FOURIER TRANSFORMS OVER FINITE QUANTUM GROUPS

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1. Introduction

In this note we want to clarify the notion of an integral for arbitrary Hopf algebras that has been introduced a long time ago [2, 6]. The relation between the integral on a Hopf algebra and integrals in functional analysis has only been hinted at in several publications. With the strong interest in quantum groups, i.e. non-commutative and non-cocommutative Hopf algebras, we wish to show in which form certain transformation rules for integrals occur in quantum groups.

Our point of view will be the following. Let $G$ be a quantum group in the sense of non-commutative algebraic geometry, that is a space whose function algebra is given by an arbitrary Hopf algebra $H$ over some base field $K$. We will also have to use the algebra of linear functionals $H^* = \text{Hom}(H, K)$ with the multiplication induced by the diagonal of $H$ (called the bialgebra of $G$ in the French literature). For most of this paper we will assume that $H$ is finite dimensional. Observe that the functions in $H$ do not commute under multiplication and that they usually have no general commutation formula.

The model for this setup can be found in functional analysis. There the group $G$ is a locally compact group, $H$ the space of representative functions on $G$, and $H^*$ the space of generalized functions or distributions. Then the functions commute under multiplication.

We will also consider two special examples of our setup. For an arbitrary finite group $G$ the Hopf algebra $H = \mathbb{K}^G$ is defined to be the algebra of functions on $G$. Then $H^* = \mathbb{K}G$, the group algebra, is the linear dual of $H$.

If the finite group $G$ is Abelian and if $K$ is algebraically closed with $\text{char}(K) \not\equiv |G|$ then the corresponding Hopf algebra is as above $H = \mathbb{K}^G$ and $H^* = \mathbb{K}G$. By Pontryagin duality there is the group $\hat{G}$ of characters on $G$ such that $H = \mathbb{K}^G = \mathbb{K}\hat{G}$ and $H^* = \mathbb{K}G = \mathbb{K}\hat{G}$.

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2. INTEGRALS

Let $H$ be an arbitrary Hopf algebra [3, 6]. The linear functionals $a \in H^*$ will be considered as generalized integrals on $H$ ([5] p.123). We have an operation $H^* \otimes H \ni a \otimes f \mapsto \langle a, f \rangle \in \mathbb{K}$ that is nondegenerate on both sides.

We denote the elements of $H$ by $f, g, h \in H$, the elements of $H^*$ by $a, b, c \in H^*$; the (non-existing) elements of the quantum group $G$ by $x, y, z \in G$.

We will be interested in a special generalized integral $\int \in H^*$ satisfying

$$a \int = \langle a, 1_H \rangle \int$$

or $a \int = \varepsilon(a) \int$. Such an integral is called a left invariant integral.

In the case of a locally compact group $G$ such an element is given by the Haar integral with respect to a left invariant Haar measure [1]

$$\int_G f(x) \mu dx = \langle \int, f \rangle.$$ 

Therefore we write in the general quantum group situation

$$\int f(x) dx := \langle \int, f \rangle.$$ 

This notation has two parentheses, $\int$ and $dx$, so that the integrand $f$ is clearly separated. We also use the notation

$$\int f(x) g(x) dx := \langle \int, f g \rangle.$$ 

Observe that $f(x)$ and $g(x)$ are just parts of the whole symbol and in particular that they do not commute.

In the case of a finite group $G$ a left invariant integral in $H^* = \mathbb{K} G$ on $H = \mathbb{K} G$ is known to be

$$\int = \sum_{x \in G} x$$

since $y \sum_{x \in G} x = \sum_{x \in G} y x = \sum_{x \in G} x = \langle y, 1_H \rangle \sum_{x \in G} x$. For arbitrary $a \in \mathbb{K} G$ we have $a \sum_{x \in G} x = \varepsilon(a) \sum_{x \in G} x$. So our integral notation turns out to be

$$\int f(x) dx = \sum_{x \in G} f(x)$$

and has the property

$$\int f(x) dx = \sum_{x \in G} f(x) = \sum_{x \in G} f(y x) = \int f(y x) dx$$

for all $y \in G$. This left invariant integral turns out to be also right invariant $\int a = \int \varepsilon(a)$.
3. v. Neumann Transforms

We return to the arbitrary Hopf algebra $H$ of a quantum group. Since $H^* = \text{Hom}(H, \mathbb{K})$ and $S : H \to H$ is an algebra antihomomorphism, the dual $H^*$ is an $H$-module in four different ways:

\begin{equation}
\langle f \leftrightarrow a, g \rangle := \langle a, gf \rangle, \quad \langle (a \leftarrow f), g \rangle := \langle a, fg \rangle,
\langle (f \rightarrow a), g \rangle := \langle a, S(f)g \rangle, \quad \langle (a \rightarrow f), g \rangle := \langle a, gS(f) \rangle.
\end{equation}

If $H$ is finite dimensional then $H^*$ is a Hopf algebra. The equality $\langle (a \rightarrow f), g \rangle = \langle a, gf \rangle = \sum \langle a_{(1)}, g \rangle \langle a_{(2)}, f \rangle$ implies

\begin{equation}
(f \rightarrow a) = \sum a_{(1)} \langle a_{(2)}, f \rangle.
\end{equation}

Analogously we have

\begin{equation}
(a \leftarrow f) = \sum \langle a_{(1)}, f \rangle a_{(2)}.
\end{equation}

An easy observation about left invariant integrals on $H$ is

**Lemma 1.** The set $\text{Int}_l(H^*)$ of left invariant integrals is a two sided ideal in $H^*$.

The integral $\int f$ is left invariant iff $\forall y \in H^*: y \int f = \varepsilon(y) \int f$ iff $\forall y \in H^*, f \in H: \langle y \int f, f \rangle = \langle \int, (f \leftarrow y) \rangle = \varepsilon(y) \langle \int, f \rangle$. Since $\langle x, f \rangle = f(x)$ and $\langle x, (f \leftarrow y) \rangle = \langle yx, f \rangle = f(yx)$, the integral $\int f$ is left invariant iff

\begin{equation}
\int f(yx)dx = \varepsilon(y) \int f(x)dx.
\end{equation}

**Theorem 2.** If there exists $0 \neq \int \in \text{Int}_l(H^*)$ then the map $H \ni f \mapsto (f \leftarrow f) \in H^*$ is injective.

**Proof.** By [6] theorem 5.1.3 the following homomorphism $\text{Int}_l(H^*) \otimes H \ni \int \otimes f \mapsto (f \leftarrow f) \in H^{\text{cov}}(\subseteq H^*)$ is bijective. \hfill \Box

**Corollary 3.** If there exists $0 \neq \int \in \text{Int}_l(H^*)$ then the antipode $S : H \to H$ is injective.

**Proof.** The monomorphism $H \ni f \mapsto (f \leftarrow f) \in H^*$ is composed of $S : H \to H$ and $H \ni f \mapsto (f \rightarrow \int) \in H^*$. \hfill \Box

We call a generalized integral $a \in H^*$ a rational integral if $a$ is of the form $a = \sum \int_i \leftrightarrow f_i$.

**Corollary 4.** For every rational integral $a \in H^*$ there is a unique $g \in H$ such that

\begin{equation}
\langle a, f \rangle = \int f(x)S(g)(x)dx
\end{equation}

for all $f \in H$.

**Proof.** For every rational integral $a$ there is a unique function $g \in H$ with $a = (\int \leftarrow g)$, hence $\langle a, f \rangle = \langle f \leftarrow g, f \rangle = \langle S(g) \rightarrow f, f \rangle = \langle f, fS(g) \rangle = \int f(x)S(g)(x)dx$. \hfill \Box
One of the first to study this property of the integral $\int$ to represent other linear functionals was J. v. Neumann in [4].

If $H$ is finite dimensional then the isomorphism $\text{Int}_i(H^*) \otimes H \ni \int \otimes f \mapsto (\int \leftarrow f) \in H^{**}(\subseteq H^*)$ shows $\text{Int}_i(H^*)$ has dimension $1$.

We choose for the rest of this paper a non-zero left invariant integral $f$ whenever we are in the situation of $H$ finite dimensional.

Let $H$ be finite dimensional. Since $\int a$ is a left invariant integral and $\dim(\text{Int}_i(H^*)) = 1$ there is a unique $\text{mod}(a) \in \mathbb{K}$ with

$$\int a = \text{mod}(a)\int.$$  

One checks that $\text{mod} : H^* \to \mathbb{K}$ is an algebra homomorphism called the \textit{modulus of $H^*$}. If $\text{mod} = \varepsilon = 1_{H^*}$ then $H^*$ is called \textit{unimodular}. This is equivalent to $\int$ also being right invariant or $\text{Int}_r(H^*) = \text{Int}_r(H^*)$.

**Corollary 5.** If $H$ is finite dimensional then for every $a \in H^*$ there is a unique $g \in H$ such that $\langle a, f \rangle = \int f(x)S(g)(x)dx$ for all $f \in H$.

**Corollary 6.** If $H$ is finite dimensional then $S : H \to H$ and $H \ni f \mapsto (f \mapsto f) \in H^*$ are bijective.

If $G$ is a finite group then every generalized integral $a \in \mathbb{K}^G$ can be written with a uniquely determined $g \in H$ as

$$(11) \quad \langle a, f \rangle = \int f(x)S(g)(x)dx = \sum_{x \in G} f(x)g(x^{-1})$$

for all $f \in H$.

If $G$ is a finite Abelian group then each group element (rational integral) $y \in G \subseteq \mathbb{K}^G$ can be written as

$$y = \sum_{x \in G} \sum_{\chi \in \widehat{G}} \beta_{\chi} \langle x^{-1}, \chi \rangle x$$

since $\langle y, f \rangle = \langle \int - \sum_{x \in G} \beta_{\chi} x \rangle, f \rangle = \langle \int, fS(\sum_{x \in G} \beta_{\chi} \chi) \rangle = \sum_{x \in G} \langle x, f \rangle \sum_{\chi \in \widehat{G}} \beta_{\chi} \langle x^{-1}, \chi \rangle x, f \rangle$. In particular the matrix $(\langle x^{-1}, \chi \rangle)$ is invertible.

4. The Nakayama Automorphism

Let $H$ be finite dimensional. Since $\langle f, g \rangle = \langle (f \mapsto f), g \rangle$ as a functional on $g$ is a generalized integral, there is a unique $\nu(f) \in H$ such that

$$(12) \quad \langle f, g \rangle = \langle \int, g\nu(f) \rangle$$

or

$$(13) \quad \int f(x)g(x)dx = \int g(x)\nu(f)(x)dx.$$
Although the functions \( f, g \in H \) of the quantum group do not commute under multiplication, there is a simple commutation rule if the product is integrated.

**Proposition 7.** The map \( \nu : H \to H \) is an algebra automorphism, called the Nakayama automorphism.

**Proof.** It is clear that \( \nu \) is a linear map. We have \( \int f \nu(gh) = \int ghf = \int hf \nu(g) = \int f \nu(g) \nu(h) \) hence \( \nu(gh) = \nu(g) \nu(h) \) and \( \int f \nu(1) = \int f \) hence \( \nu(1) = 1 \). Furthermore if \( \nu(g) = 0 \) then 0 = \( \langle \int f \nu(g), \rangle = \langle \int, g \rangle = \langle (f \rightarrow \int), g \rangle \) for all \( f \in H \) hence \( \langle a, g \rangle = 0 \) for all \( a \in H^* \) hence \( g = 0 \). So \( \nu \) is injective hence bijective. \( \square \)

**Corollary 8.** The map \( H \ni f \mapsto (\int \sim f) \in H^* \) is an isomorphism.

**Proof.** We have

\[
(\int \sim f) = (\nu(f) \rightarrow \int)
\]

since \( \langle (\int \sim f), g \rangle = \langle \int, g \rangle = \langle \int, \nu(f) \rangle = \langle \nu(f) \rightarrow \int, g \rangle \). This implies the corollary. \( \square \)

If \( G \) is a finite group and \( H = \mathbb{K}^G \) then \( H \) is commutative hence \( \nu = \text{id} \).

5. The Dirac Delta Function

An element \( \delta \in H \) is called a Dirac \( \delta \)-function if \( \delta \) is a left invariant integral in \( H \) with \( \langle \int, \delta \rangle = 1 \), i.e. if \( \delta \) satisfies

\[
f \delta = \varepsilon(f) \delta \quad \text{and} \quad \int \delta(x) dx = 1
\]

for all \( f \in H \). If \( H \) has a Dirac \( \delta \)-function then we write

\[
\int^* a(x) dx = \int^* a := \langle a, \delta \rangle.
\]

**Proposition 9.**

1. If \( H \) is finite dimensional then there exists a unique Dirac \( \delta \)-function \( \delta \).
2. If \( H \) is infinite dimensional then there exists no Dirac \( \delta \)-function.

**Proof.** 1. Since \( H \ni f \mapsto (\int \sim f) \in H^* \) is an isomorphism there is a \( \delta \in H \) such that \( (\delta \rightarrow \int) = \varepsilon \). Then \( (f \delta \rightarrow \int) = (f \rightarrow (\delta \rightarrow \int)) = (f \rightarrow \varepsilon) = \varepsilon(f) \varepsilon = \varepsilon(f)(\delta \rightarrow \int) \) which implies \( f \delta = \varepsilon(f) \delta \). Furthermore we have \( \langle \int, \delta \rangle = \langle \int, 1_H \delta \rangle = \langle (\delta \rightarrow \int), 1_H \rangle = \varepsilon(1_H \delta) = 1_H \).

2. is [6] exercise V.4. \( \square \)

**Lemma 10.** Let \( H \) be a finite dimensional Hopf algebra. Then \( \int \in H^* \) is a left integral iff

\[
a(\sum \int_{(1)} \otimes S(\int_{(2)})) = (\sum \int_{(1)} \otimes S(\int_{(2)}))a
\]
iff 

(16) \[ \sum S(a) f_{(1)} \otimes f_{(2)} = \sum f_{(1)} \otimes a f_{(2)} \]

iff 

(17) \[ \sum f_{(1)}(f, f_{(2)}) = \langle f, f \rangle_{1_H}. \]

Proof. Let \( f \) be a left integral. Then 

\[ \sum a(1) f_{(1)} \otimes S(f_{(2)}) S(a_{(2)}) = \sum (a f_{(1)} \otimes S(a f_{(2)})) = \varepsilon(a)(\sum f_{(1)} \otimes S(f_{(2)})) \]

for all \( a \in H \). Hence 

\[ (\sum f_{(1)} \otimes S(f_{(2)})) a = \sum \varepsilon(a_{(1)})(f_{(1)} \otimes S(f_{(2)})) a_{(2)} \]

\[ = \sum a(1) f_{(1)} \otimes S(f_{(2)}) S(a_{(2)}) a_{(3)} \]

\[ = \sum a(1) f_{(1)} \otimes S(f_{(2)}) \varepsilon(a_{(2)}) = a(\sum f_{(1)} \otimes S(f_{(2)})). \]

Conversely \( a(\sum f_{(1)} \varepsilon(S(f_{(2)}))) = (\sum f_{(1)} \varepsilon(S(f_{(2)}))) = \varepsilon(a)(\sum f_{(1)} \varepsilon(S(f_{(2)}))), \)

hence \( f = \sum f_{(1)} \varepsilon(S(f_{(2)})) \) is a left integral.

Since \( S \) is bijective the following holds 

\[ \sum S(a) f_{(1)} \otimes f_{(2)} = \sum S(a) f_{(1)} \otimes S^{-1}(S(f_{(2)})) \]

\[ = \sum f_{(1)} \otimes S^{-1}(S(f_{(2)})) S(a) = \sum f_{(1)} \otimes a f_{(2)}. \]

The converse follows easily.

If \( f \in \text{Int}_{l}(H) \) is a left integral then \( \sum \langle a, f_{(1)} \rangle \langle f, f_{(2)} \rangle = \langle a f, f \rangle = \langle a, 1_{H} \rangle \langle f, f \rangle. \)

Conversely if \( \lambda \in H^{*} \) with (17) is given then \( \langle a \lambda, f \rangle = \sum \langle a, f_{(1)} \rangle \langle \lambda, f_{(2)} \rangle = \langle a, 1_{H} \rangle \langle \lambda, f \rangle \) hence \( a \lambda = \varepsilon(a) \lambda. \) \( \square \)

If \( G \) is a finite group then 

(18) \[ \delta(x) = \begin{cases} 0 & \text{if } x \neq e; \\ 1 & \text{if } x = e. \end{cases} \]

In fact since \( \delta \) is left invariant we get \( f(x) \delta(x) = f(e) \delta(x) \) for all \( x \in G \) and \( f \in \mathbb{K}^{G}. \)

Since \( G \subset H^{*} = \mathbb{K}^{G} \) is a basis, we get \( \delta(x) = 0 \) if \( x \neq e. \) Furthermore \( \int \delta(x) dx = \sum_{x \in G} \delta(x) = 1 \) implies \( f(e) = 1. \)

If \( G \) is a finite Abelian group we get \( \delta = \alpha \sum_{\chi \in \hat{G}} \chi \) for some \( \alpha \in \mathbb{K}. \) The evaluation gives \( 1 = \alpha \langle f, \delta \rangle = \alpha \sum_{x \in G} \langle \chi, x \rangle \). Now let \( \lambda \in \hat{G}. \) Then \( \sum_{\chi \in \hat{G}} \langle \lambda, x \rangle = \sum_{\chi \in \hat{G}} \langle x, \lambda \rangle. \) Since for each \( x \in G \setminus \{e\} \) there is a \( \lambda \) such that \( \langle \lambda, x \rangle \neq 1 \) and we get 

\[ \sum_{\chi \in \hat{G}} \langle \chi, x \rangle = |G| \delta_{e,x}. \]
Hence $\sum_{x \in G, \chi \in \hat{G}} \langle \chi, x \rangle = |G| = \alpha^{-1}$ and

$$\delta = |G|^{-1} \sum_{\chi \in \hat{G}} \chi.$$  

6. Fourier Transforms

Let $H$ be finite dimensional for the rest of this paper. In Corollary 8 we have seen that the map $H \ni f \mapsto (\hat{f} \leftarrow f) \in H^*$ is an isomorphism. This map will be called the Fourier transform.

**Theorem 11.** The Fourier transform $H \ni f \mapsto \hat{f} \in H^*$ is bijective with

$$\hat{f} = (f \leftarrow f) = \sum \langle f_{(1)}, f \rangle f_{(2)}$$

The inverse Fourier transform is defined by

$$\hat{a} = \sum S^{-1}(\delta_{(1)}) \langle a, \delta_{(2)} \rangle.$$  

Since these maps are inverses of each other the following formulas hold

$$\langle \hat{f}, g \rangle = \int f(x)g(x)dx \quad \langle a, \hat{b} \rangle = \int S^{-1}(a)(x)b(x)dx$$

$$f = \sum S^{-1}(\delta_{(1)}) \langle \hat{f}, \delta_{(2)} \rangle \quad a = \sum \langle f_{(1)}, \hat{a} \rangle f_{(2)}.$$  

**Proof.** We use the isomorphisms $H \to H^*$ defined by $\hat{f} := \hat{f} = (f \leftarrow f) = \sum \langle f_{(1)}, f \rangle f_{(2)}$ and $H^* \to H$ defined by $\hat{a} := (a \to \delta) = \sum \delta_{(1)} \langle a, \delta_{(2)} \rangle$. Because of

$$\langle a, \hat{b} \rangle = \langle a, (b \to \delta) \rangle = \langle ab, \delta \rangle$$  

and

$$\langle \hat{f}, g \rangle = \langle (f \leftarrow f), g \rangle = \langle \hat{f}, fg \rangle$$

we get for all $a \in H^*$ and $f \in H$

$$\langle a, \hat{f} \rangle = \langle a \hat{f}, \delta \rangle = \sum \langle a, \delta_{(1)} \rangle \langle \hat{f}, \delta_{(2)} \rangle = \sum \langle a, \delta_{(1)} \rangle \langle \hat{f}, f \delta_{(2)} \rangle \quad (\text{by Lemma 10})$$

$$= \sum \langle a, S(f) \delta_{(1)} \rangle \langle \hat{f}, \delta_{(2)} \rangle = \langle a, S(f) \rangle \langle \hat{f}, \delta \rangle = \langle a, S(f) \rangle.$$  

This gives $\hat{f} = S(f)$. So the inverse map of $H \to H^*$ with $\hat{f} = (f \leftarrow f) = \hat{f}$ is $H^* \to H$ with $S^{-1}(\hat{a}) = \sum S^{-1}(\delta_{(1)}) \langle a, \delta_{(2)} \rangle = \hat{a}$. Then the given inversion formulas are clear.  

If $G$ is a finite group and $H = \mathbb{K}^G$ then

$$\hat{f} = \sum_{x \in G} f(x) x.$$
Since $\Delta(\delta) = \sum_{x \in G} x^{-1} \otimes x^*$ where the $x^* \in K^G$ are the dual basis to the $x \in G$, we get

$$\tilde{a} = \sum_{x \in G} \langle a, x^* \rangle x^*.$$  

If $G$ is a finite Abelian group then the groups $G$ and $\hat{G}$ are isomorphic so the Fourier transform induces a linear automorphism $\sim: K^G \to K^G$ and we have

$$\tilde{a} = |G|^{-1} \sum_{\chi \in \hat{G}} \langle a, \chi \rangle \chi^{-1}.$$  

By substituting the formulas for the integral and the Dirac $\delta$-function (4) and (19) we get

$$\begin{align*}
\tilde{f} &= \sum_{x \in G} f(x) x, \\
f &= |G|^{-1} \sum_{\chi \in \hat{G}} \tilde{f}(\chi) \chi^{-1}, \\
a &= \sum_{x \in G} a(x) x.
\end{align*}$$  

This implies

$$\begin{align*}
\tilde{f}(\chi) &= \sum_{x \in G} f(x) \chi(x) = \int f(x) \chi(x) \, dx \\
\text{with inverse transform} \\
\tilde{a}(x) &= |G|^{-1} \sum_{\chi \in \hat{G}} \chi(a) \chi^{-1}(x).
\end{align*}$$  

**Lemma 12.** The Fourier transforms of the left invariant integrals in $H$ and $H^*$ are

$$\begin{align*}
\tilde{\delta} &= \varepsilon \nu^{-1} \in H^* \\
\tilde{1} &= 1 \in H.
\end{align*}$$  

**Proof.** We have $\langle \tilde{\delta}, f \rangle = \langle f, \delta f \rangle = \langle f, \nu^{-1}(f) \delta \rangle = \varepsilon \nu^{-1}(f) \langle f, \delta \rangle = \varepsilon \nu^{-1}(f)$ hence $\tilde{\delta} = \varepsilon \nu^{-1}$ and $\langle a, \tilde{f} \rangle = \sum \langle a, S^{-1}(\delta_{(1)}) \rangle \langle f, \delta \rangle = \langle a, S^{-1}(1) \rangle \langle f, \delta \rangle = \langle a, 1 \rangle$, hence $\tilde{f} = 1$.  

**Proposition 13.** Define a convolution multiplication on $H^*$ by

$$\langle a \ast b, f \rangle := \sum \langle a, S^{-1}(\delta_{(1)}) f \rangle \langle b, \delta_{(2)} \rangle.$$  

Then the following transformation rule holds for $f, g \in H$:

$$\begin{align*}
\tilde{f} g &= \tilde{f} \ast \tilde{g}.
\end{align*}$$  

In particular $H^*$ with the convolution multiplication is an associative algebra with unit $1_H = \tilde{f}$, i.e.

$$\begin{align*}
\tilde{f} \ast a &= a \ast \tilde{f} = a.
\end{align*}$$
Proof. Given \( f, g, h \in H^* \). Then
\[
\langle fg, h \rangle = \langle f, gfh \rangle = \langle f, f S^{-1}(1_H)gh \rangle \langle f, \delta \rangle
\]
\[
= \sum \langle f, f S^{-1}(\delta(1))gh \rangle \langle f, \delta(2) \rangle = \sum \langle f, f S^{-1}(\delta(1))h \rangle \langle f, g\delta(2) \rangle
\]
\[
= \sum \langle f, S^{-1}(\delta(1))h \rangle \langle \delta(1), \delta(2) \rangle = \langle \hat{f} * \tilde{g}, h \rangle.
\]
From (28) we get \( \hat{1}_H = f \). So we have \( \hat{f} = \hat{1} * \hat{f} = \hat{1} * f = \hat{f} \). \( \square \)

If \( G \) is a finite Abelian group and \( a, b \in H^* = \mathbb{K}^\mathbb{G} \). Then
\[
(a * b)(\mu) = |G|^{-1} \sum_{\chi \lambda \in G, \lambda = \mu} a(\chi)b(\chi).
\]
In fact we have
\[
(a * b)(\mu) = \langle a * b, \mu \rangle = \sum_{\chi \lambda \in G} \langle a, S^{-1}(\delta(1))\mu \rangle \langle b, \delta(2) \rangle
\]
\[
= |G|^{-1} \sum_{\chi \lambda \in G} \langle a, \chi \mu \rangle \langle \chi, \lambda \rangle = |G|^{-1} \sum_{\chi \lambda \in \mathbb{G}, \lambda = \mu} a(\chi)b(\chi).
\]

7. The Plancherel Formula

One of the most important formulas for Fourier transforms is the Plancherel formula on the invariance of the inner product under Fourier transforms. We have

**Theorem 14.** (The Plancherel formula)
\[
\langle a, f \rangle = \langle \hat{f}, \nu(\tilde{a}) \rangle.
\]

**Proof.** First we have
\[
\langle a, f \rangle = \sum \langle f(1), \tilde{a} \rangle \langle f(2), S^{-1}(\delta(1)) \rangle \langle \tilde{f}, \delta(2) \rangle = \sum \langle f, \tilde{a} S^{-1}(\delta(1)) \rangle \langle \tilde{f}, \delta(2) \rangle
\]
\[
= \sum \langle f, S^{-1}(\delta(1)) \nu(\tilde{a}) \rangle \langle f, \delta(2) \rangle = \sum \langle f, S^{-1}(\nu(\tilde{a}) \delta(1)) \rangle \langle \tilde{f}, \delta(2) \rangle
\]
\[
= \sum \langle f, S^{-1}(\delta(1)) \rangle \langle \nu(\tilde{a}), \delta(2) \rangle = \sum \langle f, S^{-1}(\delta(1)) \rangle \langle \nu(\tilde{a}), S(\nu(\tilde{a}) S^{-1}(\delta(1))) \rangle
\]
\[
= \langle f, S^{-1}(\delta) \rangle \langle \hat{f}, \nu(\tilde{a}) \rangle.
\]
Apply this to \( \langle f, \delta \rangle \). Then we get
\[
1 = \langle f, \delta \rangle = \langle f, S^{-1}(\delta) \rangle \langle \tilde{f}, \nu(\tilde{f}) \rangle = \langle f, S^{-1}(\delta) \rangle \nu(1) = \langle f, S^{-1}(\delta) \rangle.
\]
Hence we get \( \langle a, f \rangle = \langle \hat{f}, \nu(\tilde{a}) \rangle \). \( \square \)

**Corollary 15.** If \( H \) is unimodular then \( \nu = S^2 \).

**Proof.** \( H \) unimodular means that \( \delta \) is left and right invariant. Thus we get
\[
\langle a, f \rangle = \sum \langle f(1), \tilde{a} \rangle \langle f(2), S^{-1}(\delta(1)) \rangle \langle \tilde{f}, \delta(2) \rangle
\]
\[
= \sum \langle f, \tilde{a} S^{-1}(\delta(1)) \rangle \langle \tilde{f}, \delta(2) \rangle = \sum \langle f, S^{-1}(\delta(1)) S(\tilde{a}) \rangle \langle \tilde{f}, \delta(2) \rangle
\]
\[
= \sum \langle f, S^{-1}(\delta(1)) \rangle \langle \tilde{f}, S^2(\tilde{a}) \rangle \quad \text{(since } \delta \text{ is right invariant)}
\]
\[
= \langle f, S^{-1}(\delta) \rangle \langle \tilde{f}, S^2(\tilde{a}) \rangle = \langle f, S^2(\tilde{a}) \rangle.
\]
Hence \( S^2 = \nu \). \( \square \)
We also get a special representation of the inner product \( H^* \otimes H \to \mathbb{K} \) by both integrals:

**Corollary 16.**

\[
\langle a, f \rangle = \int \tilde{a}(x)f(x)dx = \int^* S^{-1}(a)(x) \tilde{f}(x)dx.
\]

**Proof.** We have the rules for the Fourier transform. From (24) we get \( \langle a, f \rangle = \int \tilde{a}(x)f(x)dx \) and from (23) we get

\[
\langle a, f \rangle = \langle S^{-1}(a)\tilde{f}, \delta \rangle = \int^* S^{-1}(a)(x) \tilde{f}(x)dx.
\]

\[\square\]

The Fourier transform leads to an interesting integral transform on \( H \) by double application.

**Proposition 17.** The double transform \( \tilde{\tilde{f}} := (\delta \leftarrow (\int \leftarrow f)) \) defines an automorphism \( H \to H \) with

\[
\tilde{\tilde{f}}(y) = \int f(x)\delta(xy)dx.
\]

**Proof.** We have

\[
\langle y, \tilde{\tilde{f}} \rangle = \langle y, (\delta \leftarrow (\int \leftarrow f)) \rangle = \langle (\int \leftarrow f)y, \delta \rangle
\]

\[
= \sum \langle (\int \leftarrow f), \delta_1 \rangle \langle y, \delta_2 \rangle = \sum \langle \tilde{f}, f\delta_1 \rangle \langle y, \delta_2 \rangle
\]

\[
= \sum \langle \tilde{f}_1, \tilde{f} \rangle \langle f_2, \delta_1 \rangle \langle y, \delta_2 \rangle = \sum \langle \tilde{f}_1, \tilde{f} \rangle \langle f_2, y, \delta \rangle
\]

\[
= \sum \langle \tilde{f}_1, \tilde{f} \rangle \langle f_2, (y \to \delta) \rangle = \sum \langle \tilde{f}, f(y \to \delta) \rangle
\]

\[
= \int f(x)\delta(xy)dx
\]

since \( \langle x, (y \to \delta) \rangle = \langle xy, \delta \rangle \).  

\[\square\]

**References**


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