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Functional Analysis II
Winter Semester 2017/18
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Excercise Sheet 4 for 17. 11. 2017

Let H be a separable Hilbert space.

4.1. Let $A : D(A) \to H$ be a densely defined operator satisfying $A \geq 1$. Let $Q(u, v)$ be the corresponding quadratic form defined on the form domain $Q(A)$ and let A_0 be the Friedrichs' extension. In the lecture, to prove that A_0 is self-adjoint, we have used the fact that $D(A_0^*) \subset Q(A)$. The purpose of this exercise is to prove this fact.
(i) Take $x \in D(A_0^*)$. Prove that there exists $a \in Q(A)$ such that

$$\langle x, A_0 y \rangle = Q(a, y), \quad \forall y \in D(A_0).$$

(ii) Take $x_n \in D(A)$, $x_n \to x$ in H. Prove that $x_n \rightharpoonup a$ in $Q(A)$. Deduce that $x \in Q(A)$.

4.2. Let $A : D(A) \to H$ be a self-adjoint operator such that

$$\|Ax\| \geq \|x\|, \quad \forall x \in D(A).$$

Prove that A is a bijection from $D(A)$ to H.
Hint: The proof is similar to (even simpler than) the proof of $\mathrm{Ran}(A \pm i) = H$.

4.3. Let $A : D(A) \to H$ be a self-adjoint operator. For every z in the resolvent set $\rho(A)$, denote $R(z) = (A - z)^{-1}$. Prove the resolvent identity

$$\frac{R(z_1) - R(z_2)}{z_1 - z_2} = R(z_2)R(z_1), \quad \forall z_1, z_2 \in \rho(A).$$

Deduce that

$$\frac{d}{dz}\langle x, R(z)x \rangle = \langle x, R(z)^2 x \rangle, \quad \forall z \in \rho(A), \forall x \in H.$$

4.4. Let $\{u_n\}_{n=1}^\infty$ be an orthonomal basis in H and let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of non-negative numbers converging to 0. What is the spectrum of the following operator?

$$Au = \sum_{n=1}^\infty \lambda_n \langle u_n, u \rangle u_n, \quad \forall u \in H.$$