

### Excercise Sheet 5 for 24. 11. 2017

**5.1.** Let  $f : \mathbb{R}^d \rightarrow \mathbb{C}$  be a continuous function with compact support and  $f \not\equiv 0$ . Prove that the multiplication operator  $M_f$  is not a compact operator on  $L^2(\mathbb{R}^d)$ .

**5.2.** Let  $A : D(A) \rightarrow H$  be a self-adjoint operator. For every  $n \in \mathbb{N}$ , define

$$A_n = (A + i)(A + in)^{-1}.$$

Prove that  $A_n$  is bounded and  $\|A_n u\| \rightarrow 0$  as  $n \rightarrow \infty$  for every  $u \in H$ .

Hint: You can use the spectral theorem.

**5.3.** Is it true that the operator  $A_n$  in Problem 5.2 converges to 0 in the operator norm? Prove it for every self-adjoint operator  $A$ , or disprove it by an example.

**5.4.** Let  $A : D(A) \rightarrow H$  be self-adjoint. Let  $B : D(B) \rightarrow H$  be symmetric. Assume that  $D(A) \subset D(B)$  and  $B$  is  $A$ -compact, namely  $B(A + i)^{-1}$  is a compact operator on  $H$ . Prove the operator norm convergence

$$\lim_{n \rightarrow \infty} \|B(A + in)^{-1}\| = 0.$$

Hint: You can use Problem 5.2 and the fact that  $\|X\| = \|X^*\|$ .