Mathematisches Institut der LMU
Prof. P. T. Nam
D.T. Nguyen

Functional Analysis II
Winter Semester 2017/18
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Excercise Sheet 8 for 15. 12. 2017

8.1. Let $1 \le p < q < r \le \infty$. Prove that for every $f \in L^q(\mathbb{R}^d)$ and $\varepsilon > 0$, we can write

$$f = f_1 + f_2$$

such that $f_1 \in L^p(\mathbb{R}^d)$, $\|f_1\|_{L^p} \le \varepsilon$ and $f_2 \in L^r(\mathbb{R}^d)$.

8.2. Let $V \in L^p(\mathbb{R}^d)$ where $1 \le p < \infty$ if $d = 1$; $1 < p < \infty$ if $d = 2$; and $d/2 \le p < \infty$ if $d \ge 3$. Let $\{u_n\}$ be a sequence in $H^1(\mathbb{R}^d)$ such that $u_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^d)$. Prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} V(x)|u_n(x)|^2 \mathrm{d}x = 0.$$

8.3. We consider another proof of Hardy's inequality. Show that

$$\int_{\mathbb{R}^3} |\nabla f(x)|^2 \, \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3} \frac{|f(x)|^2}{|x|^2} \, \mathrm{d}x = \int_{\mathbb{R}^3} \left|\nabla f(x) + \frac{x}{2|x|^2} f(x)\right|^2 \mathrm{d}x, \quad \forall f \in C_c^\infty(\mathbb{R}^3\backslash\{0\}).$$

Then using a density argument to conclude Hardy's inequality for all $f \in H^1(\mathbb{R}^3)$. Deduce also that the constant $1/4$ in Hardy's inequality is sharp.

8.4. We consider the Perron-Frobenius principle and its application to hydrogen atom.
(i) Let Ω be an open subset of $\mathbb{R}^d$. Let $V \in L^1_{\mathrm{loc}}(\Omega)$ and assume that the equation

$$(-\Delta + V)\, f(x) = 0$$

has a solution $0 < f \in C^2(\Omega)$ (in the pointwise sense). Prove that $-\Delta + V \ge 0$, namely

$$\int_\Omega |\nabla u(x)|^2 \mathrm{d}x + \int_\Omega V(x)|u(x)|^2 \mathrm{d}x \ge 0, \quad \forall u \in C_c^1(\Omega).$$

Hint: You can denote $u = fg$ and rewrite the quadratic form in terms of f and g.
(ii) Prove that $-\Delta - |x|^{-1}$, which is a self-adjoint operator on $H^2(\mathbb{R}^3)$, has the lowest eigenvalue $-1/4$ with the eigenvector $f(x) = e^{-|x|/2}$.
Hint: You can use (i) with $\Omega = \{x \in \mathbb{R}^3, x \neq 0\}$ and a density argument.