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Functional Analysis II
Winter Semester 2017/18
19. 1. 2018

Excercise Sheet 12 for 26. 1. 2018

12.1. We prove the Tauberian lemma. Let $0 \le \mu_1 \le \mu_2 \le ...$ be an increasing sequence. Denote $S(\Lambda) := \sum_{i=1}^{\infty} [\mu_i - \Lambda]_-$ and denote $N(\Lambda)$ the number of $\mu_i \le \Lambda$.

(i) Prove that

$$\frac{S(\Lambda + h) - S(\Lambda)}{h} \ge N(\Lambda) \ge \frac{S(\Lambda) - S(\Lambda - h)}{h}, \quad \forall \Lambda > 0, h > 0.$$

(ii) Let $a, b > 0$ be two constants. Prove that the following two statements are equivalent:

$$S(\Lambda) = b\Lambda^{a+1} + o(\Lambda^{a+1})_{\Lambda \to \infty} \iff N(\Lambda) = (a+1)b\Lambda^a + o(\Lambda^a)_{\Lambda \to \infty}.$$

12.2. Let Ω be an open bounded set in $\mathbb{R}^d$. Let μ_i be the i-th min-max value of the Dirichlet Laplacian on Ω. We use the convention $a_\pm = \max(\pm a, 0)$.

(i) Use the Brezin-Li-Yau inequality to prove that for all $\Lambda > 0$,

$$-\sum_{i=1}^{\infty} [\mu_i - \Lambda]_- \ge -\int_{\mathbb{R}^d} \int_{\Omega} [|2\pi k|^2 - \Lambda]_- dk dx = -L_{\text{cl}} |\Omega| \Lambda^{1+d/2}, \quad L_{\text{cl}} = \frac{2}{2+d} \frac{|B_1|}{(2\pi)^d}.$$

(ii) Assume further that $|\partial\Omega| < \infty$ (in the sense of Minkowski content). Show that

$$-\sum_{i=1}^{\infty} [\mu_i - \Lambda]_- = -L_{\text{cl}} |\Omega| \Lambda^{1+d/2} + o(\Lambda^{1+d/2})_{\Lambda \to \infty}.$$

(iii) Deduce Weyl's law for the number of Dirichlet eigenvalues $\le \Lambda$:

$$N(\Lambda) = \frac{|\Omega||B_1|}{(2\pi)^d} \Lambda^{d/2} + o(\Lambda^{d/2})_{\Lambda \to \infty}.$$

12.3. Let $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^a)$ and $v_n \rightharpoonup v$ in $L^2(\mathbb{R}^b)$. Show that

$$u_n \otimes v_n \rightharpoonup u \otimes v \quad \text{in} \quad L^2(\mathbb{R}^{a+b}).$$

Here recall that $(u \otimes v)(x, y) = u(x)v(y)$ with $x \in \mathbb{R}^a, y \in \mathbb{R}^b$.

12.4. Take a constant $Z > 0$ and consider the 2-body Hamiltonian

$$H = -\Delta_{x_1} - \Delta_{x_2} - \frac{Z}{|x_1|} - \frac{Z}{|x_2|} + \frac{1}{|x_1 - x_2|}$$

on $L^2((\mathbb{R}^3)^2)$ (here $x_i \in \mathbb{R}^3$). We know that H is a self-adjoint operator on $H^2((\mathbb{R}^3)^2)$.

(i) Use the HVZ theorem to compute $\sigma_{\text{ess}}(H)$.

(ii) Explain why the multiplication operator $|x_1 - x_2|^{-1}$ is *not* $(-\Delta_{\mathbb{R}^6})$-compact.