Mathematisches Institut der LMU
Prof. P. T. Nam
D.T. Nguyen

Functional Analysis II
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Excercise Sheet 9 for 22. 12. 2017

In this exercise, we use the Birman-Schwinger method to prove that $-\Delta + V$ has finite number of negative eigenvalues if V is sufficiently nice.

First, assume $0 \geq V \in L^{3/2}(\mathbb{R}^3)$. Then we know that $-\Delta + V$ is a self-adjoint operator on $H^2(\mathbb{R}^3)$ and $\sigma_{\text{ess}}(-\Delta + V) = [0, \infty)$. Moreover, for every $e > 0$, $-\Delta + V$ has an eigenvalue $-e$ if and only if K_e has an eigenvalue 1, where

$$K_e := \sqrt{|V|}(-\Delta + e)^{-1}\sqrt{|V|}$$

is a non-negative, Hilbert-Schmidt operator on $L^2(\mathbb{R}^3)$ with the integral kernel

$$K_e(x, y) := \sqrt{|V(x)|}G_e(x - y)\sqrt{|V(y)|}$$

where

$$G_e(x) = \frac{1}{4\pi|x|}\exp(-\sqrt{e}|x|)$$

(or equivalently $\widehat{G}_e(k) = (4\pi^2 k^2 + e)^{-1}$).

9.1. Let $\mu_j(e)$ be the j-th largest eigenvalue of K_e. Prove that for every $j \in \mathbb{N}$, the mapping $e \mapsto \mu_j(e)$ is continuous and monotonically decreasing on $(0, \infty)$.

Hint: You can use the min-max principle for $-K_e$.

9.2. Let N_e be the number of negative eigenvalues of $-\Delta + V$ which are $< -e$. Prove that N_e is equal to the number of eigenvalues of K_e which are > 1. Deduce the Birman-Schwinger bound

$$N_e \leq \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |K_e(x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y.$$

9.3. Now consider a general potential $V : \mathbb{R}^3 \to \mathbb{R}$ with $V_+ \in L^1_{\text{loc}}(\mathbb{R}^3)$ and $V_- \in L^{3/2}(\mathbb{R}^3)$, where $V_\pm(x) = \max(\pm V(x), 0)$. Since $-\Delta + V$ is bounded from below on $C_c^\infty(\mathbb{R}^3)$, it can be extended to be a self-adjoint operator by Friedrichs method. Prove that the number of negative eigenvalues of $-\Delta + V$ is not larger than $C\|V_-\|^2_{L^{3/2}}$, where C is a universal constant independent of V.