Mathematisches Institut der LMU
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Functional Analysis II
Winter Semester 2017/18
3. 11. 2017

# **Excercise Sheet 3** for 10. 11. 2017

Let $H$ be a separable Hilbert space. Let $A : D(A) \to H$ be an operator which is densely defined and bounded from below, i.e. $A \geq -C$ with a constant $C > 0$.

**3.1.** Recall that the quadratic form $Q(u, v) = \langle u, Av \rangle$, first defined for all $u, v \in D(A)$, can be extended to all $u, v \in Q(A) = \overline{D(A)}^{\|\cdot\|_Q}$ where

$$\|u\|_Q^2 = \langle u, Au \rangle + (C + 1)\|u\|^2.$$

Recall that the Friedrichs' extension $A_0$ of $A$ is defined on

$$D(A_0) = \Big\{ u \in Q(A), \sup_{v \in Q(A), \|v\| \leq 1} \langle u, v \rangle_Q < \infty \Big\}$$

by

$$\langle A_0 u, v \rangle = \langle u, v \rangle_Q, \quad \forall u \in D(A_0), \forall v \in Q(A).$$

Use Riesz representation theorem to explain why $A_0 u$ is well-defined for all $u \in D(A_0)$. Prove that the quadratic form domain $Q(A_0)$ of $A_0$ coincides with $Q(A)$. Deduce that the Friedrichs' extension of $A_0$ coincides with $A_0$.

**3.2.** Recall that the min-max value of $A$ is defined by

$$\mu_n(A) = \inf_{M \subset D(A), \dim M = n} \max_{u \in M, \|u\| = 1} \langle u, Au \rangle.$$

Prove that $\mu_n(A) = \mu_n(A_0)$ for every $n \in \mathbb{N}$.

**3.3.** Prove that for every $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} \mu_n(A) = \inf \Big\{ \sum_{n=1}^{N} \langle u_n, Au_n \rangle : u_i \in D(A),\ \{u_1, \ldots, u_N\} \text{ an orthonormal family in } H \Big\}.$$

**3.4.** Prove that if $\mu_n(A) \to +\infty$ as $n \to \infty$, then $(A_0 + C + 1)^{-1}$ is a compact operator.