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Functional Analysis II
Winter Semester 2017/18
26. 1. 2018

Excercise Sheet 13 for 2. 2. 2018

13.1. Prove that for every constant $0 < \alpha < 1$ and for every $N \in \mathbb{N}$, the operator

$$H_N = \sum_{j=1}^{N} \left(-\Delta_{x_j} - |x_j|^{-\alpha}\right) + \sum_{1 \le i < j \le N} \frac{1}{|x_i - x_j|}, \quad x_j \in \mathbb{R}^3,$$

has a ground state on $H^2((\mathbb{R}^3)^N)$. (Note: The result also holds in the symmetric case and anti-symmetric case.)

Hint: You can use the HVZ theorem and minic the proof of Zhislin's theorem.

13.2. Let $N \in \mathbb{N}$ and let $\{u_j\}_{j=1}^N$ be an orthonormal family in $L^2(\mathbb{R}^d)$. Consider the Slater determinant

$$\Psi_N(x_1, x_2, ..., x_N) = (u_1 \wedge u_2 \wedge ... \wedge u_N)(x_1, x_2, ..., x_N) = \frac{1}{\sqrt{N!}} \det[u_i(x_j)]_{1 \le i,j \le N}.$$

(a) Prove that Ψ_N is an anti-symmetric function in $L^2((\mathbb{R}^d)^N)$ and $\|\Psi_N\|_{L^2} = 1$.

(b) Prove that the one-body density matrix of Ψ_N is the projection:

$$\gamma = \sum_{j=1}^{N} |u_j\rangle\langle u_j|.$$

13.3. For every $N \in \mathbb{N}$, consider the operator

$$H_N = \sum_{j=1}^{N} \Big(-\Delta_{x_j} + |x_j|^2 \Big), \quad x_j \in \mathbb{R}^3,$$

on the anti-symmetric space $L^2_a((\mathbb{R}^3)^N)$. Show that $H_N \ge CN^{4/3}$ with a constant $C > 0$ independent of N.

Hint: You can estimate the sum of negative eigenvalues of $-\Delta + |x|^2 - L$ with $L > 0$.

13.4. Take $Z > 0$ (not necessarily an integer) and $N \in \mathbb{N}$. Consider the atomic Hamiltonian

$$H_N = \sum_{j=1}^{N} \left(-\Delta_{x_j} - Z|x_j|^{-1}\right) + \sum_{1 \le i < j \le N} \frac{1}{|x_i - x_j|}, \quad x_j \in \mathbb{R}^3,$$

on the anti-symmetric space $L^2_a((\mathbb{R}^3)^N)$.

(a) Prove that $H_N \ge -CZ^{7/3}$ with a constant $C > 0$ independent of Z and N.

(b) Assume that $N \ge Z + 1$ and H_N has a ground state Ψ_N. Let $R > 0$ be the "radius" of Ψ_N, namely

$$\int_{|x| \le R} \rho_{\Psi_N}(x)\, \mathrm{d}x = N - 1$$

where ρ_{Ψ_N} is the one-body density of Ψ_N. Prove that

$$CR \ge Z^{-1/3}.$$

Hint: You can first show that $\int Z|x|^{-1} \rho_{\Psi_N}(x)\, \mathrm{d}x \le CZ^{7/3}$.