The Unruh effect without observer

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1 Introduction

In 1974 Paul C.W. Davies published the paper "Scalar production in Schwarzschild and Rindler metrics" [6], in which he argued that a uniformly accelerated observer would see a thermal distribution of particles for the temperature $T = \frac{a}{2\pi}$, where a is the magnitude of acceleration. He was the first to put forth quantum field theoretic arguments to conclude that a uniformly accelerated observer in flat spacetime would see thermal radiation. His work was heavily motivated by the results of Hawking on the thermal radiation of black holes, although not yet published at that time.
In 1976 William G. Unruh wrote his famous paper "Notes on black-hole evaporation" [18], where he discussed a model of a black hole to investigate Hawking radiation. Having in mind the idea of a detector fixed at constant spacial distance to a black hole, he discusses a particle detector in flat spacetime, which is constantly accelerated. The situation is said to be similar to a detector fixed next to a black hole, since in the mindset of general relativity local inertial motion corresponds to geodesic motion. Therefore, a detector at fixed distance to a black hole experiences acceleration due to its very special wordline. Its acceleration is uniform in time, since the detectors deviation from geodesic motion does not change with time. Hence, the situation is static, where as the uniformly accelerated detector in Minkowski spacetime also experiences uniform acceleration.
In his discussion he draws the conclusion that a uniformly accelerated detector will get excited at a rate which is proportional to the Bose-Einstein distribution $\frac{1}{e^{\beta \Delta E}-1}$, where the temperature is given by $T = \frac{1}{\beta} = \frac{a}{2\pi}$ and ΔE is the energy difference of the detectors eigenstates.
Unruh's work has the advantage of conceptual clarity, because he speaks of very precise detector models, which get excited due to their motion and interaction with a scalar quantum field, rather than the fuzzy notion of observers that are somehow attached to coordinate frames.
Since then a myriad of physicists and mathematicians have been investigating the subject and we can only give an overview of a tiny part of the literature available. The term Unruh effect has been used synonymously to refer to either quantum field theoretic arguments that conclude that the vacuum as described by an inertial observer looks like a thermal state if described by an accelerated observer, or the fact that accelerated detectors coupling to the vacuum of a scalar quantum field click at thermal rates.
For example Padmanabhan[13] discussed slightly more general types of motion, for which he also presented quantisation schemes similar to the one in the uniformly accelerated case. He also introduces a particle detector analogously to the pointlike two level detector, called Unruh DeWitt detector, which is abundantly used in the description of the Unruh effect. The scheme of second quantisation can be used to define what a particle is, by stating that all excitations of the field around the vacuum are called particle. One can also use the detector to define a particle to

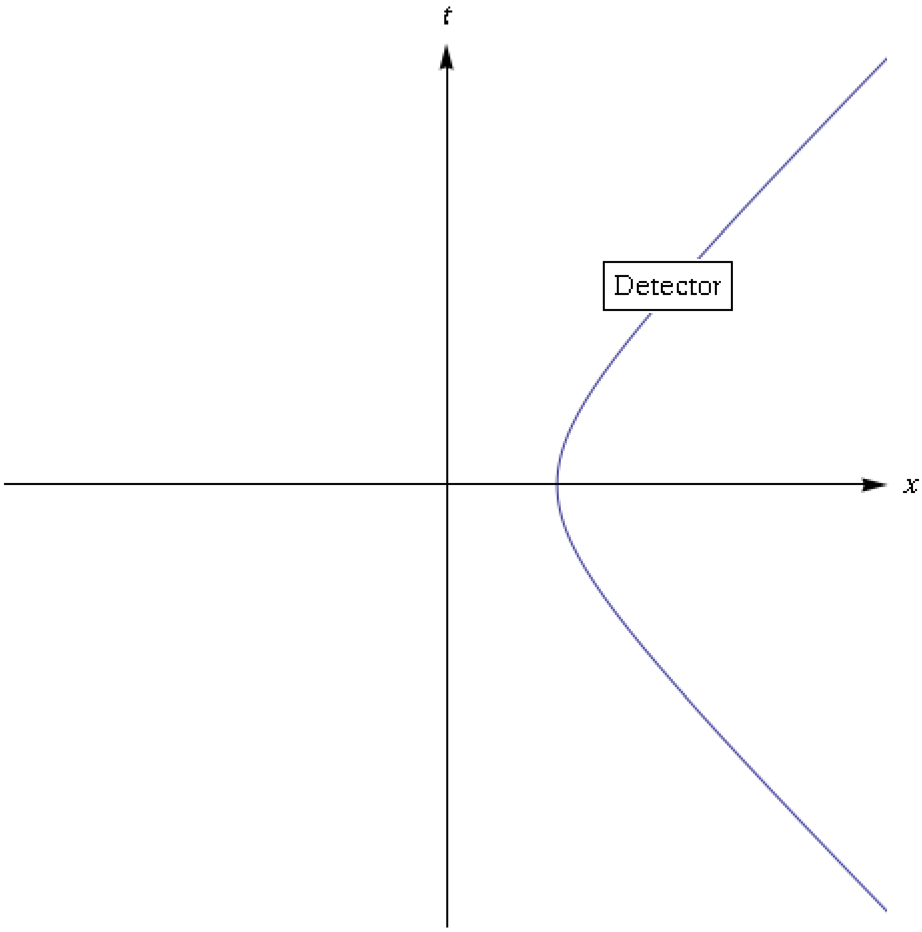

Figure 1: Uniformly accelerated detector

be defined by whatever makes the detector click. The two notions of particles do not agree for certain types of motion of the detector, which is why he arrives at the conclusion that at least one of them can not be trusted to describe the physical situation.

To clarify which of the two notions we should be using, he then presses on to improve the understanding of the modeled detector in his paper "Why does an accelerated detector click?"[14]. Instead of imposing the trajectory and the time evolution of the properties of the detector to be of a certain kind, he models the detector as a simple quantum mechanical system which is subjected to the action of an external potential. This treatment has the advantage that it is perfectly obvious why the detector switches to an excited state in the first place and where the energy for this process comes from. Before this work it has been claimed several times that the accelerating agent provides the energy to the detector to switch states and thereby causes the detection process, but it has never been demonstrated so clearly before.

In 1982 J. Bell and J. Leinaas [1] introduced a new setting to look for thermal aspects of accelerated systems. They looked at electrons in storage rings following circular trajectories. As they pointed out, the Unruh effect for linearly accelerated detectors is far to small to be detected directly, but if one uses spin flip statistics

of electrons in such a storage ring the expected effect is much larger. However the resulting statistics is complicated. One can define an effective temperature but the distribution is far from being completely characterized by this number alone.
S. Takagi published the review article "Vacuum noise and stress induced by uniform acceleration Hawking Unruh effect in Rindler manifold of arbitrary dimension"[17] in 1986. He generalized Unruh's discussion to universes which have D spacial dimensions and one time dimension. He also showed that the standard Unruh DeWitt detector will respond as if it was immersed in a thermally distributed bosons for D odd and as if it was immersed in a thermally distributed fermions if D is even. This holds even though one of the standard ways to discuss the Unruh effect, which we will employ in section 2.1.1, does not encounter a Fermi-distribution for any dimension. He also shows that one can arrive at the Plack-formula for the Unruh-DeWitt detectors response to the accelerated motion without using the second quantisation procedure in the Rindler frame at all. Furthermore, he discusses a detector immersed in a Dirac and an electromagnetic field instead of the usual Klein-Gordon field and many more generalizations of the usual treatment of the Unruh-effect.
In 2012 S. Lyle [11] discusses the general role acceleration has in the theories of special and general relativity. He argues that there are general principles built into the theory on how to deal with accelerated motion in order to extract information from the theories, namely the strong and weak equivalence principles. The weak equivalence principle is built into general relativity such that:
"In the usual formulation of general relativity, for any selected event, one can always find a locally inertial frame in which the metric takes the Minkowski form at that event and changes only very slowly as one moves away from that event."
Whereas the strong equivalence principle says:
"Our theories of non-gravitational physical phenomena are then shipped into the curved spacetime context by saying that they must look roughly as they do in flat spacetime when expressed relative to such locally inertial frames"
However, neither in special nor in general relativity there is a principle that tells us how to interpret non-inertial frames, which are not even locally inertial. He therefore concludes that any calculation depending on the use of the accelerated frame ought to be omitted.
The first aim of this thesis is to clear up some of the confusion that still haunts current literature, such as the article at scholarpedia.org on the Unruh effect by S. A. Fulling and G. E. A. Matsas [7] in 2014. This article starts with the sentence "The Unruh effect is a surprising prediction of quantum field theory: From the point of view of an accelerating observer or detector, empty space contains a gas of particles at a temperature proportional to the acceleration", hopefully after reading this thesis it will be clear to the reader why this way of thinking is very problematic.
The second goal is to shed light on parallels between the Unruh-effect and classical electromagnetism that to my knowledge has not received the appropriate attention yet.
I hope to achieve the first objective by giving a detailed description of what is

typically referred to as the Unruh effect in section 2, discuss conceptual difficulties that arise with this definition in chapter 3, and finally introduce the reader to the progress that has been made so far in relieving these difficulties. I hope to reach the second aim by drawing parallels to classical electromagnetism throughout sections 3 to 5.

First, in section 2.1.1, I will formulate what some call the natural description of spacetime by a uniformly accelerated observer, a second quantisation of the quantum field with respect to a different parametrisation of Minkowski spacetime, and try to find the relation of this picture to the one obtained by second quantisation in an arbitrary inertial frame of reference.

Afterwards, in section 2.1.2, I give a summary of the treatments, how to calculate the detector response to uniformly accelerated motion using only quantities defined in the inertial frame of reference.

In section 2.2 I talk about detectors subjected to circular motion as opposed to linear acceleration and electrons in a storage ring. From the point of view of experiments, this setting is much more interesting since the effect turns out to be much larger than the usual Unruh effect.

In chapter 3 problems of the treatment so far, such as quantisation in non-inertial frames, are discussed. Also the role of the accelerating agent, energy conservation and the interpretation of physical quantities in accelerated frames will play an important part in concluding that one ought to try to do better.

In chapter 4 I introduce a much less technical view of the situation that is also conceptually much more clear cut. The detector is modelled by a simple harmonic oscillator that couples to the relativistic quantum field. I try to derive the behaviour of this detector and identify the Unruh effect in this simpler setting without the conceptual issues that haunt earlier treatments.

2 Standard Approach

In this chapter we will follow the presentation by Takagi [17], before we criticise it in section 3. In section 2.1.1 we will first take a closer look at the Klein-Gordon equation, then we will construct a second quantised field for two different ways of parametrising Minkowski spacetime. We will then try to find a relation connecting the two vacua associated with the two Fock space constructions in order to compare the two descriptions of the same field. This strategy is inspired by the intuition that what happens to an accelerated observer ought to be most easily understood by using non-inertial coordinate frames.
In section 2.1.2 we will move on to a more pragmatic point of view. It will no longer be important for our analysis what observers might or might not consider a natural way to describe their perceptions, but we will discover whether or not a detector of a given dynamical description will click or not if it is forced to on a uniformly accelerating trajectory.

2.1 Linear acceleration

2.1.1 Comparing vacua

The idea in this subsection is not, as it is in Quantum Electrodynamics, that the different quantum field representations belong to distinct Fock spaces because the Fock spaces are build on top of the field generated by charges of different states of motion. In Quantum Electrodynamics this is a crucial point, because the Coulomb field of a stationary charge cannot be transformed into the Coulomb field of a uniformly moving charge by means that would allow a common description in the same Fock space. The idea here is that the field present, through which the detector is moving, really is the vacuum corresponding to the standard Fock space obtained by second quantisation with respect to an inertial coordinate frame. However, since from General Relativity we have this idea that there are coordinate frames fitted to non-inertially moving observers, we are tempted to describe the situation of a uniformly accelerated observer with coordinates that are adapted to their motion. We are also tempted to describe the quantum field by the same means and for this motion we are even lucky enough that the coordinates usually associated with uniform acceleration, the Rindler coordinates, even allow the standard quantisation procedure to work properly.
If we assume that this description of the quantum field is the natural one for a uniformly accelerated observer, we need this description to be equivalent to the description by inertial means. If the two vacua corresponding to these two descriptions

turn out to be different, the two descriptions will disagree on what to call radiation, which would make it very difficult to talk about the actual physics happening in such a setting.
As it turns out, the situation for our uniformly accelerated observer is even more awkward.

Coordinates

In order to speak about different frames, we need to introduce three sets of coordinates on Minkowski spacetime. The first set consists of the standard Minkowski coordinates with the standard line element.

$$\begin{aligned} \{x^\alpha\}_\alpha &= \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \qquad \alpha = 0,1,2,3 \\ \mathrm{d}s^2 &= \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \end{aligned} \tag{1}$$

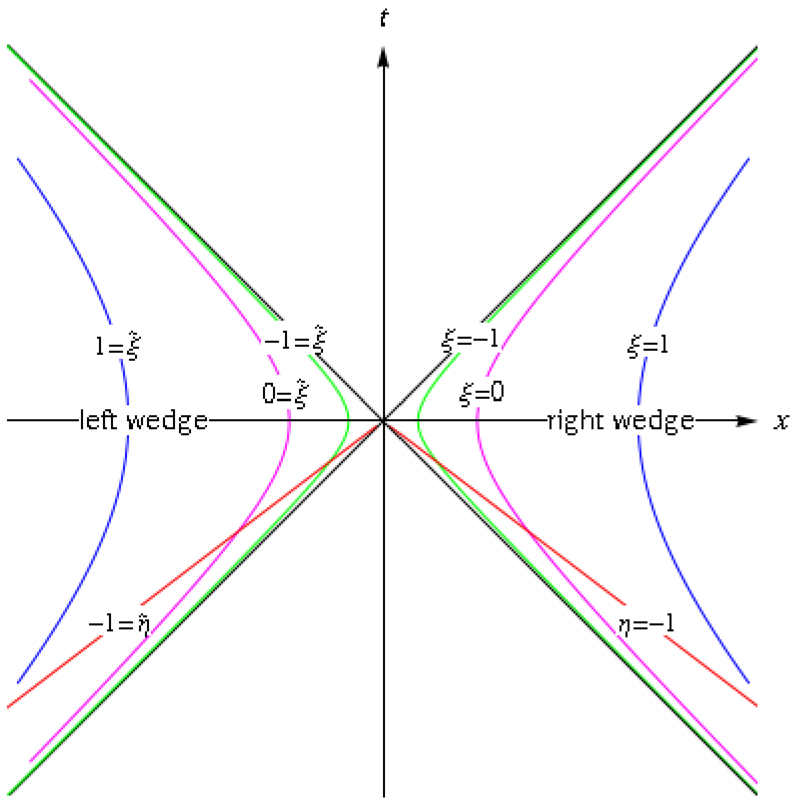

Figure 2: Left and right Rindler wedge. Lines of constant ξ, resp.$\tilde{\xi}$ in blue, pink and green. Lines of constant η, resp. $\tilde{\eta}$ in orange.

The other two sets of coordinates are chosen such that uniformly accelerated motion and the line element looks very simple if expressed by them. They have to be defined for the left and right Rindler wedge separately (see figure 2)and are essentially what is usually called Rindler coordinates and will be called Rindler coordinates henceforth:

$$\begin{aligned} x^0 &=: \mathfrak{f} e^{\xi} \sinh(\eta), \qquad \mathfrak{f} \in \mathbb{R}^+ \\ x^1 &=: \mathfrak{f} e^{\xi} \cosh(\eta) \\ \mathrm{d}s^2 &= \mathfrak{f}^2 e^{2\xi}(\mathrm{d}\eta^2 - \mathrm{d}\xi^2) - \mathrm{d}y^2 - \mathrm{d}z^2 \end{aligned} \tag{2}$$

$$\begin{aligned} x^0 &=: \mathfrak{f} e^{\tilde{\xi}} \sinh(\tilde{\eta}) \\ x^1 &=: -\mathfrak{f} e^{\tilde{\xi}} \cosh(\tilde{\eta}) \\ \mathrm{d}s^2 &= \mathfrak{f}^2 e^{2\tilde{\xi}}(\mathrm{d}\tilde{\eta}^2 - \mathrm{d}\tilde{\xi}^2) - \mathrm{d}y^2 - \mathrm{d}z^2 \end{aligned} \tag{3}$$

Where $\mathfrak{f}$(pronounced just as f) is some positive number. y and z are the same as for the inertial coordinates. The new time coordinate therefore is η respectively $\tilde{\eta}$ and the new spacial coordinate is ξ respectively $\tilde{\xi}$.The relation between the actual Rindler coordinates found in the literature and the ones we picked is: $\mathfrak{X}_{\text{Rindler}} = \mathfrak{f} e^{\xi}$ for $\mathfrak{X}_{\text{Rindler}} \geq 0$ and $\mathfrak{X}_{\text{Rindler}} = -\mathfrak{f} e^{\tilde{\xi}}$ otherwise. Where $\mathfrak{X}$ (Pronounced just as X) is the Rindler coordinate that replaces our coordinates ξ and $\tilde{\xi}$. Obviously (2) only covers the right wedge, where as (3) only covers the left wedge.

Klein-Gordon equation

In order to proceed with the quantisation scheme, we need to look at the field equation. The equation for a scalar field with mass m is the Klein-Gordon equation. In Minkowski and respectively in the new coordinates it takes the following form:

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + m_f^2\right) \phi(t,x,y,z) &= 0 \\ \left(\frac{e^{-2\xi}}{\mathfrak{f}^2}\left(\frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \xi^2}\right) - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + m_f^2\right) \phi(\eta,\xi,y,z) &= 0 \\ \left(\frac{e^{-2\xi}}{\mathfrak{f}^2}\left(\frac{\partial^2}{\partial \tilde{\eta}^2} - \frac{\partial^2}{\partial \tilde{\xi}^2}\right) - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + m_f^2\right) \phi(\tilde{\eta},\tilde{\xi},y,z) &= 0, \end{aligned} \tag{4}$$

where m_f is the mass associated with the field. If one does not like to think of the field as something continuous but rather as being a compound object, one can picture m_f as something like the inertial mass of these compounds. The standard way to solve a wave equation like this is by an exponential ansatz. Since the dependence on ξ is not the same as in other wave equations we make this ansatz only for the other coordinates. We are only looking for positive frequency solutions since we get the negative frequency solutions simply by complex conjugation. The time coordinate in the left wedge $\tilde{\eta}$ has the opposite direction of time compared with t and η, which is why we switched the sign in the argument of the plane wave. The normalisation is chosen for later convenience. The new parameters $k_1, k_2, k_3, \omega_k, \Omega \in \mathbb{R}$ are all invers length dimensional except for Ω which is dimensionless.

$$\begin{aligned}
\phi^M_{\vec{k}}(t,x,y,z) &= \frac{1}{\sqrt{2\omega_k(2\pi)^3}} e^{i(k_1x+k_2y+k_3z-\omega_k t)} \\
\phi^R_{\Omega,k_2,k_3}(\eta,\xi,y,z) &= \frac{1}{\sqrt{2\Omega(2\pi)^2}} h^R_k(\xi) e^{i(k_2y+k_3z-\Omega\eta)} \\
\phi^L_{\Omega,k_2,k_3}(\tilde{\eta},\tilde{\xi},y,z) &= \frac{1}{\sqrt{2\Omega(2\pi)^2}} h^L_k(\tilde{\xi}) e^{i(k_2y+k_3z+\Omega\tilde{\eta})},
\end{aligned} \tag{5}$$

where M stands for Minkowski and L and R stand for left respectively right Rindler-wedge. This ansatz immediately solves the equation for the standard coordinates as long as the correct dispersion relations is fulfilled:

$$\omega_k = \sqrt{m_f^2 + k_1^2 + k_2^2 + k_3^2} \tag{6}$$

For the left and right Rindler wedge the remaining equations yield:

$$\begin{aligned}
\left(-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \mathfrak{f}^2\mu_k^2 e^{2\xi}\right) h^R_k(\xi) &= \Omega^2 h^R_k(\xi) \\
\left(-\frac{\mathrm{d}^2}{\mathrm{d}\tilde{\xi}^2} + \mathfrak{f}^2\mu_k^2 e^{2\tilde{\xi}}\right) h^L_k(\tilde{\xi}) &= \Omega^2 h^L_k(\tilde{\xi})
\end{aligned}, \tag{7}$$

where we defined $\mu_k = \sqrt{m^2 + k_2^2 + k_3^2}$. This equation is solved by modified Bessel functions of the second kind $K_{i\Omega}$(see [8] for more information about them):

$$\begin{aligned}
h^R_k(\xi) &= \sqrt{\frac{2}{\pi}} \frac{A^R_k}{\Gamma(i\Omega)} \left(\frac{\mathfrak{f}\mu_k}{2}\right)^{i\Omega} K_{i\Omega}(\mu_k\xi) \\
h^L_k(\tilde{\xi}) &= \sqrt{\frac{2}{\pi}} \frac{A^L_k}{\Gamma(i\Omega)} \left(\frac{\mathfrak{f}\mu_k}{2}\right)^{i\Omega} K_{i\Omega}(\mu_k\tilde{\xi})
\end{aligned} \tag{8}$$

A^R_k and A^L_k are phase factors that can be chosen arbitrarily, Γ is the gamma function, and K is the Bessel Function of the second kind. In the following it will be important that these solutions for Minkowski coordinates are orthonormalized with respect to the indefinite sesquilinearform:

$$\langle\psi|\phi\rangle_M := i\int_{\mathbb{R}^3} \mathrm{d}^3x \left(\psi^*(t,x,y,z)\frac{\partial}{\partial t}\phi(t,x,y,z) - \left(\frac{\partial}{\partial t}\psi^*(t,x,y,z)\right)\phi(t,x,y,z)\right) \tag{9}$$

That is:

$$\begin{aligned}
\left\langle \phi^M_{\vec{k}} | \phi^M_{\vec{p}} \right\rangle_M &= \delta(k_1-p_1)\delta(k_2-p_2)\delta(k_3-p_3) \\
\left\langle \phi^M_{\vec{k}} | \phi^{M*}_{\vec{p}} \right\rangle_M &= 0 \\
\left\langle \phi^{M*}_{\vec{k}} | \phi^{M*}_{\vec{p}} \right\rangle_M &= -\delta(k_1-p_1)\delta(k_2-p_2)\delta(k_3-p_3),
\end{aligned} \tag{10}$$

whereas the solutions for the left and respectively right Rindler wedge are orthonomalized with respect to the analogously defined indefinite inner product:

$$
\begin{aligned}
\langle\psi|\phi\rangle_R &:= i\int_{\mathbb{R}^3} \mathrm{d}\xi\mathrm{d}y\mathrm{d}z\left(\psi^*(\eta,\xi,y,z)\frac{\partial}{\partial\eta}\phi(\eta,\xi,y,z) - \left(\frac{\partial}{\partial\eta}\psi^*(\eta,\xi,y,z)\right)\phi(\eta,\xi,y,z)\right)\\
\langle\psi|\phi\rangle_L &:= i\int_{\mathbb{R}^3} \mathrm{d}\tilde{\xi}\mathrm{d}y\mathrm{d}z\left(\psi^*(\tilde{\eta},\tilde{\xi},y,z)\frac{\partial}{\partial\tilde{\eta}}\phi(\tilde{\eta},\tilde{\xi},y,z) - \left(\frac{\partial}{\partial\tilde{\eta}}\psi^*(\tilde{\eta},\tilde{\xi},y,z)\right)\phi(\tilde{\eta},\tilde{\xi},y,z)\right)
\end{aligned}
\tag{11}
$$

That is for all $k_1, k_2, p_1, p_2, \Omega, \Lambda \in \mathbb{R}$:

$$
\begin{aligned}
\left\langle\phi^R_{\Omega,k_1,k_2}|\phi^R_{\Lambda,p_1,p_2}\right\rangle_R &= \delta(\Omega-\Lambda)\delta(k_2-p_2)\delta(k_3-p_3)\\
\left\langle\phi^R_{\Omega,k_1,k_2}|{\phi^R_{\Lambda,p_1,p_2}}^*\right\rangle_R &= 0\\
\left\langle{\phi^R_{\Omega,k_1,k_2}}^*|{\phi^R_{\Lambda,p_1,p_2}}^*\right\rangle_R &= -\delta(\Omega-\Lambda)\delta(k_2-p_2)\delta(k_3-p_3)
\end{aligned}
\tag{12}
$$

The analogous equation to (12) also holds on the left wedge. If one now extends the definition of all functions defined in (5) trivially to all of Minkowski spacetime one can write slightly more generally:

$$
\begin{aligned}
&\langle\phi|\psi\rangle := \langle\phi|\psi\rangle_R + \langle\phi|\psi\rangle_L\\
&\left\langle\phi^Q_{\Omega,k_1,k_2}|\phi^W_{\Lambda,p_1,p_2}\right\rangle = \delta_{Q,W}\delta(\Omega-\Lambda)\delta(k_2-p_2)\delta(k_3-p_3)\\
&\left\langle\phi^Q_{\Omega,k_1,k_2}|{\phi^W_{\Lambda,p_1,p_2}}^*\right\rangle = 0\\
&\left\langle{\phi^Q_{\Omega,k_1,k_2}}^*|{\phi^W_{\Lambda,p_1,p_2}}^*\right\rangle = -\delta_{Q,W}\delta(\Omega-\Lambda)\delta(k_2-p_2)\delta(k_3-p_3),
\end{aligned}
\tag{13}
$$

where Q and W can be either L or R.
The appearance of this indefinite scalar product is not at all a coincidence, but rather a consequence of the fact that one can construct a current that is conserved by solutions of the Klein-Gordon equation. However, this fact does not concern us here, so we will not go into more details. The interested reader is referred to textbooks on quantum field theory such as the one by Schweber [15].

Second quantisation of the Klein-Gordon equation

Now we will try to find a second quantised field fulfilling the Klein-Gordon equation in the different coordinate systems. We will then end up with different second quantised fields and in the next subsection we will try to relate those fields.
The first step to second quantise a differential equation is usually to find the solutions to the equation. Afterwards one expands the general solution in Fourier modes and then defines a new object that looks formally the same. So one writes this object to be a Fourier transform of modes that solve the equation but creation and annihilation operators replace the Fourier coefficients.

We will first deal with the Minkowski coordinate case. A general solution to the equation (4) is given by:

$$\psi(t,x,y,z) = \int_{\mathbb{R}^3} \mathrm{d}^3k \left(a(\vec{k})\phi^M_{\vec{k}}(t,x,y,z) + \bar{a}(\vec{k})^* \phi^{M^*}_{\vec{k}}(t,x,y,z) \right), \tag{14}$$

with arbitrary square integrable functions a and $\bar{a}$. So the second quantized field is given by:

$$\Psi_M(t,x,y,z) = \int_{\mathbb{R}^3} \mathrm{d}^3k \left(a_{\vec{k}}\phi^M_{\vec{k}}(t,x,y,z) + \bar{a}^\dagger_{\vec{k}} \phi^{M^*}_{\vec{k}}(t,x,y,z) \right), \tag{15}$$

where the a and $\bar{a}$ are defined to act on the Fockspace as annihilation operators $F_1 := \mathcal{F}(\mathcal{L}^2(\mathbb{R}^3))$. The dagger symbol $\dagger$ denotes taking the adjoint of an operator. The adjoint operators of annihilation operators are called creation operators. The creation and annihilation operators fulfil the so called canonical commutation relations.

$$\begin{aligned} &[\bar{a}_{\vec{k}}, \bar{a}^\dagger_{\vec{p}}] = [a_{\vec{k}}, a^\dagger_{\vec{p}}] = \delta(k_1 - p_1)\delta(k_2 - p_2)\delta(k_3 - p_3) \\ &[a_{\vec{k}}, a_{\vec{p}}] = [\bar{a}_{\vec{k}}, \bar{a}_{\vec{p}}] = [a_{\vec{k}}, \bar{a}^\dagger_{\vec{p}}] = [a_{\vec{k}}, \bar{a}_{\vec{p}}] = 0 \end{aligned} \tag{16}$$

This is all that physicists usually say about the quantisation itself. From a mathematical point of view the question of what these operators do is of course still open. A very short introduction can be found in section 6.2 or in the book by Schweber[15]. The expression in (15) is the quantum field operator for the Klein-Gordon field as an expansion of the positive frequency solutions to the Klein-Gordon equation (4), with respect to the usual Minkowski coordinates. We now follow the analogous procedure to get an expression for the field operator with respect to the Rindler coordinates. A general solution to the Klein-Gordon equation in both wedges is given by:

$$\begin{aligned} \psi(t,x,y,z) = \int_{\mathbb{R}^+} \mathrm{d}\Omega \int_{\mathbb{R}^2} \mathrm{d}^2k \, \big(& b^R(\Omega,k_2,k_3)\phi^R_{\Omega,k_2,k_3}(\eta,\xi,y,z) + \\ & \bar{b}^{R*}(\Omega,k_2,k_3)\phi^{R*}_{\Omega,k_2,k_3}(\eta,\xi,y,z) + b^L(\Omega,k_2,k_3)\phi^L_{\Omega,k_2,k_3}(\tilde{\eta},\tilde{\xi},y,z) + \\ & \bar{b}^{L*}(\Omega,k_2,k_3)\phi^{L*}_{\Omega,k_2,k_3}(\tilde{\eta},\tilde{\xi},y,z) \big) \end{aligned} \tag{17}$$

Where the eigenfunctions have been trivially extended to all of Minkowski space and the b^L, b^R,$\bar{b}^L$,$\bar{b}^R$ are arbitrary elements of $\mathcal{L}^2(\mathbb{R}^3)$. By the same formal analogy we find the quantum field Operator to be:

$$\begin{aligned} \Psi_{R/L}(t,x,y,z) = \int_{\mathbb{R}^+} \mathrm{d}\Omega \int_{\mathbb{R}^2} \mathrm{d}^2k \, \big(& b^R_{\Omega,k_2,k_3}\phi^R_{\Omega,k_2,k_3}(\eta,\xi,y,z) + \\ & \bar{b}^{R\dagger}_{\Omega,k_2,k_3}\phi^{R*}_{\Omega,k_2,k_3}(\eta,\xi,y,z) + b^L_{\Omega,k_2,k_3}\phi^L_{\Omega,k_2,k_3}(\tilde{\eta},\tilde{\xi},y,z) + \\ & \bar{b}^{L\dagger}_{\Omega,k_2,k_3}\phi^{L*}_{\Omega,k_2,k_3}(\tilde{\eta},\tilde{\xi},y,z) \Big), \end{aligned} \tag{18}$$

where the operators b^R,$\bar{b}^R$,b^L,$\bar{b}^L$ are also defined to act on $F_2 := \mathcal{F}(\mathcal{L}^2(\mathbb{R}^3))$. The reader should be cautious here. Even though elements of the two Fock spaces are all

functions of $\mathbb{R}^3$ i.e. a slice of Minkowski spacetime at a point in time the way spacetime is divided into equal time slices is very different for the two frames. Therefore the two Fock spaces describe different objects.
The operators b^R,$\bar{b}^R$,b^L,$\bar{b}^L$ also fulfil the canonical commutation relations:

$$
\begin{aligned}
&[\bar{b}^Q_{\Omega,k_2,k_3}, \bar{b}^{W\dagger}_{\Lambda,p_2,p_3}] = [b^Q_{\Omega,k_2,k_3}, b^{W\dagger}_{\Lambda,p_2,p_3}] = \delta_{Q,W}\delta(\Omega-\Lambda)\delta(k_2-p_2)\delta(k_3-p_3) \\
&[\bar{b}^Q_{\Omega,k_2,k_3}, \bar{b}^{W}_{\Lambda,p_2,p_3}] = [b^Q_{\Omega,k_2,k_3}, b^{W}_{\Lambda,p_2,p_3}] = [\bar{b}^Q_{\Omega,k_2,k_3}, b^{W}_{\Lambda,p_2,p_3}] = [\bar{b}^Q_{\Omega,k_2,k_3}, b^{W\dagger}_{\Lambda,p_2,p_3}] = 0
\end{aligned}
\tag{19}
$$

The delta distributions that appear in equations (19) and (16) lead to not normalisable states. For the Minkowski coordinates this is simply due to the fact that the creation operators create plane waves, which are not square integrable functions. For the Rindler coordinates the issue is quite similar. As these potentially problematic creation operators could lead to further difficulties in our line of reasoning, we introduce a trick to circumvent this problem.

Introducing a countable basis

For this reason we introduce a countable orthonormal basis of $\mathcal{L}^2(\mathbb{R}^3)$ and represent everything in terms of this basis.
In Fourier space it has the form:

$$
f_{m,l,\rho}(k) := e^{-2\pi i l \frac{k}{\rho}} \frac{1}{\sqrt{\rho}} \mathbb{1}_{[(m-\frac{1}{2})\rho,(m+\frac{1}{2})\rho]}(k), \tag{20}
$$

where $\mathbb{1}_A$ is the indicator function of the set A and ρ a positive number, which is inverse length dimensional. The functions are localized around m with wavenumber l in Fourier space, so in real space they will be localized around l with a wavenumber m. This set of functions is orthonormal and complete with respect to the standard $\mathcal{L}^2$ norm. Orthonormality simply follows from the definition.

$$
\int_{\mathbb{R}} \mathrm{d}k f^*_{m,l,\rho}(k) f_{\tilde{m},\tilde{l},\rho}(k) = \delta_{m\tilde{m}} \frac{1}{\rho} \int_{(m-\frac{1}{2})\rho}^{(m+\frac{1}{2})\rho} \mathrm{d}k e^{2\pi i \frac{(l-\tilde{l}}{\rho})} = \delta_{m\tilde{m}}\delta_{l\tilde{l}}, \tag{21}
$$

where as completeness follows from the completeness of the plane waves on an interval:

$$
\begin{aligned}
&\sum_{l\in\mathbb{Z}} f_{m,l,\rho}(k) f_{m,l,\rho}(\tilde{k}) = \mathbb{1}_{[(m-\frac{1}{2})\rho,(m+\frac{1}{2})\rho]}(k)\mathbb{1}_{[(m-\frac{1}{2})\rho,(m+\frac{1}{2})\rho]}(\tilde{k})\frac{1}{\rho}\sum_{l\in\mathbb{Z}} e^{-i2\pi l\frac{k-\tilde{k}}{\rho}} \\
&= \mathbb{1}_{[(m-\frac{1}{2})\rho,(m+\frac{1}{2})\rho]}(k)\mathbb{1}_{[(m-\frac{1}{2})\rho,(m+\frac{1}{2})\rho]}(\tilde{k})\frac{1}{\rho}\sum_{l\in\mathbb{Z}} 2\pi\delta\left(2\pi\left(\frac{k-\tilde{k}}{\rho}-l\right)\right) \\
&= \mathbb{1}_{[(m-\frac{1}{2})\rho,(m+\frac{1}{2})\rho]}(k)\mathbb{1}_{[(m-\frac{1}{2})\rho,(m+\frac{1}{2})\rho]}(\tilde{k})\frac{1}{\rho}2\pi\delta\left(2\pi\frac{k-\tilde{k}}{\rho}\right) \\
&= \mathbb{1}_{[(m-\frac{1}{2})\rho,(m+\frac{1}{2})\rho]}(k)\delta(k-\tilde{k}),
\end{aligned}
\tag{22}
$$

which is why

$$\sum_{m,l\in\mathbb{Z}} f_{m,l,\rho}(k) f_{m,l,\rho}(\tilde{k}) = \delta(k-\tilde{k}) \tag{23}$$

By means of this basis one can rewrite the positive frequency solutions to the Klein-Gordon equation.

$$\begin{aligned}
\phi^M_{\vec{m},\vec{l},\vec{\rho}}(t,x,y,z) &:= \int_{\mathbb{R}^3} \mathrm{d}^3k f_{m_1,l_1,\rho_1}(k_1) f_{m_2,l_2,\rho_2}(k_2) f_{m_3,l_3,\rho_3}(k_3) \phi^M_{\vec{k}}(t,x,y,z) \\
\phi^R_{\vec{m},\vec{l},\vec{\rho}}(\eta,\xi,y,z) &:= \int_{\mathbb{R}^+} \mathrm{d}\Omega \int_{\mathbb{R}^2} \mathrm{d}^2k f_{m_1,l_1,\rho_1}(\Omega) f_{m_2,l_2,\rho_2}(k_2) f_{m_3,l_3,\rho_3}(k_3) \phi^R_{\Omega,k_2,k_3}(\eta,\xi,y,z) \\
\phi^L_{\vec{m},\vec{l},\vec{\rho}}(\tilde{\eta},\tilde{\xi},y,z) &:= \int_{\mathbb{R}^+} \mathrm{d}\Omega \int_{\mathbb{R}^2} \mathrm{d}^2k f_{m_1,l_1,\rho_1}(\Omega) f_{m_2,l_2,\rho_2}(k_2) f_{m_3,l_3,\rho_3}(k_3) \phi^L_{\Omega,k_2,k_3}(\tilde{\eta},\tilde{\xi},y,z),
\end{aligned}$$

(24)

where $\vec{m},\vec{l}\in\mathbb{Z}^3$. Please note that since Ω is restricted to positive values in (24) the corresponding m_1 can effectively also be restricted to values in $\mathbb{N}_0$, since the integral gives zero for negative m_1. The orthogonality relations (13) now become:

$$\begin{aligned}
\left\langle \phi^M_{\vec{m},\vec{l},\vec{\rho}}, \phi^M_{\vec{n},\vec{q},\vec{\rho}} \right\rangle &= \delta_{m_1,n_1}\delta_{m_2,n_2}\delta_{m_3,n_3}\delta_{l_1,q_1}\delta_{l_2,q_2}\delta_{l_3,q_3} \\
\left\langle \phi^{M*}_{\vec{m},\vec{l},\vec{\rho}}, \phi^{M*}_{\vec{n},\vec{q},\vec{\rho}} \right\rangle &= -\delta_{m_1,n_1}\delta_{m_2,n_2}\delta_{m_3,n_3}\delta_{l_1,q_1}\delta_{l_2,q_2}\delta_{l_3,q_3} \\
\left\langle \phi^M_{\vec{m},\vec{l},\vec{\rho}}, \phi^{M*}_{\vec{n},\vec{q},\vec{\rho}} \right\rangle &= 0 \\
\left\langle \phi^W_{\vec{m},\vec{l},\vec{\rho}}, \phi^Q_{\vec{n},\vec{q},\vec{\rho}} \right\rangle &= \delta_{Q,W}\delta_{m_1,n_1}\delta_{m_2,n_2}\delta_{m_3,n_3}\delta_{l_1,q_1}\delta_{l_2,q_2}\delta_{l_3,q_3} \\
\left\langle \phi^{W*}_{\vec{m},\vec{l},\vec{\rho}}, \phi^{Q*}_{\vec{n},\vec{q},\vec{\rho}} \right\rangle &= -\delta_{Q,W}\delta_{m_1,n_1}\delta_{m_2,n_2}\delta_{m_3,n_3}\delta_{l_1,q_1}\delta_{l_2,q_2}\delta_{l_3,q_3} \\
\left\langle \phi^W_{\vec{m},\vec{l},\vec{\rho}}, \phi^{Q*}_{\vec{n},\vec{q},\vec{\rho}} \right\rangle &= 0
\end{aligned} \tag{25}$$

Using the completeness of the new basis one can easily invert (24):

$$\begin{aligned}
\phi^M_{\vec{k}}(t,x,y,z) &:= \sum_{\vec{m},\vec{l}\in\mathbb{Z}^3} f^*_{m_1,l_1\rho_1}(k_1) f^*_{m_2,l_2,\rho_2}(k_2) f^*_{m_3,l_3,\rho_3}(k_3) \phi^M_{\vec{m},\vec{l},\vec{\rho}}(t,x,y,z) \\
\phi^R_{\Omega,k_2,k_3}(\eta,\xi,y,z) &:= \sum_{\vec{m},\vec{l}\in\mathbb{Z}^3} f^*_{m_1,l_1,\rho_1}(\Omega) f^*_{m_2,l_2,\rho_2}(k_2) f^*_{m_3,l_3,\rho_3}(k_3) \phi^R_{\vec{m},\vec{l},\vec{\rho}}(\eta,\xi,y,z) \\
\phi^L_{\Omega,k_2,k_3}(\tilde{\eta},\tilde{\xi},y,z) &:= \sum_{\vec{m},\vec{l}\in\mathbb{Z}^3} f^*_{m_1,l_1,\rho_1}(\Omega) f^*_{m_2,l_2,\rho_2}(k_2) f^*_{m_3,l_3,\rho_3}(k_3) \phi^L_{\vec{m},\vec{l},\vec{\rho}}(\tilde{\eta},\tilde{\xi},y,z)
\end{aligned}$$

(26)

Inserting this into the expression for the quantum field operator (15) and (18) gives:

$$\begin{aligned}\Psi_M(t,x,y,z) =& \sum_{\vec{m},\vec{l}\in\mathbb{Z}^3} \Big(a_{\vec{m},\vec{l},\vec{\rho}} \phi^M_{\vec{m},\vec{l},\vec{\rho}}(t,x,y,z) + \bar{a}^\dagger_{\vec{m},\vec{l},\vec{\rho}} \phi^{M*}_{\vec{m},\vec{l},\vec{\rho}}(t,x,y,z) \Big) \\ \Psi_{R/L}(t,x,y,z) =& \sum_{\vec{m},\vec{l}\in\mathbb{Z}^3} \Big(b^R_{\vec{m},\vec{l},\vec{\rho}} \phi^R_{\vec{m},\vec{l},\vec{\rho}}(\eta,\xi,y,z) + \bar{b}^{R\dagger}_{\vec{m},\vec{l},\vec{\rho}} \phi^{R*}_{\vec{m},\vec{l},\vec{\rho}}(\eta,\xi,y,z) + \\ & b^L_{\vec{m},\vec{l},\vec{\rho}} \phi^L_{\vec{m},\vec{l},\vec{\rho}}(\tilde{\eta},\tilde{\xi},y,z) + \bar{b}^{L\dagger}_{\vec{m},\vec{l},\vec{\rho}} \phi^{L*}_{\vec{m},\vec{l},\vec{\rho}}(\tilde{\eta},\tilde{\xi},y,z) \Big), \end{aligned} \tag{27}$$

where the appearing Fock space operators are given by:

$$\begin{aligned} a_{\vec{m},\vec{l},\vec{\rho}} &:= \int_{\mathbb{R}^3} \mathrm{d}^3k f^*_{m_1,l_1,\rho_1}(k_1) f^*_{m_2,l_2,\rho_2}(k_2) f^*_{m_3,l_3,\rho_3}(k_3) a_{\vec{k}} \\ \bar{a}_{\vec{m},\vec{l},\vec{\rho}} &:= \int_{\mathbb{R}^3} \mathrm{d}^3k f^*_{m_1,l_1,\rho_1}(k_1) f^*_{m_2,l_2,\rho_2}(k_2) f^*_{m_3,l_3,\rho_3}(k_3) \bar{a}_{\vec{k}} \\ b^Q_{\vec{m},\vec{l},\vec{\rho}} &:= \int_{\mathbb{R}^+} \mathrm{d}\Omega \int_{\mathbb{R}^2} \mathrm{d}^2k f^*_{m_1,l_1,\rho_1}(\Omega) f^*_{m_2,l_2,\rho_2}(k_2) f^*_{m_3,l_3,\rho_3}(k_3) b^Q_{\Omega,k_2,k_3} \\ \bar{b}^Q_{\vec{m},\vec{l},\vec{\rho}} &:= \int_{\mathbb{R}^+} \mathrm{d}\Omega \int_{\mathbb{R}^2} \mathrm{d}^2k f^*_{m_1,l_1,\rho_1}(\Omega) f^*_{m_2,l_2,\rho_2}(k_2) f^*_{m_3,l_3,\rho_3}(k_3) \bar{b}^Q_{\Omega,k_2,k_3} \end{aligned} \tag{28}$$

The new Fock space operators fulfill commutation relations which do not involve the use of Dirac distributions but only real numbers.

$$\begin{aligned} &[\bar{a}_{\vec{m},\vec{l},\vec{\rho}}, \bar{a}^\dagger_{\vec{n},\vec{q},\vec{\rho}}] = [a_{\vec{m},\vec{l},\vec{\rho}}, a^\dagger_{\vec{n},\vec{q},\vec{\rho}}] = \delta_{m_1,n_1}\delta_{m_2,n_2}\delta_{m_3,n_3}\delta_{l_1,q_1}\delta_{l_2,q_2}\delta_{l_3,q_3} \\ &[a_{\vec{m},\vec{l},\vec{\rho}}, a_{\vec{n},\vec{q},\vec{\rho}}] = [\bar{a}_{\vec{m},\vec{l},\vec{\rho}}, \bar{a}_{\vec{n},\vec{q},\vec{\rho}}] = [a_{\vec{m},\vec{l},\vec{\rho}}, \bar{a}^\dagger_{\vec{n},\vec{q},\vec{\rho}}] = [a_{\vec{m},\vec{l},\vec{\rho}}, \bar{a}_{\vec{n},\vec{q},\vec{\rho}}] = 0 \\ &[\bar{b}^Q_{\vec{m},\vec{l},\vec{\rho}}, \bar{b}^{W\dagger}_{\vec{n},\vec{q},\vec{\rho}}] = [b^Q_{\vec{m},\vec{l},\vec{\rho}}, b^{W\dagger}_{\vec{n},\vec{q},\vec{\rho}}] = \delta_{Q,W}\delta_{m_1,n_1}\delta_{m_2,n_2}\delta_{m_3,n_3}\delta_{l_1,q_1}\delta_{l_2,q_2}\delta_{l_3,q_3} \\ &[\bar{b}^Q_{\vec{m},\vec{l},\vec{\rho}}, \bar{b}^W_{\vec{n},\vec{q},\vec{\rho}}] = [b^Q_{\vec{m},\vec{l},\vec{\rho}}, b^W_{\vec{n},\vec{q},\vec{\rho}}] = [\bar{b}^Q_{\vec{m},\vec{l},\vec{\rho}}, b^W_{\vec{n},\vec{q},\vec{\rho}}] = [\bar{b}^Q_{\vec{m},\vec{l}\vec{\rho}}, b^{W\dagger}_{\vec{n},\vec{q},\vec{\rho}}] = 0 \end{aligned} \tag{29}$$

They also produce normalised states, whereas the old operators acting on the corresponding vacuum produced states, which strictly speaking were not even in the Fock space:

$$\begin{aligned} \langle 0_M | \, a_{\vec{m},\vec{l},\vec{\rho}}, a^\dagger_{\vec{m},\vec{l},\vec{\rho}} | 0_M \rangle &= 1 \\ \langle 0_M | \, \bar{a}_{\vec{m},\vec{l},\vec{\rho}}, \bar{a}^\dagger_{\vec{m},\vec{l},\vec{\rho}} | 0_M \rangle &= 1 \\ \langle 0_R | \, b^Q_{\vec{m},\vec{l},\vec{\rho}}, b^{Q\dagger}_{\vec{m},\vec{l},\vec{\rho}} | 0_R \rangle &= 1 \\ \langle 0_R | \, \bar{b}^Q_{\vec{m},\vec{l},\vec{\rho}}, \bar{b}^{Q\dagger}_{\vec{m},\vec{l},\vec{\rho}} | 0_R \rangle &= 1 \end{aligned} \tag{30}$$

Bogoliubov transformation

Now that we found two expressions for the quantum field operator, one is valid in all of Minkowski spacetime and the other is only valid in the two Rindler wedges, we would like to find a relation between the two representations where they are both valid. For this reason we set the two equations in (27) equal and make use of the orthonormality relations (25) by multiplying both sides by the projection of the positive frequency solution onto the basis of the one-particle Hilbert space $\phi^{Q*}_{\vec{m}\vec{l},\vec{\rho}}$ forming the indefinite scalar product over both wedges. You get in this way.

$$b^{Q}_{\vec{m}\vec{l},\vec{\rho}} = \sum_{\vec{n},\vec{q}\in\mathbb{Z}^3} \left(a_{\vec{n},\vec{q},\vec{\tilde{\rho}}} \ \ \epsilon^{Q}_{\vec{m},\vec{l},\vec{\rho},\vec{n},\vec{q},\vec{\tilde{\rho}}} + \bar{a}^{\dagger}_{\vec{n},\vec{q},\vec{\tilde{\rho}}} \ \ \zeta^{Q}_{\vec{m},\vec{l},\vec{\rho},\vec{n},\vec{q},\vec{\tilde{\rho}}} \right) \tag{31}$$

with ϵ^Q and ζ^Q given by:

$$\begin{aligned} \epsilon^{Q}_{\vec{m},\vec{l},\vec{\rho},\vec{n},\vec{q},\vec{\tilde{\rho}}} &:= \left\langle \phi^{Q}_{\vec{m},\vec{l},\vec{\rho}}, \phi^{M}_{\vec{n},\vec{q},\vec{\tilde{\rho}}} \right\rangle \\ \zeta^{Q}_{\vec{m},\vec{l},\vec{\rho},\vec{n},\vec{q},\vec{\tilde{\rho}}} &:= \left\langle \phi^{Q}_{\vec{m},\vec{l},\vec{\rho}}, \phi^{M*}_{\vec{n},\vec{q},\vec{\tilde{\rho}}} \right\rangle \end{aligned} \tag{32}$$

Also using $\langle \phi, \psi \rangle^* = \langle \psi, \phi \rangle$ and $\langle \phi^*, \psi^* \rangle = -\langle \psi, \phi \rangle$ which follows directly from the definition of $\langle \phi, \psi \rangle_{R/L}$ and therefore for $\langle \phi, \psi \rangle$ you get:

$$\bar{b}^{Q}_{\vec{m}\vec{l},\vec{\rho}} = \sum_{\vec{n},\vec{q}\in\mathbb{Z}^3} \left(a^{\dagger}_{\vec{n},\vec{q},\vec{\tilde{\rho}}} \ \ \zeta^{Q}_{\vec{m},\vec{l},\vec{\rho},\vec{n},\vec{q},\vec{\tilde{\rho}}} + \bar{a}_{\vec{n},\vec{q},\vec{\tilde{\rho}}} \ \ \epsilon^{Q}_{\vec{m},\vec{l},\vec{\rho},\vec{n},\vec{q},\vec{\tilde{\rho}}} \right) \tag{33}$$

Now calculating the integrals representing ϵ^Q and ζ^Q is a very laborious task, which does not give much insight. A detailed description of this calculation can be found in §2.6 in the paper by Takagi [17]. The result is:

$$\begin{aligned} &\epsilon^{R}_{\vec{m},\vec{l},\vec{\rho},\vec{n},\vec{q},\vec{\tilde{\rho}}} := \int_{\mathbb{R}^+} \mathrm{d}\Omega \int_{\mathbb{R}} \mathrm{d}p_1 \int_{\mathbb{R}^4} \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}p_2 \mathrm{d}p_3 f^{*}_{m_1,l_1,\rho_1}(\Omega) f_{n_1,q_1,\tilde{\rho}_2}(p_1) \\ &f^{*}_{m_2,l_2,\rho_2}(k_2) f^{*}_{m_3,l_3,\rho_3}(k_3) f_{m_2,l_2,\tilde{\rho}_2}(p_2) f_{m_3,l_3,\tilde{\rho}_3}(p_3) \epsilon^{R}_{\Omega,k_2,k_3,\vec{p}} \end{aligned} \tag{34}$$

and analogously for ζ^R.

$$\begin{aligned} \epsilon^{R}_{\Omega,k_2,k_3,\vec{p}} &:= \frac{1}{2\pi}\delta(k_2-p_2)\delta(k_3-p_3)e^{\pi\frac{\Omega}{2}} \left|\Gamma\left(i\Omega\right)\right| \left(\frac{\Omega}{\omega_k}\right)^{\frac{1}{2}} \left(\frac{\omega_k+p_1}{\omega_k-p_1}\right)^{i\Omega/2} \\ \zeta^{R}_{\Omega,k_2,k_3,\vec{p}} &:= \frac{1}{2\pi}\delta(k_2+p_2)\delta(k_3+p_3)e^{-\pi\frac{\Omega}{2}} \left|\Gamma\left(i\Omega\right)\right| \left(\frac{\Omega}{\omega_k}\right)^{\frac{1}{2}} \left(\frac{\omega_k+p_1}{\omega_k-p_1}\right)^{i\Omega/2} \end{aligned} \tag{35}$$

We set $\rho_2 = \tilde{\rho}_2, \rho_3 = \tilde{\rho}_3$, in the definition of the countable basis 20, to achieve orthonormality between the basis functions. The expression for the left Rindler wedge is almost identical, the only difference is the sign in the exponent of the last bracket.

Now we will use these two identities for the Gamma function.

$$\begin{aligned} x\Gamma(x) &= \Gamma(x+1) \\ \Gamma(ix)\Gamma(1-ix) &= \frac{-i\pi}{\sinh(\pi x)} \end{aligned} \tag{36}$$

to rewrite the absolute value of the Gamma function in the expression (39).

$$|\Gamma(i\Omega_m)| = \sqrt{\Gamma(i\Omega_m)\Gamma(-i\Omega_m)} = \sqrt{\Gamma(i\Omega_m)\frac{\Gamma(1-i\Omega_m)}{-ix}} = \sqrt{\frac{\pi}{\Omega_m \sinh(\pi\Omega_m)}} \quad (37)$$

Consequently we can make (35) look a little nicer:

$$\begin{aligned} \epsilon^R_{\Omega,k_2,k_3,\vec{p}} &:= \frac{1}{\sqrt{2\pi}}\delta(k_2-p_2)\delta(k_3-p_3)\frac{1}{\sqrt{1-e^{-2\pi\Omega}}}\frac{1}{\sqrt{\omega_k}}\left(\frac{\omega_k+p_1}{\omega_k-p_1}\right)^{i\Omega/2} \\ \zeta^R_{\Omega,k_2,k_3,\vec{p}} &:= \frac{1}{\sqrt{2\pi}}\delta(k_2+p_2)\delta(k_3+p_3)\frac{1}{\sqrt{e^{2\pi\Omega}-1}}\frac{1}{\sqrt{\omega_k}}\left(\frac{\omega_k+p_1}{\omega_k-p_1}\right)^{i\Omega/2} \end{aligned} \quad (38)$$

So for very small ρ_1 and $\tilde{\rho}_1$, in the definition 20 of the countable basis of square integrable functions, where ρm_1 and $\tilde{\rho} n_1$ is of order 1 this approaches the following expression:

$$\begin{aligned} \epsilon^R_{\vec{m},\vec{l},\vec{\rho},\vec{n},\vec{q},\vec{\tilde{\rho}}} :=& \sqrt{\frac{\rho_1\tilde{\rho}_1}{2\pi}}\Theta(m_1+\frac{1}{2})\delta_{q_1,0}\delta_{l_1,0}\delta_{m_2,n_2}\delta_{m_3,n_3}\delta_{l_2,q_2}\delta_{l_3,q_3} \\ & \frac{1}{\sqrt{1-e^{-2\pi\Omega_{m_1}}}}\frac{1}{\sqrt{\omega_n}}\left(\frac{\omega_n+n_1\tilde{\rho}_1}{\omega_n-n_1\tilde{\rho}_1}\right)^{i\Omega_{m_1}/2} \\ \zeta^R_{\vec{m},\vec{l},\vec{\rho},\vec{n},\vec{q},\vec{\tilde{\rho}}} :=& \sqrt{\frac{\rho_1\tilde{\rho}_1}{2\pi}}\Theta(m_1+\frac{1}{2})\delta_{q_1,0}\delta_{l_1,0}\delta_{m_2,-n_2}\delta_{m_3,-n_3}\delta_{l_2,-q_2}\delta_{l_3,-q_3} \\ & \frac{1}{\sqrt{e^{2\pi\Omega_{m_1}}-1}}\frac{1}{\sqrt{\omega_n}}\left(\frac{\omega_n+n_1\tilde{\rho}_1}{\omega_n-n_1\tilde{\rho}_1}\right)^{i\Omega_{m_1}/2} , \end{aligned} \quad (39)$$

where Θ is the Heavyside function and Ω_m and ω_n are given by:

$$\begin{aligned} \Omega_{m_1} &:= m_1\rho \\ \omega_n &:= \sqrt{(n_1\rho_1)^2+(n_2\rho_2)^2+(n_3\rho_3)^2+m_f^2} \end{aligned} \quad (40)$$

The error of the expression (39) can be estimated by Taylor expanding the integrand of (34) up to zeroth order and using the remainder term of the Taylor series found by Lagrange.

$$\begin{aligned} \epsilon^R_{\vec{m},\vec{l},\vec{\rho},\vec{n},\vec{q},\vec{\tilde{\rho}}} =& \sqrt{\frac{\rho_1\tilde{\rho}_1}{2\pi}}\Theta(m_1+\frac{1}{2})\delta_{q_1,0}\delta_{l_1,0}\delta_{m_2,n_2}\delta_{m_3,n_3}\delta_{l_2,q_2}\delta_{l_3,q_3} \\ & \frac{1}{\sqrt{1-e^{-2\pi\Omega_{m_1}}}}\frac{1}{\sqrt{\omega_n}}\left(\frac{\omega_n+n_1\tilde{\rho}_1}{\omega_n-n_1\tilde{\rho}_1}\right)^{i\Omega_{m_1}/2} + \mathrm{Rest}_{\text{Lagrange, first order}} \end{aligned} \quad (41)$$

and analogously for $\zeta^Q_{\vec{m},\vec{l},\vec{\rho},\vec{n},\vec{q},\vec{\tilde{\rho}}}$.
We find in this way that the error is smaller than:

$$
\begin{aligned}
&\Delta\epsilon^R_{\vec{m},\vec{l},\vec{\rho},\vec{n},\vec{q},\vec{\tilde{\rho}}} := \rho_1\sqrt{\frac{\rho_1\tilde{\rho}_1}{2\pi}}\delta_{m_2,n_2}\delta_{m_3,n_3}\delta_{l_2,q_2}\delta_{l_3,q_3}\frac{1}{\sqrt{1-e^{-2\pi\rho_1(m-\frac{1}{2})}}}\\
&\frac{1}{\sqrt{\omega_n}}\left[\frac{\pi}{2}\frac{1}{e^{2\pi\rho(m-\frac{1}{2})}-1}+\frac{1}{8}\ln\left(\frac{\omega_{\tilde{\rho}(n+\frac{1}{2})}+\tilde{\rho}(n+\frac{1}{2})}{\omega_{\tilde{\rho}(n+\frac{1}{2})}-\tilde{\rho}(n+\frac{1}{2})}\right)\right]+\\
&\tilde{\rho}_1\sqrt{\frac{\rho_1\tilde{\rho}_1}{2\pi}}\delta_{m_2,n_2}\delta_{m_3,n_3}\delta_{l_2,q_2}\delta_{l_3,q_3}\frac{1}{\sqrt{1-e^{-2\pi\rho_1(m-\frac{1}{2})}}}\frac{1}{\sqrt{\omega_n}}\\
&\left[\frac{\tilde{\rho}_1}{8\omega^2_{\tilde{\rho}_1(n-\frac{1}{2})}}(n+\frac{1}{2})+\frac{\rho_1}{4}(m+\frac{1}{2})\frac{1}{\omega_{\tilde{\rho}(n-\frac{1}{2})}}\right]\\
&\Delta\zeta^R_{\vec{m},\vec{l},\vec{\rho},\vec{n},\vec{q},\vec{\tilde{\rho}}} := \rho_1\sqrt{\frac{\rho_1\tilde{\rho}_1}{2\pi}}\delta_{m_2,-n_2}\delta_{m_3,-n_3}\delta_{l_2,-q_2}\delta_{l_3,-q_3}\frac{1}{\sqrt{e^{-2\pi\rho_1(m-\frac{1}{2})}-1}}\\
&\frac{1}{\sqrt{\omega_n}}\left[\frac{\pi}{2}\frac{1}{1-e^{2\pi\rho(m-\frac{1}{2})}}+\frac{1}{8}\ln\left(\frac{\omega_{\tilde{\rho}(n+\frac{1}{2})}+\tilde{\rho}(n+\frac{1}{2})}{\omega_{\tilde{\rho}(n+\frac{1}{2})}-\tilde{\rho}(n+\frac{1}{2})}\right)\right]+\\
&\tilde{\rho}_1\sqrt{\frac{\rho_1\tilde{\rho}_1}{2\pi}}\delta_{m_2,-n_2}\delta_{m_3,-n_3}\delta_{l_2,-q_2}\delta_{l_3,-q_3}\frac{1}{\sqrt{e^{-2\pi\rho_1(m-\frac{1}{2})}-1}}\frac{1}{\sqrt{\omega_n}}\\
&\left[\frac{\tilde{\rho}_1}{8\omega^2_{\tilde{\rho}_1(n-\frac{1}{2})}}(n+\frac{1}{2})+\frac{\rho_1}{4}(m+\frac{1}{2})\frac{1}{\omega_{\tilde{\rho}(n-\frac{1}{2})}}\right]
\end{aligned}
\tag{42}
$$

So the error terms (42) are of order ρ, respectively $\tilde{\rho}$ compared to the original terms (39). Nevertheless the transformation given by (39) is clearly not invertible in l_1 and q_1 and hence not a Bogoliubov transformation so we will only use it to approximate certain expressions and not treat it as the full transformation.
If one now absorbs the factors dependent on $\vec{n}$ and the respective sums into a and $\bar{a}$ and restricts the dependence on m_1 to values in $\mathbb{N}_0$ the transformation can be written in very compact form:

$$
\begin{pmatrix} \mathrm{b}^R_{\vec{m},q,l_2,l_3,\vec{\rho}} \\ \bar{\mathrm{b}}^{L\dagger}_{\vec{\tilde{m}},q,\tilde{l}_1,\tilde{l}_2,\vec{\rho}} \end{pmatrix} = \delta_{q,0}\delta_{l_1,0}\begin{pmatrix} \sqrt{1+\mathrm{N}_{m_1}} & \sqrt{\mathrm{N}_{m_1}} \\ \sqrt{\mathrm{N}_{m_1}} & \sqrt{1+\mathrm{N}_{m_1}} \end{pmatrix}\begin{pmatrix} \mathrm{d}^R_{\vec{m},l_1,l_2,l_3,\vec{\rho}} \\ \bar{\mathrm{d}}^{L\dagger}_{\vec{\tilde{m}},l_1,\tilde{l}_2,\tilde{l}_3,\vec{\rho}} \end{pmatrix}
\tag{43}
$$

Where:

$$
\begin{aligned}
\vec{\tilde{m}} &= \begin{pmatrix} m_1 \\ -m_2 \\ -m_3 \end{pmatrix}\\
\vec{\tilde{l}} &= \begin{pmatrix} l_1 \\ -l_2 \\ -l_3 \end{pmatrix}
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
d^R_{\vec{m},\vec{l},\vec{\rho}} &= \sum_{n_1\in\mathbb{Z}} \frac{\sqrt{\rho\tilde{\rho}}}{\sqrt{2\pi\omega_n}} \left(\frac{\omega_n + n_1\tilde{\rho}_1}{\omega_n - n_1\tilde{\rho}_1}\right)^{i\Omega_{m_1}/2} a_{\vec{n},\vec{q},\vec{\tilde{\rho}}} \\
d^L_{\vec{m},\vec{l},\vec{\rho}} &= \sum_{n_1\in\mathbb{Z}} \frac{\sqrt{\rho\tilde{\rho}}}{\sqrt{2\pi\omega_n}} \left(\frac{\omega_n + n_1\tilde{\rho}_1}{\omega_n - n_1\tilde{\rho}_1}\right)^{-i\Omega_{m_1}/2} a_{\vec{n},\vec{q},\vec{\tilde{\rho}}} \\
\bar{d}^R_{\vec{m},\vec{l},\vec{\rho}} &= \sum_{n_1\in\mathbb{Z}} \frac{\sqrt{\rho\tilde{\rho}}}{\sqrt{2\pi\omega_n}} \left(\frac{\omega_n + n_1\tilde{\rho}_1}{\omega_n - n_1\tilde{\rho}_1}\right)^{i\Omega_{m_1}/2} \bar{a}_{\vec{n},\vec{q},\vec{\tilde{\rho}}} \\
\bar{d}^L_{\vec{m},\vec{l},\vec{\rho}} &= \sum_{n_1\in\mathbb{Z}} \frac{\sqrt{\rho\tilde{\rho}}}{\sqrt{2\pi\omega_n}} \left(\frac{\omega_n + n_1\tilde{\rho}_1}{\omega_n - n_1\tilde{\rho}_1}\right)^{-i\Omega_{m_1}/2} \bar{a}_{\vec{n},\vec{q},\vec{\tilde{\rho}}}
\end{aligned}
\tag{45}
$$

$$
N_{m_1} = \frac{1}{e^{2\pi\Omega_{m_1}} - 1} \tag{46}
$$

Equation (46) is the Bose-Einstein distribution. We encounter this distribution in equation (38) for the first time. At this point it is only a factor in the transformation from one set of annihilation operators to another, so we will comment on possible interpretations later on. The operators d^Q and $\bar{d}^Q$ annihilate the Minkowski vacuum. One can try to find expressions for the expected number of Rindler excitations in the Minkowski vacuum. In doing so one relies on the possibility to write the Minkowski vacuum as a state that is an element of the Fock space constructed from the Rindler vacuum. This essentially means that the Minkowski vacuum can be constructed from the Rindler vacuum by applying linear combinations of finitely many $b^\dagger$ and $\bar{b}^\dagger$ operators to it. We will try to find out whether or not this hypothesis is plausible. The number of excitations can be calculated to be:

$$
\begin{aligned}
\langle 0_M | \, N^R_{\vec{m},\vec{l},\vec{\rho}} | 0_M \rangle &= \langle 0_M | \, b^{R\dagger}_{\vec{m},\vec{l},\vec{\rho}} b^R_{\vec{m},\vec{l},\vec{\rho}} | 0_M \rangle \\
&= \langle 0_M | \, N_{m_1} \frac{\rho\tilde{\rho}}{2\pi} \delta_{l_1,0} \sum_{n_1\in\mathbb{Z}} \frac{1}{\sqrt{\omega_n}} \left(\frac{\omega_n + n_1\tilde{\rho}_1}{\omega_n - n_1\tilde{\rho}_1}\right)^{-i\Omega_{m_1}/2} \bar{a}_{\vec{n},\vec{q},\vec{\tilde{\rho}}} \sum_{\nu_1\in\mathbb{Z}} \frac{1}{\sqrt{\omega_\nu}} \\
&\qquad \left(\frac{\omega_\nu + \nu_1\tilde{\rho}_1}{\omega_\nu - \nu_1\tilde{\rho}_1}\right)^{-i\Omega_{m_1}/2} \bar{a}^\dagger_{\vec{\nu},\vec{\sigma},\vec{\tilde{\rho}}} | 0_M \rangle \\
&= \langle 0_M | \, N_{m_1} \frac{\rho\tilde{\rho}}{2\pi} \delta_{l_1,0} \sum_{n_1\in\mathbb{Z}} \frac{1}{\omega_n} | 0_M \rangle = \frac{1}{e^{2\pi\Omega_{m_1}} - 1} \frac{\rho\tilde{\rho}}{2\pi} \delta_{l_1,0} \sum_{n_1\in\mathbb{Z}} \frac{1}{\omega_n}
\end{aligned}
\tag{47}
$$

Therefore the expected number of Rindler excitations to be found in Minkowski vacuum is proportional to the Bose-Einstein distribution.

The appearance of the density of states for bosons for the expected number of particles is a surprising result and is often interpreted as the Minkowski vacuum being seen as a thermal state in the view of an accelerated observer. One often proceeds further and finds a grand canonical ensemble for the Minkowski vacuum if one focuses ones attention to one wedge and traces out the degrees of freedom of the other wedge (see [5], [17]). However, we shall end our considerations at this point.

Since the sum in (47) does not converge and every summand is independent of m_2, m_3, l_2, l_3 we can see already at this point that the Minkowski vacuum is not an element of the Fockspace we constructed with the help of the Rindler vacuum. One can easily convince oneself of this fact since the expected number of particles can be interpreted as the norm squared of the vector you get when acting with the Rindler annihilation operator on the Minkowski vacuum.

$$\langle 0_M | \, b^{R\dagger}_{\vec{m},\vec{l},\vec{\rho}} b^{R}_{\vec{m},\vec{l},\vec{\rho}} |0_M\rangle = \left\| b^{R}_{\vec{m},\vec{l},\vec{\rho}} |0_M\rangle \right\|^2 \tag{48}$$

Now due to this vector having infinite norm the vector $|0_M\rangle$ cannot lie in the domain of $b^{R}_{\vec{m},\vec{l},\vec{\rho}}$. Moreover, since the expected number of excitations with numbers $m_1, m_2, m_3, l_1, l_2, l_3$ turns out to be independent of m_2, m_3, l_2, l_3 which we remember to be associated with a transverse momentum of $\left(k^2, k^3\right)$. A state occupying an equal number of excitations for each transverse momentum k^2, k^3 can not have finite Fock space norm, since there are infinitely many different transverse momenta. Since the errors $\Delta\epsilon$ and $\Delta\zeta$ are of higher order in ρ and $\tilde{\rho}$ respectively compared to the transformation given by (39), terms of higher order may be dependent on m and l but this will not cancel the independent part. Therefore no matter how many corrections we consider, the transformation will never map the Minkowski vacuum to a vector in the Rindler Fock space. Hence we conclude that there is no well defined canonical transformation in the form of (31), i.e. the two Fockspaces of excitations around the Minkowski vacuum and of excitations around the Rindler vacuum are unitary inequivalent.

Compared to the derivation of [17], we deviated from his line of reasoning after (47). Takagi proceeds by inverting (43) and explicitly deriving a formal expression for the Minkowski vacuum in terms of the Rindler vacuum, which is not well defined neither. However, he also ends his argument by stating that the two Fock spaces are unitary non-equivalent. We did not pursue this line of argument since equation (43) is not invertible in the indices q and l_1.

We will now summarize what has been done in this chapter, since the notation has been quite cramped and it is therefore easy to miss out on some details if one reads this chapter for the first time.

In the beginning of this chapter we defined two different frames of reference. The first one arbitrary inertial frame equipped with the usual Minkowski coordinates and the second is the frame obtained by continuously boosting the an inertial frame of reference into the instantaneous rest frame of a uniformly accelerated particle at proper time η. In the second frame we picked spacial coordinates, which almost coincide with the continuously boosted Minkowski coordinates, where the difference between the boosted coordinates and the ones we picked is simply a reparametrisation of the spacial coordinates. We did this, because the Unruh effect is often characterized as "A uniformly accelerated observer sees a thermal bath of particles". In order to verify this assertion we would like to describe spacetime in the way an uniformly accelerated observer might see it.

The "thermal bath of particles" refers to excitations of a second quantised Klein-Gordon field, thus we have to study the Klein-Gordon equation. This is what we did in the following paragraph. We have to do this for both inertial frame and for the accelerated frame in order to be able to compare the two. As expected, the Klein-Gordon equations takes a different form in the accelerated frame, so the generalised eigenfunctions also take a different form there. Nevertheless, the line element in the accelerated frame is static, so it is still possible to split the generalized eigenfunctions into positive and negative frequency part. This means that the support of the Fourier transform of the eigenfunctions with respect to the temporal coordinate has a definite sign.

Now since we want to compare the notions of the second quantized Klein-Gordon field of the two observers we have to second quantise the field we study in section 2.1.1, this is done in 2.1.1. The procedure here is perform Fourier transformation to the general solutions of the Klein-Gordon equation and replace the appearing coefficients by operators on the Fock space over $\mathcal{L}^2$ and impose certain commutation relations for the new operators. If one uses formal analogies like this, one should always bare in mind that ultimately the meaning of the objects obtained in this way is very unclear and has to be verified anew for every new instance.

Next, we introduced a countable basis for the space of square integrable functions in order to get rid of some infinite expressions that appear otherwise. This was mainly done, because I wanted to make sure that the fact that the expected particle number is infinite cannot be blamed on using a basis like plane waves which are not square integrable. I would like the reader to recognise that this infinity is of a different nature.

Finally in the last paragraph of this chapter we tried to find an explicit expression connecting the Fock spaces used for the two different descriptions. We arrived at the unexpected result that the number Rindler excitations of the Minkowski vacuum with respect to the Rindler vacuum is independent of their wavenumber. Since there are obviously infinitely many different possible wavenumbers for the excitations we drew the conclusion that apparently there is no Bogoliubov transformation meaning that the two Fock spaces describe physically distinct fields.

As mentioned in the beginning of this chapter the whole situation for our uniformly accelerated observer seems to be quite awkward if in her natural description of spacetime and quantum field theory she finds the quantum field to be in an excited state even though an inertial observer would call the state the vacuum. As it turns out, not only is the description of the two observers of the same state quite different, the description of the uniformly accelerated observer is actually completely unfit to speak about the Minkowski vacuum using his "natural" Fock space construction. So we conclude that we went astray when we assumed that this particular Fock space construction is what a uniformly accelerated observer would call "natural". Indeed it seems that one needs to be very careful in interpreting calculations based on non-inertial frames of reference. Thus the intuition that what happens to accelerated observers ought to be most easily understood with respect to non-inertial frames of reference turns out to be wrong.

2.1.2 Calculating the detector response

We turn our attention now to a situation that is easier to interpret. We consider a concrete model for a detector that is supposed to measure excitations of the state of the Klein-Gordon quantum field. This detector is then uniformly accelerated to check whether or not it will click. The standard detector-field system which is considered in the literature involves a two level detector which follows an externally given trajectory that corresponds to uniformly accelerated motion. So it is forced on the trajectory given by (See figure 1):

$$\left(\frac{1}{a}\sinh(\tau a), 0, 0, \frac{1}{a}\cosh(\tau a)\right) \tag{49}$$

The detector is a point like object, which couples linearly with its monopole moment to the field operator evaluated at the position of the detector. Since the detector only has two states the wave function of the whole system is an element of $\mathbb{R}^2 \otimes \mathcal{F}$, where $\mathcal{F}$ denotes the Fock space over $L^2(\mathbb{R}^3)$. The Hamiltonian is given by:

$$\begin{aligned} H = H_D \otimes \mathbb{1} + \mathbb{1} \otimes \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \omega_{\vec{p}} a^\dagger_{\vec{p}} a_{\vec{p}} + \mathbb{1} \otimes \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} \omega_{\vec{p}} \bar{a}^\dagger_{\vec{p}} \bar{a}_{\vec{p}} \\ - e^{-s|\tau|} M(0) \otimes \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + \bar{a}^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}}\right) \end{aligned} \tag{50}$$

where H_D is the Hamiltonian of the free two level detector i.e. $H_D |E_l\rangle = E_l |E_l\rangle$ for $l = 1, 2$. $\omega_{\vec{p}} = \sqrt{m_f^2 + \vec{p}^2}$, a,$\bar{a}$ are annihilation operators of the Fock space associated with the quantum field and $M(0)$ is the Monopole moment of the detector at time 0. The exponential function in front of the last integral is meant to adjust the interaction such that it is slowly switched on for early τ and slowly turns off again for late τ. The rate of switching s is understood as an extremely small positive number. We will let s tend to zero at the end of the calculation. This mechanism is used to eliminate excitation of the detector due to switching it on and off, while letting the detector be active over a finite time interval. The last integral is the Klein-Gordon field operator constructed by second quantisation of the Klein-Gordon equation in an inertial frame, (see section 2.1.1 for more details).
We switch to the interaction picture to investigate the behavior of the system starting at the vacuum for late times using first order in perturbation theory. The Hamiltonian then takes the form:

$$\begin{aligned} H(\tau) &= -e^{-s|\tau|} M(\tau) \otimes \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} e^{-ip_\alpha x^\alpha(\tau)} + \bar{a}^\dagger_{\vec{p}} e^{ip_\alpha x^\alpha(\tau)}\right) \\ &= -e^{-s|\tau|} M(\tau) \otimes \phi(\tau) \end{aligned} \tag{51}$$

We would like to calculate the rate for a transition of the whole system from a state given by $|E_1\rangle \otimes |0\rangle$ to any other state $|E_2\rangle \otimes |\psi\rangle$, where $|0\rangle$ denotes the vacuum of the Fock space and $|\psi\rangle$ any other Fock space vector. For the calculation itself we refer to the appendix 6.1. For a massless field the transition rate is given by:

$$R_{1\to 2} = \left(\left|\langle E_2|\, M(0)\, |E_1\rangle\right|^2 + \left|\langle E_2|\, M^{\dagger}(0)\, |E_1\rangle\right|^2 \right) \frac{E_2 - E_1}{2\pi} \frac{1}{e^{\frac{E_2-E_1}{T}} - 1}, \tag{52}$$

where T is the Davies-Unruh-Temperature.

$$T = \frac{2\pi}{a} \tag{53}$$

For the massive case the situation is vastly more complicated. It is very hard to analyse the formulas one obtains for the transition rate. In certain limits Takagi finds expressions, which however do not look like the Planck formula anymore. For example for large m_f he obtains asymptotically [17]:

$$R_{1\to 2} = \left(\left|\langle E_2|\, M(0)\, |E_1\rangle\right|^2 + \left|\langle E_2|\, M^{\dagger}(0)\, |E_1\rangle\right|^2 \right) \frac{a}{8\pi} e^{-\frac{(E_2-E_1)2+m_f/\pi)}{T}} \tag{54}$$

Summarizing one can say that an almost point like detector forced on the uniformly accelerated trajectory immersed in a Klein-Gordon field will get excited. For a massless field the rate of excitation is given by the Planck formula, which is rather surprising. However for a massive field, the rate is not given by anything simple. One can still show that these rates fulfil the abstract criterion of the KMS condition, meaning that somehow one can view it still as a thermal result. For more information about the KMS condition see [4]. Usually, one does not talk much about the massive case in this concrete setting involving the Unruh-DeWitt detector. Instead one finds that in the massless case the excitation rate coincides with the Planck formula and concludes that the detector is interacting with a thermal gas of Rindler particles.
The fact that even the complicated transition rate for the massive case fulfills the KMS condition is often considered to proof that the Minkowski vacuum is in fact a thermal gas of Rindler particles. However, there is no need for the state of the quantum field to be thermal simply because it takes a stationary form. The detector-field system is bound to become stationary, simply because of the setting we chose. The KMS condition is built such that it singles out stationary expressions, because thermal distributions are stationary distributions. But since not all stationary distributions are thermal, there is no need to interpret the distribution of excitations arising in the Rindler frame as a thermal gas of particles. In fact the Rindler frame plays no role in this calculation whatsoever and it failed completely to give an adequate description of the situation without detector. Whereas in the picture actually used, namely the second quantisation of the Klein-Gordon field with respect to an inertial frame, we stay in the one particle regime so we certainly found no thermal distributions of Minkowski particles. That is why I advocate to omit any words that imply an interpretation of Rindler particles such as "thermal gas" as long as no concrete arguments in favour of such an interpretation are given.

2.2 Circular motion

Additionally to the linear accelerated detector discussed so far, the physical literature often also discusses detectors in circular motion. Takagi [17] considers a setting like this and relates it to another setting quite similar to the usual uniformly accelerated detector. For both settings he is interested in the behavior of an Unruh DeWitt detector coupled to a massless scalar field. In the first setting the detector follows a circular motion with velocity v. In the second setting the detector is uniformly accelerated in one direction and additionally has a constant velocity v, with respect to its proper time, in a direction perpendicular to its acceleration. Takagi shows that the transition rates of the Unruh DeWitt detectors in the two settings coincide for $v \to c$. The treatment is again up to first order perturbation theory and the starting points for finding the transition rates are the same as in section 2.1.2: One finds the world lines associated with the two kinds of motion, writes down the Hamiltonian (50) and tries to evaluate the expressions that define the excitation rates (93).
The world lines are given by:

$$x_{\text{circular}}(\tau) := \begin{pmatrix} \gamma\tau \\ R\cos(\omega\gamma\tau) \\ R\sin(\omega\gamma\tau) \\ 0 \end{pmatrix} \tag{55}$$

$$x_{\text{drift}}(\tau) := \begin{pmatrix} a^{-1}\sinh(\tau\gamma a) \\ a^{-1}\cosh(\tau\gamma a) \\ v\gamma\tau \\ 0 \end{pmatrix}, \tag{56}$$

where $\gamma := \frac{1}{\sqrt{1-v^2}}$ and v either is the constant velocity of the circular motion or the velocity perpendicular to the acceleration for the uniformly accelerated motion. Also the angular frequency is given by $\omega := \frac{v}{R}$, as usual for circular motion. The drifting motion of (56) is not to be confused with constant motion in a different direction than the acceleration induced by a Lorentz boost. The motion considered here refers to constant motion with respect to the proper time of the detector. In an inertial frame the drifting detector will therefore move infinitely slow for very early and very late times.
To make the two systems more readily comparable Takagi picks $\frac{\gamma^2 v^2}{R}$ for the circular motion and $\gamma^2 a$ for the uniformly accelerated motion both to be equal to the constant value $\mathfrak{a}$.
For the details of the calculation I refer to §12.2 of [17]. The two point functions for the two cases are obtained in an analogous manner to (95). For circular motion Takagi gets:

$$
\begin{aligned}
\langle 0| \phi_{\text{circular}}(t)\phi^{\dagger}_{\text{circular}}(\tilde{t}) |0\rangle &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 k}{(2\pi)^3 2\sqrt{\omega_k}} e^{-ik\cdot(x_{\text{circular}}(\tau)-x_{\text{circular}}(\tilde{\tau}))} \\
&= \frac{1}{4\pi^2} \frac{1}{-\gamma^2(\tau-\tilde{\tau}-i\epsilon)^2 + 4\gamma^4 \frac{v^4}{\mathfrak{a}^2}\sin^2\left[\frac{\mathfrak{a}}{2v\gamma}(\tau-\tilde{\tau}-i\epsilon)\right]},
\end{aligned} \tag{57}
$$

whereas for the drifting motion Takagi arrives at:

$$
\begin{aligned}
\langle 0| \phi_{\text{drift}}(t)\phi^{\dagger}_{\text{drift}}(\tilde{t}) |0\rangle &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 k}{(2\pi)^3 2\sqrt{\omega_k}} e^{-ik\cdot(x_{\text{drift}}(\tau)-x_{\text{drift}}(\tilde{\tau}))} \\
&= \frac{1}{4\pi^2} \frac{1}{-\gamma^2 v^2(\tau-\tilde{\tau}-i\epsilon)^2 - 4\frac{\gamma^4}{\mathfrak{a}^2}\sinh^2\left[\frac{\mathfrak{a}}{2\gamma}(\tau-\tilde{\tau}-i\epsilon)\right]}
\end{aligned} \tag{58}
$$

The resulting spectra of excitation of the Unruh DeWitt detector are hard to discuss without making use of strong simplifications. However the two point functions converge for velocities very close to the speed of light to the same limit:

$$
\langle 0| \phi_{\infty}(t)\phi^{\dagger}_{\infty}(\tilde{t}) |0\rangle := \frac{1}{4\pi^2} \frac{1}{-(\tau-\tilde{\tau}-i\epsilon)^2 - \frac{\mathfrak{a}^2}{12}(\tau-\tilde{\tau}-i\epsilon)^4} \tag{59}
$$

The resulting spectrum, which is obtained by plugging the two-point function into euqation (93), is:

$$
F(\omega) = \frac{1}{8\sqrt{3}\pi}\mathfrak{a} e^{-2\sqrt{3}\frac{\omega}{\mathfrak{a}}} \tag{60}
$$

One can even show that the corresponding spectra converge to (60) and the error for finite γ is of order γ^{-2}.
I think it is worth mentioning that the two point functions (57), (58) and (59) do not fulfil the KMS condition [17]. Thus, not only do the corresponding spectra not look like the Planck formula, but they also do not fulfil the general abstract criterion that characterizes thermal states. Even though one might be able to define an effective temperature in these two cases the spectrum is far from being fully characterized by this parameter alone.
Bell and Leinaas ([1] and [2]) consider a more realistic version of the circular Unruh effect. In both papers they consider electrons in a storage ring. In [1] the main idea is that even though the usual Unruh effect is way too small to be verified experimentally the situation turns out to be different if one uses the magnetic moment of electrons in storage rings instead of the usual Unruh DeWitt detector. In a storage ring the magnetic momenta of the electrons tend to align themselves with the external magnetic field, but not fully so. The resulting effect turns out to be much larger than the linear Unruh effect, but the resulting spectrum is rather complicated and not of a recognizable thermal form. In their second paper on this subject [2] they also consider quantum corrections to the trajectory of the electrons. Only the corrections perpendicular to the plane of rotation turn out to be thermally distributed to a

temperature given by $\frac{13}{93}\sqrt{3}\mathfrak{a}$, slightly higher than the Unruh temperature for the same absolute value of the acceleration.
Summarizing one might say that, although some parts of systems subject to circular movement might have spectra similar to or even equal to a thermal spectrum. However, the situation is more complicated and one should not expect a system that exhibits a constant acceleration, relative to some notion, to behave thermally.

3 Reasons to be unsatisfied

In this section I present arguments against the treatment of sections 2.1.1 and 2.1.2. Much of this has already been mentioned, but I rephrase these points and add some new ones.
First things first, the general strategy in section 2.1.1 was to find a description of the quantum field in the frame of the accelerated observer and try to relate this description to the inertial one. On the way the Einstein-Bose distribution pops up and one tries to write the vacuum relative to the inertial frame in the description of the accelerated frame as a grand canonical ensemble.
First of all this approach is extremely indirect. Even if we were successful it would be quite hard to construct a coherent story out of this. Essential points are completely left out of the picture, such as the role of the accelerating agent. She obviously should have a central position in this description, being the only possible source of energy for ultimately all processes that happen in this setting.
Speaking of energy, the notion of energy in the accelerated frame is actually a quite unclear one, as Lyle argues in [10] and [11]. For two inertial frames the Theory of General Relativity guarantees us that the two frames give equivalent descriptions of the physical reality involving different notions of energy. Clearly, these notions are connected via the Lorentz transformation that connects the two frames. For non-inertial frames no such guarantee exists, all General Relativity Theory has to say about such frames is that if a frame locally looks inertial, i.e. the metric takes Minkowskian form and the Christoffel symbols vanish at this point, physics locally looks roughly like it does in inertial frames. Of course one can always use the coordinates one wants to calculate certain quantities, but the only way one is guaranteed to use the correct interpretation of these quantities is to transform them into an inertial or a locally inertial frame. This subtle point is often ignored, since in the literature one seldom specifically states the Equivalence Principle one uses and just assumes that everyone has the same intuition about it. Since the fact that the interpretation of mathematical quantities is unclear in non-inertial frames is quite general, one should be very careful with using second quantisation schemes in such frames.
There is a prominent example of a physical debate that was held for over half a century, one of the reasons why it has not been settled after a few years is that many authors tried to clear things up by using interpretations of objects in non-inertial frames that simply have no simple interpretation there. The example I have in mind is the discussion of whether or not a constantly accelerated particle radiates. It started in 1954 with a paper by Bondi and Gold [3] or even earlier and in 2008 Lyle gave a comprehensive summary of the literature and added essential new points to the debate. There were many efforts to settle the debate once and for all: for example by showing that in certain frames the components of the electromagnet field strength tensor are static, forgetting that those components only represent the

electric and magnetic field we know how to interpret in inertial frames. considering this the whole approach of section 2.1.1 seems flawed, since in order to compare the descriptions of the inertial and the non-inertial frame one needs to have an independent interpretation of the non-inertial frame. The former arguments were relied on the fact that the strategy of the first paragraph of this chapter works out, or can be made rigorous. However, as we saw there this does not seem to be the case. We ended section 2.1.1 by concluding that the picture of the quantum field using the second quantisation in the accelerated frame of reference does not seem to be fit to even speak of the Minkowski vacuum and hence is unable to give an adequate description of the whole situation.
Furthermore, for this whole procedure to work, one has to perform the second quantisation scheme in a non-inertial frame. In order for this to be successful the solutions of the field equation (4) ought to be divided into positive and negative frequency parts, which is only possible if the metric is stationary. This means that only observers whose motions happens to be such that the metric with respect to their coordinate system turns out to be stationary are allowed to follow this scheme of second quantisation. It would be rather odd if fundamental statements like "From an accelerated point of view, the vacuum is a thermal state" were only possible to check for very special kinds of motions. Being able to perform second quantisation in the reference frame of an observer obviously also takes for granted that there is a natural coordinate system in which a non-inertial observer ought to describe his perception, since the General Relativity Theory does not provide an interpretation for any such frame it is very bold to claim there is a canonical one.
Considering the treatment of section 2.1.2 this strategy is much better. One does not talk about observers and what they might consider natural. The procedure does not fundamentally depend on the particular type of motion of the detector and since it is described within an inertial frame no fundamental problems of interpretation of the mathematical quantities kind appear either.
Having said that, there is still space for improvement. The motion is imposed on the detector in a very unnatural way, not by any kind of interaction but simply by inserting the worldline into the Hamiltonian. This lets the whole point one is trying to make appear rather dubious. Since we do not talk about the agent that causes the detector to move in this particular way, it seems odd that both the quantum field and the detector get excited during this thought experiment. Naively one would expect the internal energy of the detector plus the energy of the field to be conserved, but of course once the accelerated wordline of the detector is put into the Hamiltonian the flow of energy is not easy to keep track of anymore. The detectors movement can be a source as well as a consumer of energy. If the accelerating agent is included into the description, it is perfectly obvious that she must provide the energy for the excitation of both the detector and the quantum field, because she drags the former through the latter.
The idea behind chapter 2.2 is to widen the discussion from detectors that are constantly accelerated in one direction to detectors which are moving in a circle

with constant velocity and hence with constant absolute value of the acceleration. The discussion shows that these detectors do also get excited, but at a rate that is more complicated than a thermal rate. Since the Takagi's strategy we summarised to examine this type of motion was essentially the same as the one for linear acceleration in section 2.1.2 the problems associated with this treatment are basically the same. The motion is still artificially imposed on the detectors and the accelerating agent is completely left out of the picture.

4 Possible improvements

In this section we eliminate out the main disadvantages of the earlier approaches, which were mentioned at the end of the last section. By including the accelerating agent explicitly into the description of the thought experiment the resulting model is heuristically more clear than the ones considered so far. Apart from considering different models, the strategy of this chapter is the same as in section 2.1.2.

4.1 Introducing two toy models

We will consider two models in this chapter, one where the detector of the model couples to a Klein-Gordon field and one where the detector couples to a quantum field with non relativistic dispersion relation. Both models contain two non relativistic particles bound together by an harmonic potential. This will become important later on, because the model shows unexpected behaviour in the relativistic regime where its dispersion relation is wrong. These two particles form a primitive thermometer which interacts with a quantum field. This interaction is realised by coupling one of the particles to the field. The Hamilton operators for the two models thus assume the general form:

$$
\begin{aligned}
H := \frac{-\hbar^2}{2M}\Delta_R + \frac{-\hbar^2}{2\mu}\Delta_r + \frac{\mu}{2}\bar{\omega}^2 r^2 + \int \frac{\mathrm{d}p}{2\pi} E_{\text{kinetic, field}}(p) a_p^\dagger a_p \\
+ H_{\text{interaction}}(r, R, a_p, a_p^\dagger) + H_{\text{external}}(R) \quad (61)
\end{aligned}
$$

Where r is the relative coordinate and R is the coordinate of the center of mass. For simplicity r and R are one dimensional. M is the absolute mass of the two particles, μ the reduced mass and $\bar{\omega}$ the natural frequency of the harmonic potential binding the two particles together.

4.2 Relativistic quantum field

The first model we consider is the one where a relativistic quantum field couples to our two particle thermometer, so the kinetic energy of the quantum field $E_{\text{kinetic, field}}$ is the same as in section 2.1.1:

$$
E_{\text{kinetic, field}} = \omega_p = c\sqrt{p^2 + m_f^2 c^2} \quad (62)
$$

We choose the interaction part of the Hamiltonian to be of the form $x_2\phi(x_2)$, where x_2 is the coordinate of one of the particles and ϕ is the quantum field operator.We also switch coordinates to center of mass(R) and relative (r) coordinates, since the free motion separates nicely in these coordinates.

$$H_{\text{interaction}} := \sigma\left(R + \frac{m_2}{M}r\right)\int \frac{\mathrm{d}p}{2\pi\sqrt{\omega_p}}\left(e^{\frac{ip}{\hbar}(R+\frac{m_2}{M}r)}a_p + e^{\frac{-ip}{\hbar}(R+\frac{m_2}{M}r)}a_p^*\right), \tag{63}$$

where σ is an interaction parameter which is small enough such that perturbation theory will be successful. This is the simplest and most straight forward model to describe the Unruh effect that one can think of, which is why it has already been analysed before. Padmanabhan [13] also considers this model and we will follow his line of argument for the most part. Only after equation (68) we will deviate from his approach.
Next we look at the generator of the S-matrix of this model in the interaction picture, to be able to do time dependent perturbation theory.

$$\mathcal{H}(t) = \sigma\left(R + \frac{m_2}{M}r\right)\int \frac{\mathrm{d}p}{2\pi\sqrt{\omega_p}}\left(e^{-i\omega_p t}e^{\frac{ip}{\hbar}(R+\frac{m_2}{M}r)}a_p + e^{i\omega_p t}e^{\frac{-ip}{\hbar}(R+\frac{m_2}{M}r)}a_p^*\right) \tag{64}$$

For the expansion in perturbation terms we consider the free evolution, the evolution of the system without (63), i.e. the motion subject to the external potential in the center of mass coordinate, time evolution of the harmonic oscillator in the relative coordinate and free evolution of the quantum field.
Before we set H_{external} to be a linear potential to achieve uniform acceleration of our two particle thermometer we will first set it to zero to check whether the detector works properly.

4.2.1 Uniformly moving thermometer

For $H_{\text{external}} = 0$ we have uniform motion in the center of mass coordinate. We pick an initial wave function, which is the tensor product of solutions to the stationary Schrödinger equation of each of the components of the system. That means plane waves for the center of mass motion and Gauss distributions multiplied by Hermit polynomials for the relative motion. Since we are not interested in radiation from the detector due to relaxation to the ground state while reducing internal energy we let the system be in the ground state for very early times:

$$\begin{aligned}\Psi_0 &:= \psi_R \otimes \psi_r \otimes |0\rangle \\ \psi_R &:= \frac{1}{\sqrt{2\pi}}e^{-i\frac{\frac{\chi^2}{2M}t-\chi R}{\hbar}} \\ \psi_r &:= \sqrt[4]{\frac{\mu\bar{\omega}}{\hbar\pi}}e^{-i\frac{\bar{\omega}t}{2}-\frac{\bar{\omega}\mu r^2}{2\hbar}},\end{aligned} \tag{65}$$

where $|0\rangle$ as always is the vacuum of the quantum field. For very early times the absolute motion of our thermometer is given by uniform motion with some arbitrary momentum χ. We want to compute the transition rate up to first order from this state to any other state where the thermometer is excited, i.e. not the ground state.

$$
\begin{aligned}
\Psi_f &:= \psi_R \otimes \psi_r \otimes a_g^* |0\rangle \\
\psi_R &:= \frac{1}{\sqrt{2\pi}} e^{-i \frac{\frac{\bar{\chi}^2}{2M} t - \bar{\chi} R}{\hbar}} \\
\psi_r &:= \sqrt[4]{\frac{\mu\bar{\omega}}{\hbar\pi}} \frac{1}{\sqrt{2^n n!}} e^{-i\bar{\omega}t\left(\frac{1}{2}+n\right) - \frac{\bar{\omega}\mu r^2}{2\hbar}} H_n\left(\sqrt{\frac{\bar{\omega}\mu}{\hbar}} r\right),
\end{aligned}
\tag{66}
$$

where g is the momentum of the excitation of the quantum field, $\bar{\chi}$ the momentum of the center of mass wave function after the transition and H_n is the nth Hermit polynomial. We pick these final wave functions because they evolve in a simple manner with respect to the free dynamics and other wave functions can be constructed as linear combinations of them.

The reader should be aware that by choosing initial and final wave functions that are not square integrable for the center of mass motion we are bound to get squares of delta distributions for the transition rates. So there is of course still further room for improvement. However, since this can be eliminated by choosing properly normalised wave packets, this should not raise serious concerns about the end results.

The transition amplitude of this process is then given by:

$$\begin{aligned}
M &= i\int_{\mathbb{R}} \mathrm{d}t \,\langle\text{final state}|\,\mathcal{H}(t)\,|\text{initial state}\rangle \\
&= i\int_{\mathbb{R}} \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}R \int_{\mathbb{R}} \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}p \frac{1}{2\pi\sqrt{\omega_p}} \langle 0|\, a_g \sigma \left(R + \frac{m_2}{M} r\right) e^{i\omega_p t} \\
&\qquad e^{\frac{-ip}{\hbar}(R+\frac{m_2}{M}r)} a_p^* \frac{1}{\sqrt{2\pi}} e^{i\frac{\frac{\bar{\chi}^2}{2M}t-\bar{\chi}R}{\hbar}} \frac{1}{\sqrt{2\pi}} e^{-i\frac{\frac{\chi^2}{2M}t-\chi R}{\hbar}} \sqrt[4]{\frac{\mu\bar{\omega}}{\hbar\pi}} \\
&\qquad e^{i\bar{\omega}t(\frac{1}{2}+n)-\frac{\bar{\omega}\mu r^2}{2\hbar}} H_n\left(\sqrt{\frac{\bar{\omega}\mu}{\hbar}} r\right) |0\rangle \sqrt[4]{\frac{\mu\bar{\omega}}{\hbar\pi}} \frac{1}{\sqrt{2^n n!}} e^{-i\frac{\bar{\omega}t}{2}-\frac{\bar{\omega}\mu r^2}{2\hbar}} \\
&= i\delta\left(\omega_g + \frac{\bar{\chi}^2}{2M\hbar} - \frac{\chi^2}{2M\hbar} + \bar{\omega}n\right) \frac{\sigma}{\sqrt{\omega_g}} \int_{\mathbb{R}} \mathrm{d}R \int_{\mathbb{R}} \mathrm{d}r \left(R + \frac{m_2}{M} r\right) \\
&\qquad e^{\frac{-ig}{\hbar}(R+\frac{m_2}{M}r)} \frac{1}{2\pi} e^{i\frac{(\chi-\bar{\chi})R}{\hbar}} \sqrt{\frac{\mu\bar{\omega}}{\hbar\pi}} \frac{1}{\sqrt{2^n n!}} e^{-\frac{\bar{\omega}\mu r^2}{\hbar}} H_n\left(\sqrt{\frac{\bar{\omega}\mu}{\hbar}} r\right) \\
&= i\sqrt{\frac{\mu\bar{\omega}}{\hbar\pi}} \frac{1}{\sqrt{2^n n!}} \delta\left(\omega_g + \frac{\bar{\chi}^2}{2M\hbar} - \frac{\chi^2}{2M\hbar} + \bar{\omega}n\right) \frac{\sigma}{\sqrt{\omega_g}} \\
&\qquad \Bigg[\delta\left(\frac{\chi-\bar{\chi}-g}{\hbar}\right) \frac{m_2}{M} \int_{\mathbb{R}} \mathrm{d}r r e^{\frac{-igm_2 r}{\hbar M}} e^{-\frac{\bar{\omega}\mu r^2}{\hbar}} H_n\left(\sqrt{\frac{\bar{\omega}\mu}{\hbar}} r\right) \\
&\qquad -i\delta'\left(\frac{\chi-\bar{\chi}-g}{\hbar}\right) \int_{\mathbb{R}} \mathrm{d}r e^{\frac{-igm_2 r}{\hbar M}} e^{-\frac{\bar{\omega}\mu r^2}{\hbar}} H_n\left(\sqrt{\frac{\bar{\omega}\mu}{\hbar}} r\right)\Bigg], \qquad (67)
\end{aligned}$$

where δ' denotes the the derivative of the delta distribution.
At this point we notice that for any transition to happen the two conditions dictated by the delta distributions have to be fulfilled a the same time. The argument for the derivative of the delta distribution and the usual distribution in the first summand of the bracket are the same so they introduce the same constraint.
Now we will show that within the boundaries where we trust our model to give transition rates, that are close to the rates one would measure in such an experiment, no transitions are possible.
The two constraints provided by the delta distributions expressed in the following equations:

$$\begin{aligned}
&\chi - \bar{\chi} = g \\
&\frac{1}{2M}\chi^2 - \frac{1}{2M}\bar{\chi}^2 - \hbar n\bar{\omega} - \sqrt{c^4 m_f^2 + c^2 g^2} = 0
\end{aligned} \qquad (68)$$

It can be seen that all solutions of these coupled equations fulfil $|\chi| > Mc$ and are therefore in the relativistic regime, where the dispersion relation of the two particles that constitute the thermometer is no longer correct.
As a first step to proof this claim we plug the first equation into the second and arrange a few common factors such that the remaining equation can be written in

dimensionless form:

$$2Mc^2\left(\left(\frac{\chi}{2Mc}\right)^2-\left(\frac{\bar{\chi}}{2Mc}\right)^2-\frac{\hbar n\bar{\omega}}{2Mc^2}\right)=2Mc^2\sqrt{\frac{m_f^2}{4M^2}+\left(\frac{\chi-\bar{\chi}}{2Mc}\right)^2} \tag{69}$$

Now the dimensionless variables and constants can easily be identified to be:

$$\begin{aligned}
\mathfrak{d} &:= \frac{\chi-\bar{\chi}}{2Mc}\neq 0\\
\mathfrak{e} &:= \frac{\chi+\bar{\chi}}{2Mc}\\
\alpha &:= \frac{m_f}{2M}\geq 0\\
\beta &:= \frac{\hbar n\bar{\omega}}{2Mc^2}\geq 0
\end{aligned} \tag{70}$$

We can $\mathfrak{d}$ assume to be nonzero, because (68) obviously has only the trivial solution where nothing changes for $\chi=\bar{\chi}$.
Now the equation in question looks far more welcoming:

$$\mathfrak{d}\mathfrak{e}-\beta=\sqrt{\alpha^2+\mathfrak{d}^2} \tag{71}$$

Therefore $\mathfrak{e}$ is:

$$\mathfrak{e}=\frac{\beta}{\mathfrak{d}}+\frac{1}{\mathfrak{d}}\sqrt{\alpha^2+\mathfrak{d}^2} \tag{72}$$

Now to estimate the absolute value of χ:

$$\begin{aligned}
|\chi|=Mc|\mathfrak{d}+\mathfrak{e}|=Mc\left|\mathfrak{d}+\frac{\beta}{\mathfrak{d}}+\frac{1}{\mathfrak{d}}\sqrt{\alpha^2+\mathfrak{d}^2}\right|\\
\geq Mc\sqrt{\frac{\alpha^2}{\mathfrak{d}^2}+1}>Mc,
\end{aligned} \tag{73}$$

which proves the claim that there are no transitions in first order in the non-relativistic regime.
For $H_{\text{external}}=0$ we saw that there are in fact transitions, where the detector gets excited, but only in the strongly relativistic regime, for which this model was not built. If we can show that in the non-relativistic regime there are transitions, we can say that this simple model experiences the Unruh effect.
Before we move on and consider the constantly accelerated case I want to give a few comments on the literature for this chapter:
Padmanabhan [14] did not arrive exactly at equation (67), since he used other approximations. However, he arrived at a result involving an equivalent product of delta distributions. He concluded that the system of equations (68) has no solutions. We have just proved that it does, in fact, have solutions, but they do not matter for the validity of our model, due to the fact that they only happen in a regime where we do not trust our model anyway.

4.2.2 Constantly accelerated thermometer

This section will be analogous to the previous one where we showed that our two particle detector will not make a transition from the ground state to an excited state unless it is moving with ultra relativistic momentum. Only now we are talking about constantly accelerated motion. Since the discussion of the corresponding one particle time-independent Schrödinger equation is a bit more involved than the discussion of free evolution of the last section, the following remarks are in order: We choose the Hamiltonian and the corresponding potential to induce non-relativistic constantly accelerated motion $H_{\text{external}} = R\mathfrak{g}M$. Where $\mathfrak{g}$ the constant regulating the acceleration of the system. One should bare in mind that even in classical mechanics there is a difference between setting the trajectory of the system to be that of a constantly accelerated particle and letting a constant force act on the particle. Namely everything else the system might interact with, will have no effect on the motion of the system if we postulate the trajectory to be that of a constantly accelerated particle. If we let the detector be under the influence of a linear potential but also to a field, the motion of the detector will probably not be exactly like that of a constantly accelerated particle. This point will hardly affect our calculations since they are only correct for very early and very late times anyway, but it is always useful to have a clear intuition of what is going to happen in the system under consideration.

We have the time independent Schrödinger equation for the center of mass motion:

$$E\psi = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial R^2}\psi + MR\mathfrak{g}\psi \tag{74}$$

The general solution of this is given by a linear combination of Airy functions:

$$\psi(R) = \mathfrak{m}_1 \mathrm{Ai}\left[\sqrt[3]{\frac{2}{M\hbar^2\mathfrak{g}^2}}\,(-E + M\mathfrak{g}R)\right] + \mathfrak{m}_2 \mathrm{Bi}\left[\sqrt[3]{\frac{2}{M\hbar^2\mathfrak{g}^2}}\,(-E + M\mathfrak{g}R)\right], \tag{75}$$

where E is the eigenvalue of the eigenfunctions Ai and Bi. I would like to stress at this point that even though there is a clear ground state for uniform motion and for the harmonic oscillator and many more relevant quantum mechanical systems, (74) does not admit a ground state, i.e. for every $E \in \mathbb{R}$ and every $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathbb{C}$ the solution given by (75) does in fact solve the differential equation (74). This is analogous to the classical mechanical situation, with a particle subject to a linear potential. There is no trajectory with minimal energy, since moving the particle in negative $\mathfrak{g}$ direction while not changing its momentum will always provide another solution with lower absolute energy.

The definitions of Airy functions in terms of integrals are:

$$\begin{aligned} \mathrm{Ai}(R) &:= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}\mathfrak{t}\, e^{i\left(R\mathfrak{t}+\frac{\mathfrak{t}^3}{3}\right)} \\ \mathrm{Bi}(R) &:= \frac{1}{\pi}\int_{\mathbb{R}+} \mathrm{d}\mathfrak{t}\left[e^{R\mathfrak{t}-\frac{\mathfrak{t}^3}{3}} + \sin\left(R\mathfrak{t}+\frac{\mathfrak{t}^3}{3}\right)\right] \end{aligned} \tag{76}$$

These functions are real for real arguments. For large arguments they behave roughly like:

$$
\begin{aligned}
\mathrm{Ai}(R) &\sim \frac{e^{-\frac{2}{3}R^{\frac{3}{2}}}}{2\sqrt{\pi}\sqrt[4]{R}} && \text{for } R \gg 1 \\
\mathrm{Bi}(R) &\sim \frac{e^{\frac{2}{3}R^{\frac{3}{2}}}}{\sqrt{\pi}\sqrt[4]{R}} && \text{for } R \gg 1 \\
\mathrm{Ai}(R) &\sim \frac{\sin\left(\frac{2}{3}(-R)^{\frac{3}{2}}+\frac{\pi}{4}\right)}{\sqrt[4]{-R}} && \text{for } -R \gg 1 \\
\mathrm{Bi}(R) &\sim \frac{\cos\left(\frac{2}{3}(-R)^{\frac{3}{2}}+\frac{\pi}{4}\right)}{\sqrt[4]{-R}} && \text{for } -R \gg 1
\end{aligned}
\tag{77}
$$

We see that in order to analyse transition rates, Bi is not an appropriate candidate, since it is simply too far from being in $\mathcal{L}^2$, whereas for the Airy Ai an analogous treatment as Fourier analysis for plane waves exists. One can easily derive the following property of the Airy functions by using (76):

$$
\int_{\mathbb{R}} \mathrm{d}\mathfrak{t}\mathrm{Ai}(E+\mathfrak{t})\mathrm{Ai}(\bar{E}+\mathfrak{t}) = \delta\left(E-\bar{E}\right) \tag{78}
$$

Therefore using Airy Ai as initial wave functions for the center of mass motion will be just as good as plane waves were for describing uniform motion.
Our choice for initial and final wave functions is the same as in equations (65) and (66), but the plane waves will now be replaced by Airy Ai:

$$
\begin{aligned}
\Psi_0 &:= \psi_R \otimes \psi_r \otimes |0\rangle \\
\psi_R &:= \sqrt[6]{\frac{2M^2\mathfrak{g}}{\hbar^2}}\mathrm{Ai}\left[\sqrt[3]{\frac{2}{M\hbar^2\mathfrak{g}^2}}\left(-E+M\mathfrak{g}R\right)\right]e^{-\frac{iEt}{\hbar}} \\
\psi_r &:= \sqrt[4]{\frac{\mu\bar{\omega}}{\hbar\pi}}e^{-i\frac{\bar{\omega}t}{2}-\frac{\bar{\omega}\mu r^2}{2\hbar}} \\
\Psi_f &:= \bar{\psi}_R \otimes \bar{\psi}_r \otimes a_g^* |0\rangle \\
\psi_R &:= \sqrt[6]{\frac{2M^2\mathfrak{g}}{\hbar^2}}\mathrm{Ai}\left[\sqrt[3]{\frac{2}{M\hbar^2\mathfrak{g}^2}}\left(-\bar{E}+M\mathfrak{g}R\right)\right]e^{-\frac{i\bar{E}t}{\hbar}} \\
\psi_r &:= \sqrt[4]{\frac{\mu\bar{\omega}}{\hbar\pi}}\frac{1}{\sqrt{2^n n!}}e^{-i\bar{\omega}t\left(\frac{1}{2}+n\right)-\frac{\bar{\omega}\mu r^2}{2\hbar}}H_n\left(\sqrt{\frac{\bar{\omega}\mu}{\hbar}}r\right),
\end{aligned}
\tag{79}
$$

which means for the transition amplitude:

$$
\begin{aligned}
M = {} & i\int_{\mathbb{R}} \mathrm{d}t \, \langle \text{final state}| \, \mathcal{H}(t) \, |\text{initial state}\rangle \\
= {} & i\int_{\mathbb{R}} \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}R \int_{\mathbb{R}} \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}p \frac{1}{2\pi\sqrt{\omega_p}} \langle 0| \, a_g \sigma(R + \frac{m_2}{M} r) e^{i\omega_p t} e^{\frac{-ip}{\hbar}(R + \frac{m_2}{M} r)} a_p^* \\
& \sqrt[3]{\frac{2M^2\mathfrak{g}}{\hbar^2}} \mathrm{Ai}\left[\sqrt[3]{\frac{2}{M\hbar^2\mathfrak{g}^2}} \left(-\bar{E} + M\mathfrak{g}R\right)\right] e^{\frac{i\bar{E}t}{\hbar}} \mathrm{Ai}\left[\sqrt[3]{\frac{2}{M\hbar^2\mathfrak{g}^2}} \left(-E + M\mathfrak{g}R\right)\right] \\
& e^{-\frac{iEt}{\hbar}} \sqrt{\frac{\mu\bar{\omega}}{\hbar\pi}} \frac{1}{\sqrt{2^n n!}} e^{i\bar{\omega}t\left(\frac{1}{2}+n\right) - \frac{\bar{\omega}\mu r^2}{2\hbar}} H_n\left(\sqrt{\frac{\bar{\omega}\mu}{\hbar}} r\right) |0\rangle \, e^{-i\frac{\bar{\omega}t}{2} - \frac{\bar{\omega}\mu r^2}{2\hbar}} \\
= {} & \frac{\sqrt{\mu\bar{\omega}}}{2\pi} \frac{1}{\sqrt{2^n n!}} \frac{i\sigma}{\sqrt{|q|\,\omega_q}} \delta\left(\omega_q + \frac{\bar{E}}{\hbar} - \frac{E}{\hbar} + n\frac{\bar{\omega}}{2}\right) \sqrt[6]{\frac{2M^2\mathfrak{g}}{\hbar^2}} \\
& e^{-i\frac{q^3}{24M^2\mathfrak{g}\hbar}} e^{i\frac{(E-\bar{E})^2}{2q\mathfrak{g}\hbar}} e^{i\frac{q(E+\bar{E})}{2M\mathfrak{g}\hbar}} \int_{\mathbb{R}} \mathrm{d}r e^{-i\frac{qm_2 r}{\hbar M}} H_n\left(\sqrt{\frac{\bar{\omega}\mu}{\hbar}} r\right) e^{-\frac{\bar{\omega}\mu r^2}{\hbar}} \left\{\frac{m_2}{M} r\right. \\
& \left. + i\hbar\left[-\frac{1}{2q} + \frac{i}{2\mathfrak{g}\hbar}\left(-\frac{q^2}{4M^2} - \frac{\left(E-\bar{E}\right)^2}{q^2} + \frac{\left(E+\bar{E}\right)}{M}\right)\right]\right\},
\end{aligned} \tag{80}
$$

where for the third equality one has to evaluate the integral involving the Airy functions, which can be obtained by using the definition of those functions (76), changing the order of integrals and using the theorem of residua:

$$
\begin{aligned}
& \int_{\mathbb{R}} \mathrm{d}R \mathrm{Ai}\left[\sqrt[3]{\frac{2}{M\hbar^2\mathfrak{g}^2}} \left(-E + M\mathfrak{g}R\right)\right] \mathrm{Ai}\left[\sqrt[3]{\frac{2}{M\hbar^2\mathfrak{g}^2}} \left(-\bar{E} + M\mathfrak{g}R\right)\right] e^{-i\frac{qR}{\hbar}} \\
& = \frac{1}{2}\sqrt{\frac{\hbar^2}{\pi\,|q|}} \sqrt[6]{\frac{\hbar^2}{2M^2\mathfrak{g}}} e^{-i\frac{q^3}{24M^2\mathfrak{g}\hbar}} e^{i\frac{(E-\bar{E})^2}{2q\mathfrak{g}\hbar}} e^{i\frac{q(E+\bar{E})}{2M\mathfrak{g}\hbar}}
\end{aligned} \tag{81}
$$

We do not worry about convergence of these integrals since we think of all the expressions occuring in (80) as distributions, which will behave the way they should once one introduces wave packets to evaluate the transition rates.
The most notable difference between equations (80) and (67) is that in the former there is only one delta distribution which enforces energy conservation, whereas in the latter there are two delta distributions enforcing conservation of momentum on top of conservation of energy. Since energy conservation is satisfied as long as the difference between the eigenvalues of the center of mass motion is given by the energy, the quantum field acquires, plus the energy of the excitation of the detector, there will be transitions between states of the center of mass motion that are very close in situations where the mass of the quantum field and $\bar{\omega}$ are small. Thus there clearly are transitions even in the non-relativistic regime, at least for small spring constants and small mass of the Klein-Gordon field.
One has to bear in mind that this is only a very crude approximation. We only consider transitions in first order of the weak coupling of our detector to the quantum

field. The reader who is familiar with second quantised models of quantum electrodynamics will have noticed that this model is very similar to the Nelson Model for scalar electrodynamics. Therefore the particle that interacts with the quantum field should build up its natural field, i.e. the analogous one to the Liénard Wiechert fields of classical electrodynamics. This does not happen in first order so we expect the second and higher order corrections to contribute many important aspects of this model. In fact it is quite clear that this model should also exhibit the behaviour of building up the natural field of the charged particle, because this effect was already proven to take place in an almost identical model without acceleration by Felix Hänle [9]. However, since we already saw a qualitative difference between the behavior with and without constant external acceleration in first order, we expect this difference to persist in a more rigorous treatment. i.e. the simple detector will get excited if it is accelerated and it will not get excited if it is not accelerated.
All of this is completely analogous to standard electrodynamics, except that in quantum mechanics it is much easier to model detectors than it is in classical physics. A uniformly moving charge does not radiate while an accelerated charge does, the whole difference is that the object interacting with the quantum field in this model, i.e. the charge, does not only radiate but it is also able to detect its own radiation. I would like the reader to remember this very simple picture of the Unruh effect, that it is essentially nothing but the radiation induced by acceleration.
There is of course the second aspect of this effect, the thermal character of the detection rates. In this thesis we showed that previous discussions on this subject are very problematic. The whole discussion of the two quantisation schemes in 2.1.1 turned out to be very problematic at almost every level of its line of argument, therefore all one should learn from this discussion is that this treatment is not fit to analyse the Unruh effect. All that remains in favour of the thermal character is that according to 2.1.2 in the limit of very small two level detectors the transition rate has the same form as the Planck formula, as long as the quantum field the detector interacts with is massless and the dimension of spacetime is two or four. For the model which we discussed in section 4.2 there is no hope of obtaining the Bose distribution in first order and it seems unlikely to appear if one includes higher orders. As this very simple and intuitive model does not exhibit any thermal character, it is not clear what a real detector might show. In fact it might very well depend on the very details of the detector that is used.

4.3 The Schrödinger-Unruh Effect

We continue with the second toy model. This model is a variation of the first one. Instead of coupling the particle to a second quantised Klein-Gordon field we couple it to a quantum field that is obtained by second quantisation of the free Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m_f}\Delta\psi \tag{82}$$

The purpose of using a second quantised Schrödinger field is to see whether a purely classical model that is analogous to the first model still behaves in the same way. The solutions one obtains by solving the time independent Schrödinger equation are:

$$\psi_k(x,t) = e^{ikx/\hbar} \tag{83}$$

Using this we Fourier transform the general solution to equation (82).

$$\psi(x) = \int_{\mathbb{R}} \mathrm{d}k \left(\psi_k(x)a_k + \psi_k^*(x)\bar{a}^\dagger\right) \tag{84}$$

Now we treat the a and $\bar{a}$ as operators on the Fock space over $\mathcal{L}^2$, fulfilling commutation relations anologous to (16):

$$\begin{aligned} [\bar{a}_k, \bar{a}_p^\dagger] &= [a_k, a_p^\dagger] = \delta(k-p) \\ [a_k, a_p] &= [\bar{a}_k, \bar{a}_p] = [a_k, \bar{a}_p^\dagger] = [a_k, \bar{a}_p] = 0 \end{aligned} \tag{85}$$

With these quantum field operators at hand we are ready to propose the new model. We choose for the Hamiltonian according to (61) with $H_{\text{external}} = 0$, $E_{\text{kinetic,field}} = \frac{p^2}{2m_f}$ and $H_{\text{interaction}}$ analogous to (63) but with the new field operator just obtained:

$$\begin{aligned} H := \frac{-\hbar^2}{2M}\Delta_R + \frac{-\hbar^2}{2\mu}\Delta_r + \frac{\mu}{2}\omega^2 r^2 + \int \frac{\mathrm{d}p}{2\pi}\frac{p^2}{2m_f}a_p^\dagger a_p + \int \frac{\mathrm{d}p}{2\pi}\frac{p^2}{2m_f}\bar{a}_p^\dagger \bar{a}_p + \\ \sigma\left(R + \frac{m_2}{M}r\right)\int \frac{\mathrm{d}p}{2\pi}\left(e^{\frac{\mathrm{i}p}{\hbar}(R+\frac{m_2}{M}r)}a_p + e^{\frac{-\mathrm{i}p}{\hbar}(R+\frac{m_2}{M}r)}\bar{a}_p^\dagger\right) \end{aligned} \tag{86}$$

such that in the interaction picture the generator of the S-matrix is generated by:

$$H_I(t) := \sigma\left(R + \frac{m_2}{M}r\right)\int \frac{\mathrm{d}p}{2\pi}\left(e^{-\mathrm{i}t\frac{p^2}{2m_f\hbar} + \frac{\mathrm{i}p}{\hbar}(R+\frac{m_2}{M}r)}a_p + e^{\mathrm{i}t\frac{p^2}{2m_f\hbar} - \frac{\mathrm{i}p}{\hbar}(y+\frac{m_2}{M}x)}\bar{a}_p^\dagger\right) \tag{87}$$

The calculation is then completely analogous to equation (67), where we also end up with two delta distributions enforcing momentum and energy conservation:

$$\begin{aligned} &\chi - \bar{\chi} = g \\ &\frac{1}{2M}\chi^2 - \frac{1}{2M}\bar{\chi}^2 - \hbar n\bar{\omega} - \frac{g^2}{2m_f} = 0 \end{aligned} \tag{88}$$

This is essentially a quadratic equation for χ and $\bar{\chi}$. This equation has real solutions whenever the following inequality is fulfilled.

$$\chi^2 \geq 2\frac{M}{m_f}(m_f + M)\hbar n\bar{\omega} \tag{89}$$

Clearly momenta fulfilling equation (89) can be chosen to be in the non-relativistic regime, or in other words, the detector will get excited if it moves fast initially even if it is not accelerated. One might wonder whether this is just the consequence of the Hamiltonian (86) not being Galilei invariant, but a simple thought experiment shows that a mechanical grid exhibits the same effect.
Consider a one dimensional quantum mechanical field constructed from a chain of atoms each connected to the previous and to the next one by a harmonic potential and by letting the spacing between the atoms tend to zero. The kinetic energy term for each atom is the usual one for non-relativistic physics. We couple this field to an external particle, again using an harmonic potential. This setting clearly is Galilei invariant since all the coupling is realised by harmonic potentials and the kinetic terms all are Galilei invariant. Even in this setting the situation is clearly very different for a moving particle than it is for a particle at rest. In one setting the external particle and the part of the chain, it is tied to, are in relative motion whereas in the other setting they are at rest relative to each other.
A fact at the very heart of relativity plays an important role for our intuition about fields. The speed of light is the same in every inertial system can be rephrased as the electromagnetic field does not have a natural rest frame. Precisely this fact is the main difference between relativistic and non relativistic fields, so in hindsight we should not be surprised to see the detector clicking even though it is not accelerated. Therefore we see that the Unruh effect, just like the fact that only accelerated particles, as opposed to uniformly moving particles, emit radiation is of relativistic nature. For fully non-relativistic fields to exhibit the same effect, one would have to construct a non-relativistic field that does not have a natural inertial frame.

5 Discussion of results and conclusion

In retrospect, what would one anticipate the Unruh effect to be from reading current and historical literature, and what did we actually find?
In the introduction we noted that the Unruh effect is commonly referred to as: "*a surprising prediction of quantum field theory: From the point of view of an accelerating observer or detector, empty space contains a gas of particles at a temperature proportional to the acceleration.*" This is not what should be concluded from the discussion above.
Let us separate this statement into two, to be able to refer to them more easily. Statement one is:

"*From the point of view of an accelerated* ***observer****, empty space contains a gas of particles at a temperature proportional to the acceleratio*" (1)

Statement two is:

"*From the point of view of an accelerated* ***detector****, empty space contains a gas of particles at a temperature proportional to the acceleration*" (2)

First of all, the notion of "*observer*" is too vague to disprove every possible interpretation of statement (1). However, most authors draw this conclusion from the fact that the Minkowski vacuum appears as a thermal state when the quantum field is described as a second quantised field in the Rindler frame of reference. This turned out to be impossible, since the Minkowski vacuum is not part of this description of the quantum field, which we showed in chapter 2 . We conclude that in the usual sense statement (1) is wrong and for the sake of clarity one should avoid using vague statements such as this one.
Concerning statement (2) the models, described in chapter 4 and the Unruh DeWitt detector described in section 2.1.2, click when accelerated, so one might say that from their point of view the vacuum contains particles. However, one might also depict this in a different way. Since a detector is made up of particles that interact with the quantum field, i.e. charged particles, these particles will generate field modes if they are accelerated just like in classical electrodynamics. These modes will then effect the detector, hence it clicks. The detector really detects the particles it created itself, and therefore the distribution of particles will generally depend strongly on the kind of detector motion and on the details of the detector.
Although we could confirm with our intuitive and simple model that constantly accelerated detectors will click, we could not confirm that the detection rate is thermal in the cases where the Unruh DeWitt detector predicts this to be the case.

There are two possible reasons for this. Firstly the Unruh DeWitt detector is far too idealised, i.e. no realistic detector model will get excited at the rate predicted by this model, or secondly our model was oversimplified.
The dispersion relation we picked for our model is non-relativistic. This might turn out to be very important, since we need to average over very long times in the Unruh DeWitt model to arrive at thermal rates. Especially since for early and late times the detector in the Unruh DeWitt model is close to the speed of light. For this model, even though the period over which the effective measurement is taking place, please keep in mind the function of the adiabatic switching factor, the detector itself has to spend most of its time close to the speed of light. Whereas for our own model, for very early and very late times the detector will be much faster than the speed of light, therefore it might be wrong to consider very large measurement times.
Furthermore, we could show with our intuitive model that the energy that is necessary to excite the detector and to emit the particle both are provided by the accelerating agent. This has been claimed before many times but only Padmanabhan [14], whose line of argument we corrected, actually tried to prove it.
All in all the situation is very analogous to classical electrodynamics where only accelerated particles, but not uniformly moving ones emit radiation. The main difference between the two settings is, that in quantum mechanics one can easily model a detector and must not only speak about individual particles. The thermal rates must be a consequence of the interaction of the detector system with the waves it created itself, it might be worthwhile to investigate the mechanism responsible for this effect in detail.
It is also a promising idea to find differences and similarities between classical systems including radiation reaction terms such as the Abraham-Lorentz-Dirac equation and the system of a quantum field interacting with an accelerated detector.

6 Appendix

6.1 Computing the transition rate of the Unruh-DeWitt detector for uniform acceleration

In this section we calculate the transition rate o the uniformly accelerated Unruh-DeWitt detector, which is immersed in the Minkowski vacuum, without use of the quantisation scheme in curved coordinates. This was first done by Meyer [12] and generalized by Takagi [17]. The detector is in an internal state $|E_1\rangle$ at early proper time τ_0 and after it clicks it will be in the internal state $|E_2\rangle$. The state of the quantum field is the vacuum $|0\rangle$ at early times $-\tau_0$ and will be some state $|\psi\rangle$ at late times $+\tau_0$. We will let τ_0 tend to infinity later on. The interaction Hamiltonian in the interaction picture is given by equation (51).
Therefore the transition amplitude up to first order is:

$$i \langle E_2| \otimes \langle\psi| \int_{-\tau_0}^{\tau_0} \mathrm{d}\tau \; e^{-s|\tau|} \left(M(\tau) \otimes \phi(\tau) + M^\dagger(\tau) \otimes \phi^\dagger(\tau)\right) |E_1\rangle \otimes |0\rangle \tag{90}$$

Using (50) we find that this is equal to:

$$\begin{aligned} &= i \langle E_2| M(0) |E_1\rangle \int_{-\tau_0}^{\tau_0} \mathrm{d}\tau \; e^{i(E_2-E_1)\tau - s|\tau|} \langle\psi| \phi(\tau) |0\rangle \\ &+ i \langle E_2| M^\dagger(0) |E_1\rangle \int_{-\tau_0}^{\tau_0} \mathrm{d}\tau \; e^{i(E_2-E_1)\tau - s|\tau|} \langle\psi| \phi^\dagger(\tau) |0\rangle . \end{aligned} \tag{91}$$

In order to arrive at the transition rate we take the absolute value squared of the amplitude and sum it over all field states $|\psi\rangle$ and divide it by $2\tau_0$. We let τ_0 tend to infinity to simulate the detector measuring for an arbitrarily long time interval. For every nonzero s the state of the detector will be disturbed by switching it on and off, in order to avoid these disturbances we take the limit $s \to 0$. We can separate the transition rate into two factors:

$$R_{1\to 2} = \left(|\langle E_2| M(0) |E_1\rangle|^2 + \left|\langle E_2| M^\dagger(0) |E_1\rangle\right|^2\right) F(E_2 - E_1), \tag{92}$$

where F is given by:

$$F(\omega) := \lim_{s\to 0} \lim_{\tau_0\to\infty} \frac{1}{2\tau_0} \int_{-\tau_0}^{\tau_0} \mathrm{d}\tau \int_{-\tau_0}^{\tau_0} \mathrm{d}\tilde{\tau} \; e^{-i\omega(\tau-\tilde{\tau}) - s|\tau| - s|\tilde{\tau}|} \langle 0| \phi(\tau)\phi^\dagger(\tilde{\tau}) |0\rangle \tag{93}$$

This term in the integral of F is the only contributing term, since:

$$\begin{aligned} &\langle 0|\, \phi^\dagger(t)\phi(\tilde{t})\, |0\rangle = \langle 0|\, \phi(t)\phi^\dagger(\tilde{t})\, |0\rangle \\ &\langle 0|\, \phi(t)\phi(\tilde{t})\, |0\rangle = 0 \end{aligned} \tag{94}$$

Since in equation (92) the dependence on the monopole moment M is completely contained in the first factor, which is also independent of the details of the field operator, we will ignore this factor and focus on F alone.
We begin by rewriting the second factor of the integrand of equation (93):

$$\langle 0|\, \phi(\tau)\phi^\dagger(\tilde{\tau})\, |0\rangle = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 k}{(2\pi)^3 2\sqrt{\omega_k}} e^{-ik\cdot(x(\tau)-x(\tilde{\tau}))} \tag{95}$$

For mathematical convenience we add an infinitesimal imaginary number $i\epsilon$ to the expression involving k^0, where ϵ is an arbitrary small positive number. This rather ad hoc seeming procedure can in fact be justified by not working with a point-like detector, but rather with a sequence of detectors whose size tend to zero. The mathematical description of this sequence then turns out to be equivalent to adding this ad hoc summand. For more details see §3.2 of [17].
The integration measure is Lorentz invariant for orthochronous Lorentz transformations as can easily be seen by transforming it into a four dimensional integral:

$$\frac{\mathrm{d}^3 k}{(2\pi)^3\sqrt{\omega_k}} = \frac{\mathrm{d}^4 k}{(2\pi)^3}\Theta(k^0)\delta(k\cdot k - m_f^2). \tag{96}$$

We make use of this fact by transforming k to $\bar{k}$.

$$\begin{aligned} \bar{k}^0 &:= \frac{\mathrm{sgn}(x^0(\tau)-x^0(\tilde{\tau}))}{\Delta}\left[\left(x^0(\tau)-x^0(\tilde{\tau})\right)k^0 - \left(x^1(\tau)-x^1(\tilde{\tau})\right)k^1\right] \\ \bar{k}^1 &:= \frac{\mathrm{sgn}(x^0(\tau)-x^0(\tilde{\tau}))}{\Delta}\left[-\left(x^1(\tau)-x^1(\tilde{\tau})\right)k^0 + \left(x^1(\tau)-x^1(\tilde{\tau})\right)k^1\right] \\ \bar{k}^2 &:= k^2 \\ \bar{k}^3 &:= k^3 \end{aligned} \tag{97}$$

We also replace k^0 by $\bar{k}^0$ in the ϵ term, because the transformation is orthochronous and we do not care for the details of this term since we take the limit $\epsilon \to 0$ at the end anyway. This simplifies the integral in equation (95)

$$\langle 0|\, \phi(t)\phi^\dagger(\tilde{t})\, |0\rangle = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 \bar{k}}{(2\pi)^3 2\omega_{\bar{k}}} e^{-\omega_{\bar{k}} z}, \tag{98}$$

where z is given by:

$$\begin{aligned} z :&= i\Delta \mathrm{sgn}\left(x^0(\tau)-x^0(\tilde{\tau})\right) + \epsilon \\ &= \frac{i2}{a}\sinh\left(\frac{a}{2}(\tau-\tilde{\tau}-i\epsilon)\right), \end{aligned} \tag{99}$$

where for the second equality the concrete form of the motion (49) and the fact that we consider all terms in ϵ that give a very small but positive first order correction to be equivalent.
Evaluating the integration of the two sphere and changing variables from $\bar{k}$ to $u := \frac{\omega_{\bar{k}}}{m_f}$ we arrive at:

$$\begin{aligned}\langle 0| \phi(t)\phi^\dagger(\tilde{t}) |0\rangle &= \frac{1}{(2\pi)^2}\int_0^\infty \frac{\mathrm{d}\bar{k}}{\omega_{\bar{k}}}\bar{k}^2 e^{-\omega_{\bar{k}} z} = \frac{m_f^2}{(2\pi)^2}\int_1^\infty \mathrm{d}\bar{u}\sqrt{u^2-1}e^{-m_f u z} \\ &= \frac{m_f}{(2\pi)^2}\frac{\mathrm{K}_1(m_f z)}{z}\end{aligned} \tag{100}$$

K_1 is a modified Bessel function of the second kind.
Now we switch to the massless case, since for $m_f \neq 0$ the full function F, given by (93), is very hard to characterise. We comment on asymptotic results derived by Takagi[17] in section 2.1.2.
For $m_f = 0$ the first integral in equation (100) is easily evaluated.

$$\langle 0| \phi(t)\phi^\dagger(\tilde{t}) |0\rangle = \frac{1}{(2\pi)^2}\int_0^\infty \frac{\mathrm{d}\bar{k}}{\omega_{\bar{k}}}\bar{k}^2 e^{-\omega_{\bar{k}} z} = \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}\bar{k}\bar{k}e^{-\bar{k}z} = \frac{1}{(z2\pi)^2} \tag{101}$$

Plugging this into equation 93 we get:

$$\begin{aligned}F(\omega) &= \lim_{s\to 0}\lim_{\tau_0\to\infty}\frac{1}{2\tau_0}\int_{-\tau_0}^{\tau_0}\mathrm{d}\tau\int_{-\tau_0}^{\tau_0}\mathrm{d}\tilde{\tau}\; e^{-i\omega(\tau-\tilde{\tau})-s|\tau|-s|\tilde{\tau}|}\frac{1}{(z2\pi)^2} \\ &= \lim_{s\to 0}\lim_{\tau_0\to\infty}\frac{1}{2\tau_0 4\pi^2}\int_{-\tau_0}^{\tau_0}\mathrm{d}\tau\int_{-\tau_0}^{\tau_0}\mathrm{d}\tilde{\tau}\;\frac{e^{-i\omega(\tau-\tilde{\tau})-s|\tau|-s|\tilde{\tau}|}}{(\frac{i2}{a}\sinh(\frac{a}{2}(\tau-\tilde{\tau}-i\epsilon)))^2} \\ &= \frac{-a^2}{16\pi^2}\int_{-\infty}^{\infty}\mathrm{d}\tau\; e^{-i\omega\tau}\frac{1}{\sinh(\frac{a}{2}(\tau-i\epsilon))^2}\end{aligned} \tag{102}$$

This integral can in fact be evaluated in terms of Euler's Beta function. It can also be obtained by the theorem of residues and the periodicity of the second factor of the integrand with period $\frac{i2\pi}{a}$:
We pick the second option, therefore we close the contour of integration around the pole to a rectangle, whose upper line is at $\Im(\tau) = \frac{2\pi}{a}$ and whose left and right lines are at $\pm R$ with $R \gg 1$. (See figure 3)
We let R tend to infinity and notice that the sides do not contribute to the integral in this limit, we find for F:

$$\begin{aligned}F(\omega) &= \frac{-a^2}{16\pi^2}\int_{-\infty}^{\infty}\mathrm{d}\tau\; e^{-i\omega\tau}\frac{1}{\sinh(\frac{a}{2}(\tau-i\epsilon))^2} \\ &= \frac{-a^2}{16\pi^2}\left[2\pi i\mathrm{Res}\left(e^{-i\omega\tau}\frac{1}{\sinh(\frac{a}{2}(\tau-i\epsilon))^2}\right)\right. \\ &\left.+\int_{-\infty}^{\infty}\mathrm{d}\tau\; e^{-i\omega(\tau+\frac{i2\pi}{a})}\frac{1}{\sinh(\frac{a}{2}(\tau-i\epsilon))^2}\right]\end{aligned} \tag{103}$$

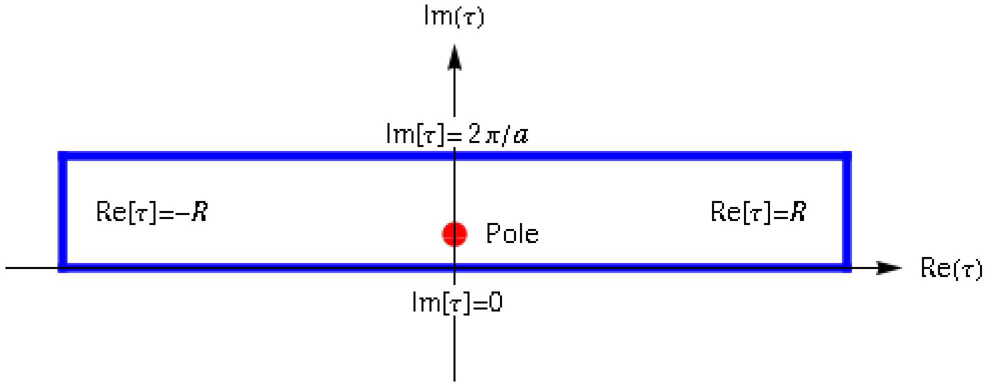

Figure 3: contour of integration

At this point we see that the integrand picks up a factor of $e^{\frac{2\pi\omega}{a}}$ and remains unchanged otherwise. This leads to:

$$F(\omega) = \frac{-a^2}{16\pi^2} 2\pi i \mathrm{Res}\left(e^{-i\omega\tau}\frac{1}{\sinh(\frac{a}{2}(\tau - i\epsilon))^2}\right) + e^{2\pi\frac{\omega}{a}} F(\omega) \tag{104}$$

$$F(\omega) = \frac{a^2}{8\pi}\frac{1}{e^{\frac{2\pi\omega}{a}} - 1} i\mathrm{Res}\left(e^{-i\omega\tau}\frac{1}{\sinh(\frac{a}{2}(\tau - i\epsilon))^2}\right) \tag{105}$$

All that is left to do is to find the value of the residue of the integrand at $i\epsilon$, which we do by using the standard formula for residues at poles of second order:

$$\begin{aligned} \mathrm{Res}\left(e^{-i\omega\tau}\frac{1}{\sinh(\frac{a}{2}(\tau - i\epsilon))^2}\right) &= \lim_{\tau \to i\epsilon} \frac{\mathrm{d}}{\mathrm{d}\tau}\left[(\tau - i\epsilon)^2 e^{-i\omega\tau}\frac{1}{\sinh(\frac{a}{2}(\tau - i\epsilon))^2}\right] \\ = -i\omega\frac{4e^{\omega\epsilon}}{a^2} \end{aligned} \tag{106}$$

Finally we let ϵ tend to zero. Overall we get:

$$F(\omega) = \frac{1}{2\pi}\frac{\omega}{e^{\frac{2\pi\omega}{a}} - 1} \tag{107}$$

6.2 Fock space construction

Since we want to work with relativistic quantum fields as well as with non relativistic quantum fields we first need to develop the necessary tools. This chapter only contains well known results and is heavily inspired by the book by Schweber. In the following we will introduce the required formalism, because the reader may not be familiar with it. We will derive the formalism for relativistic and classic fields in parallel.

Let $\{\lambda_i\}_{i\in\mathbb{N}}$ denote an orthonormal basis of the one particle Hilbert space $\mathcal{L}^2(\mathbb{R}^3)$. Since we are only interested in N particle configurations fulfilling the Bose statistics we will only consider many particle configurations in the space $\mathcal{H}_N$ generated by $\{S(\lambda_{a_1} \otimes \lambda_{a_2} \otimes \cdots \otimes \lambda_{a_N}) \, | a_i \in \mathbb{N} \, \forall i\}$. Where S denotes the symmetrisation operator. The Fock space of interest is then given by:

$$\mathcal{F}(\mathcal{H}) := \overline{\bigoplus_{N=0}^{\infty} \mathcal{H}_N} \tag{108}$$

where $\mathcal{H}_0$ is identified with $\mathbb{C}$ and the bar denotes completion with respect to the norm induced by the norms on $\mathcal{H}$ and the direct sum.
Any member of the generating set is clearly uniquely specified by specifying which λ_i occur in the tensor product and how often they occur. We therefore denote the members of the generating set of $\mathcal{F}$ by $|n_1, n_2, ..\rangle$ with $n_i \in \mathbb{N}_0$ for all i and $\sum_{i\in\mathbb{N}} n_i = N$ for some $N \in \mathbb{N}_0$.
We now introduce the annihilation operator a_i of the basis element λ_i by:

$$a_i \, |n_1, n_2, ...n_i, ...\rangle = \sqrt{n_i} \, |n_1, n_2, ..., n_i - 1, ...\rangle \tag{109}$$

Its adjoint operator, also known as the creation operator, acts as:

$$a_i^* \, |n_1, n_2, ...n_i, ...\rangle = \sqrt{n_i + 1} \, |n_1, n_2, ..., n_i + 1, ...\rangle \tag{110}$$

They obey the well-known commutation relations.

$$\begin{aligned} [a_i, a_k^*] &= \delta_{ik} \\ [a_i, a_k] &= [a_i^*, a_k^*] = 0 \end{aligned} \tag{111}$$

With these tools at hand we can now define the creation and annihilation operators which are being used in the physics literature:

$$\begin{aligned} a_{\vec{k}} &:= \sum_{l=1}^{\infty} \frac{1}{\sqrt{2\pi}^3} \int_{\mathbb{R}^3} \mathrm{d}x^3 e^{i\vec{k}\cdot\vec{x}} \lambda_l(\vec{x}) a_l \\ a_{\vec{k}}^* &:= \sum_{l=1}^{\infty} \frac{1}{\sqrt{2\pi}^3} \int_{\mathbb{R}^3} \mathrm{d}x^3 e^{-i\vec{k}\cdot\vec{x}} \lambda_l^*(\vec{x}) a_l^* \end{aligned} \tag{112}$$

Which fulfill something often called the canonical commutation relations:

$$\begin{aligned} [a_{\vec{l}}, a_{\vec{k}}^*] &= \delta^3(\vec{l} - \vec{k}) \\ [a_{\vec{l}}, a_{\vec{k}}] &= [a_{\vec{l}}^*, a_{\vec{k}}^*] = 0, \end{aligned} \tag{113}$$

where $\delta^3(\cdot)$ denotes the three dimensional Dirac delta distribution.
The number operator N is also often of great interest, its definition is very natural and it can be expressed via the usual annihilation and creation operators as follows:

$$\begin{aligned} N &:= \sum_{c=0}^{\infty} a_c^* a_c = \sum_{c=0}^{\infty} \sum_{l=0}^{\infty} \int_{\mathbb{R}^3} \mathrm{d}x^3 \int_{\mathbb{R}^3} \mathrm{d}y^3 \lambda_c^*(x) a_c^* \lambda_l(y) a_l \delta^3(\vec{x} - \vec{y}) \\ &= \int_{\mathbb{R}^3} \mathrm{d}k^3 \sum_{c=0}^{\infty} \int_{\mathbb{R}^3} \frac{\mathrm{d}x^3}{\sqrt{2\pi}^3} \lambda_c^*(\vec{x}) a_c^* e^{-i\vec{k}\cdot\vec{x}} \sum_{l=0}^{\infty} \int_{\mathbb{R}^3} \frac{\mathrm{d}y^3}{\sqrt{2\pi}^3} \lambda_l(\vec{y}) a_l e^{i\vec{k}\cdot\vec{y}} \\ &= \int_{\mathbb{R}^3} \mathrm{d}k^3 a_k^* a_k \end{aligned} \tag{114}$$

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Declaration of Authorship

I hereby confirm that I prepared this master thesis independently and on my own, by exclusive reliance on the tools and literature indicated therein.

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