, \delta_{(2)}\rangle$. Because of

$$\langle a, \widehat{b}\rangle = \langle a, (b \rightharpoonup \delta)\rangle = \langle ab, \delta\rangle \tag{23}$$

and

$$\langle\widetilde{f}, g\rangle = \langle(\textstyle\int \leftharpoonup f), g\rangle = \langle \int, fg\rangle \tag{24}$$

we get for all $a \in H^*$ and $f \in H$

$$\begin{aligned} \langle a, \widehat{\widehat{f}}\rangle &= \langle a\widehat{f}, \delta\rangle = \textstyle\sum\langle a, \delta_{(1)}\rangle\langle\widehat{f}, \delta_{(2)}\rangle = \sum\langle a, \delta_{(1)}\rangle\langle \int, f\delta_{(2)}\rangle \qquad (\text{ by Lemma 10 }) \\ &= \textstyle\sum\langle a, S(f)\delta_{(1)}\rangle\langle \int, \delta_{(2)}\rangle = \langle a, S(f)\rangle\langle \int, \delta\rangle = \langle a, S(f)\rangle. \end{aligned}$$

This gives $\widehat{\widehat{f}} = S(f)$. So the inverse map of $H \to H^*$ with $\widehat{f} = (\int \leftharpoonup f) = \widetilde{f}$ is $H^* \to H$ with $S^{-1}(\widehat{a}) = \sum S^{-1}(\delta_{(1)})\langle a, \delta_{(2)}\rangle = \widetilde{a}$. Then the given inversion formulas are clear. □

If G is a finite group and $H = \mathbb{K}^G$ then

$$\widetilde{f} = \sum_{x\in G} f(x)x.$$

Since $\Delta(\delta) = \sum_{x\in G} x^{-1*} \otimes x^*$ where the $x^* \in \mathbb{K}^G$ are the dual basis to the $x \in G$, we get

$$\widetilde{a} = \sum_{x\in G} \langle a, x^*\rangle x^*.$$

If G is a finite Abelian group then the groups G and $\widehat{G}$ are isomorphic so the Fourier transform induces a linear automorphism $\widetilde{-} : \mathbb{K}^G \to \mathbb{K}^G$ and we have

$$\widetilde{a} = |G|^{-1} \sum_{\chi\in\hat{G}} \langle a, \chi\rangle \chi^{-1}$$

By substituting the formulas for the integral and the Dirac δ-function (4) and (19) we get

$$\begin{array}{ll} \widetilde{f} = \sum_{x\in G} f(x)x, & \widetilde{a} = |G|^{-1} \sum_{\chi\in\hat{G}} a(\chi)\chi^{-1}, \\ f = |G|^{-1} \sum_{\chi\in\hat{G}} \widetilde{f}(\chi)\chi^{-1}, & a = \sum_{x\in G} \widetilde{a}(x)x. \end{array} \tag{25}$$

This implies

$$\widetilde{f}(\chi) = \sum_{x\in G} f(x)\chi(x) = \int f(x)\chi(x)dx \tag{26}$$

with inverse transform

$$\widetilde{a}(x) = |G|^{-1} \sum_{\chi\in\hat{G}} \chi(a)\chi^{-1}(x). \tag{27}$$

Lemma 12. *The Fourier transforms of the left invariant integrals in H and H^* are*

$$\widetilde{\delta} = \varepsilon\nu^{-1} \in H^* \qquad \textit{and} \qquad \widetilde{\int} = 1 \in H. \tag{28}$$

Proof. We have $\langle \widetilde{\delta}, f\rangle = \langle \int, \delta f\rangle = \langle \int, \nu^{-1}(f)\delta\rangle = \varepsilon\nu^{-1}(f)\langle \int, \delta\rangle = \varepsilon\nu^{-1}(f)$ hence $\widetilde{\delta} = \varepsilon\nu^{-1}$ and $\langle a, \widetilde{\int}\rangle = \sum \langle a, S^{-1}(\delta_{(1)})\rangle\langle \int, \delta_{(2)}\rangle = \langle a, S^{-1}(1)\rangle\langle \int, \delta\rangle = \langle a, 1\rangle$, hence $\widetilde{\int} = 1$. $\square$

Proposition 13. *Define a* convolution multiplication *on H^* by*

$$\langle a * b, f\rangle := \sum \langle a, S^{-1}(\delta_{(1)})f\rangle\langle b, \delta_{(2)}\rangle.$$

Then the following transformation rule holds for $f, g \in H$:

$$\widetilde{fg} = \widetilde{f} * \widetilde{g}. \tag{29}$$

In particular H^* with the convolution multiplication is an associative algebra with unit $\widetilde{1_H} = \int$, i.e.

$$\int * a = a * \int = a. \tag{30}$$

Proof. Given $f, g, h \in H^*$. Then

$$\begin{aligned}\langle \widetilde{fg}, h\rangle &= \langle \textstyle\int, fgh\rangle = \langle \int, fS^{-1}(1_H)gh\rangle\langle \int, \delta\rangle \\ &= \textstyle\sum\langle \int, fS^{-1}(\delta_{(1)})gh\rangle\langle \int, \delta_{(2)}\rangle = \sum\langle \int, fS^{-1}(\delta_{(1)})h\rangle\langle \int, g\delta_{(2)}\rangle \\ &= \textstyle\sum\langle \widetilde{f}, S^{-1}(\delta_{(1)})h\rangle\langle \widetilde{g}, \delta_{(2)}\rangle = \langle \widetilde{f} * \widetilde{g}, h\rangle.\end{aligned}$$

From (28) we get $\widetilde{1}_H = \int$. So we have $\widetilde{f} = \widetilde{1f} = \widetilde{1} * \widetilde{f} = \int * \widetilde{f}$. □

If G is a finite Abelian group and $a, b \in H^* = \mathbb{K}^{\hat{G}}$. Then

$$(a * b)(\mu) = |G|^{-1} \sum_{\chi,\lambda\in\hat{G},\chi\lambda=\mu} a(\lambda)b(\chi).$$

In fact we have

$$\begin{aligned}(a * b)(\mu) &= \langle a * b, \mu\rangle = \textstyle\sum\langle a, S^{-1}(\delta_{(1)})\mu\rangle\langle b, \delta_{(2)}\rangle \\ &= |G|^{-1}\textstyle\sum_{\chi\in\hat{G}}\langle a, \chi^{-1}\mu\rangle\langle b, \chi\rangle = |G|^{-1}\sum_{\chi,\lambda\in\hat{G},\chi\lambda=\mu} a(\lambda)b(\chi).\end{aligned}$$

7. The Plancherel Formula

One of the most important formulas for Fourier transforms is the Plancherel formula on the invariance of the inner product under Fourier transforms. We have

Theorem 14. *(The Plancherel formula)*

$$\langle a, f\rangle = \langle \widetilde{f}, \nu(\widetilde{a})\rangle. \tag{31}$$

Proof. First we have

$$\begin{aligned}\langle a, f\rangle &= \textstyle\sum\langle \int_{(1)}, \widetilde{a}\rangle\langle \int_{(2)}, S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle = \sum\langle \int, \widetilde{a}S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle \\ &= \textstyle\sum\langle \int, S^{-1}(\delta_{(1)})\nu(\widetilde{a})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle = \sum\langle \int, S^{-1}(S(\nu(\widetilde{a}))\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle \\ &= \textstyle\sum\langle \int, S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \nu(\widetilde{a})\delta_{(2)}\rangle = \sum\langle \int, S^{-1}(\delta)_{(2)}\rangle\langle \widetilde{f}, \nu(\widetilde{a})S(S^{-1}(\delta)_{(1)})\rangle \\ &= \langle \textstyle\int, S^{-1}(\delta)\rangle\langle \widetilde{f}, \nu(\widetilde{a})\rangle.\end{aligned}$$

Apply this to $\langle \int, \delta\rangle$. Then we get

$$1 = \langle \textstyle\int, \delta\rangle = \langle \int, S^{-1}(\delta)\rangle\langle \widetilde{\delta}, \nu(\widetilde{\int})\rangle = \langle \int, S^{-1}(\delta)\rangle\varepsilon\nu^{-1}\nu(1) = \langle \int, S^{-1}(\delta)\rangle.$$

Hence we get $\langle a, f\rangle = \langle \widetilde{f}, \nu(\widetilde{a})\rangle$. □

Corollary 15. *If H is unimodular then $\nu = S^2$.*

Proof. H unimodular means that δ is left and right invariant. Thus we get

$$\begin{aligned}\langle a, f\rangle &= \textstyle\sum\langle \int_{(1)}, \widetilde{a}\rangle\langle \int_{(2)}, S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle \\ &= \textstyle\sum\langle \int, \widetilde{a}S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}\rangle = \sum\langle \int, S^{-1}(\delta_{(1)}S(\widetilde{a}))\rangle\langle \widetilde{f}, \delta_{(2)}\rangle \\ &= \textstyle\sum\langle \int, S^{-1}(\delta_{(1)})\rangle\langle \widetilde{f}, \delta_{(2)}S^2(\widetilde{a})\rangle \quad \text{(since } \delta \text{ is right invariant)} \\ &= \langle \textstyle\int, S^{-1}(\delta)\rangle\langle \widetilde{f}, S^2(\widetilde{a})\rangle = \langle \widetilde{f}, S^2(\widetilde{a})\rangle.\end{aligned}$$

Hence $S^2 = \nu$. □

We also get a special representation of the inner product $H^* \otimes H \to \mathbb{K}$ by both integrals:

Corollary 16.

$$\langle a, f\rangle = \int \widetilde{a}(x) f(x) dx = \int^* S^{-1}(a)(x)\widetilde{f}(x)dx. \tag{32}$$

Proof. We have the rules for the Fourier transform. From (24) we get $\langle a, f\rangle = \langle \int, \widetilde{a}f\rangle = \int \widetilde{a}(x)f(x)dx$ and from (23) we get

$$\langle a, f\rangle = \langle S^{-1}(a)\widetilde{f}, \delta\rangle = \int^* S^{-1}(a)(x)\widetilde{f}(x)dx.$$

□

The Fourier transform leads to an interesting integral transform on H by double application.

Proposition 17. *The double transform $\breve{f} := (\delta \leftharpoonup (\int \leftharpoonup f))$ defines an automorphism $H \to H$ with*

$$\breve{f}(y) = \int f(x)\delta(xy)dx.$$

Proof. We have

$$\begin{aligned}
\langle y, \breve{f}\rangle &= \langle y, (\delta \leftharpoonup (\textstyle\int \leftharpoonup f))\rangle = \langle (\textstyle\int \leftharpoonup f)y, \delta\rangle \\
&= \textstyle\sum \langle (\int \leftharpoonup f), \delta_{(1)}\rangle\langle y, \delta_{(2)}\rangle = \sum\langle \int, f\delta_{(1)}\rangle\langle y, \delta_{(2)}\rangle \\
&= \textstyle\sum\langle \int_{(1)}, f\rangle\langle \int_{(2)}, \delta_{(1)}\rangle\langle y, \delta_{(2)}\rangle = \sum\langle \int_{(1)}, f\rangle\langle \int_{(2)} y, \delta\rangle \\
&= \textstyle\sum\langle \int_{(1)}, f\rangle\langle \int_{(2)}, (y \rightharpoonup \delta)\rangle = \sum\langle \int, f(y \rightharpoonup \delta)\rangle \\
&= \textstyle\int f(x)\delta(xy)dx
\end{aligned}$$

since $\langle x, (y \rightharpoonup \delta)\rangle = \langle xy, \delta\rangle$. □

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Mathematisches Institut der Universität München, Germany

E-mail address: `pareigis@rz.mathematik.uni-muenchen.de`