Mathematisches Institut der LMU
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Functional Analysis II
Winter Semester 2017/18
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Excercise Sheet 7 for 8. 12. 2017

7.1. Let $u \in H^2(\mathbb{R}^d)$ and $\chi \in C_c^\infty(\mathbb{R}^d)$. Prove that $\chi u \in H^2(\mathbb{R}^d)$ and

$$\Delta(\chi u) = (\Delta \chi) u + 2\nabla \chi \cdot \nabla u + \chi(\Delta u)$$

in the sense of weak derivatives.

7.2. Verify the following identity (which was used in the proof of Newton's theorem)

$$\int_{-1}^{1} \frac{1}{\sqrt{1 + r^2 - 2rs}} \mathrm{d}s = \frac{2}{\max(1, r)}, \quad \forall r > 0.$$

7.3. Let μ_1, μ_2 be two Borel, *radial*, probability measures on $\mathbb{R}^3$. Prove that

$$\frac{1}{|z_1 - z_2|} \ge \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\mathrm{d}\mu_1(x - z_1)\mathrm{d}\mu_2(y - z_2)}{|x - y|}, \quad \forall z_1, z_2 \in \mathbb{R}^3.$$

7.4. Assume that $f_n \rightharpoonup f$ weakly in $H^1(\mathbb{R}^d)$ and there exist constants $C > 0$, $s > 0$ such that

$$|f_n(x)| \le \frac{C}{|x|^s}, \quad \forall x \neq 0, \forall n \in \mathbb{N}.$$

Prove that $f_n \to f$ strongly in $L^p(\mathbb{R}^d)$ for all p satisfying: $2 < p < 2d/(d-2)$ if $d \ge 3$; $2 < p < \infty$ if $d = 2$; and $2 < p \le \infty$ if $d = 1$.

Note: In the usual Sobolev embedding, L^p-strong convergence only holds in bounded sets.