Mathematisches Institut der LMU
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Functional Analysis II
Winter Semester 2017/18
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Excercise Sheet 6 for 1. 12. 2017

6.1. Given $p, q, r \geq 1$ such that
$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$
Prove that if $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$ then $f * g \in L^r(\mathbb{R}^d)$ and
$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$
Hint: You can use the Riezs-Thorrin interpolation inequality. In the lecture, we have proved the cases $r = p$ and $r = \infty$.

6.2. Prove that for all $f \in L^p(\mathbb{R}^d)$, $1 \leq p < \infty$ we have
$$\int_{\mathbb{R}^d} |f(x)|^p \, \mathrm{d}x = p \int_0^\infty a^{p-1} |\{x : |f(x)| > a\}| \, \mathrm{d}a.$$
Hint: You can use the layer-cake representation and Fubini theorem.

6.3. Given $g(x) = |x|^{-2}$, $x \in \mathbb{R}^3$. Compute the convolution $g * g$.

6.4. Let $d \in \mathbb{N}$, $0 < s < \min(d/2, 1)$ and $p = 2d/(d - 2s)$. Prove the fractional Sobolev inequality
$$\langle f, (-\Delta)^s f \rangle \geq C \|f\|^2_{L^p(\mathbb{R}^d)}$$
for all $f \in H^s(\mathbb{R}^d)$. Here $C = C(d, s) > 0$ is a constant independent of f.
Hint: You can use the Hardy-Littlewood-Sobolev inequality.