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Functional Analysis II
Winter Semester 2017/18
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Excercise Sheet 11 for 19. 1. 2018

11.1. Let $G \in C_c^\infty(\mathbb{R}^d)$, $G(x) = G(-x)$, $\|G\|_{L^2} = 1$ and define the coherent states

$$F_{k,y}(x) = e^{2\pi i k\cdot x} G(x-y), \quad \forall (k,y) \in \mathbb{R}^d \times \mathbb{R}^d.$$

Prove that for every $f \in L^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |\langle F_{k,y}, f\rangle|^2 \, \mathrm{d}y = (|\widehat{G}|^2 * |\widehat{f}|^2)(k).$$

11.2. Consider Weyl's asymptotic upper bound

$$\limsup_{\lambda\to+\infty} -\frac{1}{\lambda^{1+d/2}} \mathrm{Tr}[-\Delta + \lambda V]_- \le -\iint_{\mathbb{R}^d\times\mathbb{R}^d} [|2\pi k|^2 + V(y)]_- \, \mathrm{d}k \, \mathrm{d}y = -L_{\mathrm{cl}} \int_{\mathbb{R}^d} V_-^{1+d/2}.$$

Here we use the convention $a_\pm = \max(\pm a, 0)$. In the lecture, we have proved the upper bound for all $V \in C_c^\infty(\mathbb{R}^d)$. Argue by a density argument that the upper bound holds for all potentials $V \in L^{1+d/2}(\mathbb{R}^d)$ (in fact, the result holds if $V_- \in L^{1+d/2}$).

11.3. Let $-\Delta_{\mathrm{D}}$ be the Dirichlet Laplacian on the unit cube $[0,1]^d$ in $\mathbb{R}^d$. We know that the eigenvalues of $-\Delta_{\mathrm{D}}$ are of the form $|\pi k|^2$ with $k \in \mathbb{N}^d$. Let μ_N be the N-th lowest eigenvalue (with multiplicity). Verify Pólya's conjecture in this case:

$$\mu_N \ge \frac{4\pi^2}{|B_1|^{2/d}} \cdot N^{2/d}, \quad \forall N \in \mathbb{N}.$$

Here $|B_1|$ is the volume of the unit ball in $\mathbb{R}^d$.

11.4. With the Dirichlet eigenvalues on the unit cube $[0,1]^d$, show that

$$K_{\mathrm{cl}} N^{1+2/d} \le \sum_{n=1}^N \mu_n \le K_{\mathrm{cl}} N^{1+2/d} + o(N^{1+2/d})_{N\to\infty}$$

where

$$K_{\mathrm{cl}} = \frac{d}{d+2} \cdot \frac{4\pi^2}{|B_1|^{2/d}}.$$